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LOCAL ISOMETRIC EMBEDDINGS OF SURFACES INTO A 3-SPACE**

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Abstract

In the paper, the authors show that any abstract smooth surface can be locally isometrically embedded into a class of 3-dimensional spaces N_{ρ_0} ($\rho_0 > 0$) with the non-positively sectional curvature being fixed sufficiently small.

Keywords Local isometric embeddings, Smooth surface, Curvature
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§1. Introduction

Let M^2 be a surface with a metric given by $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. The basic question in differential geometric is whether such a metric can be realized locally by a map into a (given) three dimensional Riemannian manifold N (such as S^3, R^3, H^3 which are spaces with constant curvature +1, 0 and −1 respectively)? More precisely, we want to find three C^2 (local) functions: $x^1(u,v)$, $x^2(u,v)$ and $x^3(u,v)$ such that the map $x : M^2 \to N$ defined by $x = (x^1(u,v), x^2(u,v), x^3(u,v))$ satisfies

$$h_{ij}(x)dx^i dx^j = Edu^2 + 2Fdudv + Gdv^2 \tag{1.1}$$

in a neighborhood of a fixed $p \in M^2$, where $h = h_{ij}dx^i dx^j$ is the metric of N.

The most important case is $N = R^3$, and it has been studied classically for a long time. The answer is positive when the metric ds^2 is analytic or the Gauss curvature K of ds^2 is nonvanishing at the point $p \in M^2$. A partial negative answer was given by Pogorelov[14] who showed a $C^{2,1}$ metric (of which the Gauss curvature K is nonnegative) with no C^2 isometric embedding into R^3. A remarkable theorem is due to C. S. Lin[10,11] who showed that a C^k ($k \geq 10$) metric with non-negative Gauss curvature (or a C^6 metric with $K(p) = 0$ and $\nabla K(p) \neq 0$) can be C^{k-6} (or C^6) locally isometrically embedded into R^3. Also Jacobowitz[8] and Poznyak[15] independently showed that any abstract surface can be locally isometrically embedded into the Euclidean 4-space. It should also be mentioned that a series of global isometric embedding results were obtained by Weyl, Nirenberg, Heinz, Hong and Zuily, Guan and Li and others[16,12,6,7,8].

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On the other hand, A. D. Alexandrov, Pogorelov, Nirenberg and others also considered the similar local isometric embedding problem in the case that N is a 3-dimensional sphere $S^3(1)$ or hyperbolic space $H^3(1)$ (see [12] and the references therein).

For such a local isometric embedding problem into a 3-dimensional space with constant curvature, the common argument is to solve the equivalent 2-order Darboux equation. Difficulties arise when one wants to embed locally a surface M^2 (around $p \in M^2$) with the Gauss curvature $K(p) = 1$ into $S^3(1)$ or $K(p) = 0$ into R^3 and $K(p) = -1$ into $H^3(1)$. It seems that if one wants to embed isometrically an abstract surface into a fixed 3-dimensional ambient manifold N, then the difference of the curvature of the surface and the curvature of the ambient manifold will strongly influenced the solvability of the isometric embedding equation. So the problem arises: is there a 3-dimensional ambient manifold N which is very closed to R^3 such that any abstract surface (M^2, ds^2) can be locally isometrically embedded into it? To the authors knowledge, this problem has not been answered completely yet.

In this paper, we will give an affirmative answer to the last question. Namely, we have

Main Theorem. *There exists a class of 3-dimensional Riemannian manifolds N_{ρ_0} (see section 2 below) parametered by $\rho_0 > 0$ with sectional curvature $K_{N_{\rho_0}}$ satisfying*

$$-\frac{1}{\rho_0^2} \le K_{N_{\rho_0}} \le 0$$

such that any smooth abstract surface can be locally isometric embedded into N_{ρ_0} for arbitrary $\rho_0 > 0$.

Corollary 1.1. *Any smooth abstract surface can be locally isometrically embedded into a 3-dimensional Riemannian manifold N with the sectional curvature K_N to be negatively small.*

Proof. The conclusion comes directly from Main Theorem by fixing ρ_0 sufficiently large.

The above corollary can be regarded as an approximate solution to the local isometric embedding problem in 3-Euclidean space R^3.

The paper is organized as follows: In §2 some preliminaries in isometric embedding problem will be reviewed. We will also describe a class of new model spaces and derive the isometric embedding equation in these model spaces. In §3 the local solvability of the isometric embedding equation will be showed for any given abstract surface.

§2. The Isometric Embedding Equation

First of all we recall a well-known basic result as follows, for the detailed proof one may refer to [13].

Lemma 2.1. *Let $ds^2 = g_{ij}du^i du^j$ be a Riemannian metric over an open domain $\Omega \subset R^2$ and $\phi \in C^\infty(\overline{\Omega}, R)$. If $d\overline{s}^2 = ds^2 - d\phi^2$ is also a Riemannian metric over Ω. Then*

$$\widetilde{K} = \frac{1}{1 - |\nabla\phi|^2}\Big[K - \frac{\det(\nabla_{ij}\phi)}{\det(g_{ij})(1 - |\nabla\phi|^2)}\Big], \tag{2.1}$$

where $\widetilde{K}$ and K are the Gauss curvature of the metric $d\overline{s}^2$ and ds^2 respectively, ∇ denotes the gradient operator of ds^2 and ∇_{ij} the covariant derivative in i, j direction with respect to ds^2.

Next, we review carefully some well-known ideas appearing in the study of the local isometric embedding problem in R^3. The ideas will stimulate us in solving the local isometric embedding problem in certain N. In the study of the local isometric embedding problem in $R^3 = \{(x,y,z) \mid ds^2 = dx^2 + dy^2 + dz^2\}$, we mainly solve the Monge-Ampere equation satisfied by a coordinate, say z,

$$\det(\nabla_{ij} z) = K(EG - F^2 - Ez_2^2 - Gz_1^2 + 2Fz_1 z_2), \tag{2.2}$$

instead of solving the one-order nonlinear embedding equation corresponding to (1.1) or more precisely the equation

$$dx^2 + dy^2 + dz^2 = Edu^2 + 2Fdudv + Gdv^2. \tag{2.3}$$

(2.2) is also called the Darboux equation of the isometric embedding (2.3).

There is another form of the Darboux equation which is obtained by introducing the variable $\rho = \frac{1}{2}\langle x, x\rangle$ (where $x = (x,y,z) \in R^3$). By straightforward computation, one obtains the following equation (see [4])

$$\det((\rho_{ij}) - I) = (2\rho - |\nabla\rho|^2)K, \tag{2.4}$$

where ρ_{ij} stands for the covariant derivation of ρ in the i,j direction. The equation (2.4) is equivalent to the fact that $(Edu^2 + 2Fdudv + Gdv^2 - 2\rho^{-1}(d\rho)^2)(2\rho)^{-1}$ has curvature one.

So a different model for the Euclidean 3-space R^3 leads to different (but equivalent each other) Darboux equations (2.2) and (2.4). The trick in the present paper is to find a class of model 3-spaces and apply them in the study of isometric embedding problem.

The usual model spaces $\widetilde{M}^n = \{(\rho,\ th) \in R_+ \times S^{n-1} \mid ds^2 = d\rho^2 + f^2(\rho)d\theta^2\}$, where (ρ,θ) is the polar coordinate system of R^n and $d\theta^2$ denotes the standard spherical metric, were studied carefully by Greene and Wu[5] in their study of the functional properties of a complete Riemannian manifold with a pole. Instead of the above model spaces, we introduce another class of the model spaces as follows

$$\widetilde{N}^n = \{(\rho,\theta) \in R_+ \times H^{n-1}(1) \mid ds^2 = d\rho^2 + f^2(\rho)d\theta^2\}, \tag{2.5}$$

where $d\theta^2$ denotes the standard metric of hyperbolic $(n-1)$-space. For the purpose of the present paper, we only restrict our attention to these model 3-spaces

$$\widetilde{N}^3 = \{(\rho,\theta,\phi) \in R_+ \times H^2(1) \mid ds_f^2 = d\rho^2 + f^2(\rho)(d\theta^2 + \mathrm{ch}^2\theta d\phi^2)\}, \tag{2.6}$$

where $f \in C^\infty(R_+)$ with $f > 0$ everywhere.

Proposition 2.1. *The curvature matrix of* $\widetilde{N}^3$ ((2.6)) *is*

$$\Omega = \begin{pmatrix} 0 & \frac{f''}{f}\omega_1 \wedge \omega_2 & \frac{f''}{f}\omega_1 \wedge \omega_3 \\ -\frac{f''}{f}\omega_1 \wedge \omega_2 & 0 & \frac{1+(f')^2}{f}\omega_2 \wedge \omega_3 \\ -\frac{f''}{f}\omega_1 \wedge \omega_3 & -\frac{1+(f')^2}{f}\omega_2 \wedge \omega_3 & 0, \end{pmatrix},$$

where $\omega_1 = d\rho$, $\omega_2 = f(\rho)d\theta$, $\omega_3 = f(\rho)\mathrm{ch}\theta d\phi$ *the co-framing of* ds_f^2, *i.e.,* $ds_f^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$.

Proof. It follows from a direct computation.

The following corollary indicates that R^3 can be "approximated" by these spaces.

Corollary 2.1. *If* $f(\rho) = \rho_0 + \rho$, *where* ρ_0 *is a fixed positive number, then the Riemannian curvature tensor* R_{ijkl} *of* ds_f^2 *can be expressed as follows*

$$R_{ijkl} = \begin{cases} -\frac{2}{(\rho_0+\rho)^2}, & \textit{when } i = k = 2, \quad j = l = 3, \\ 0, & \textit{else.} \end{cases}$$

Thus the sectional curvature K of this metric satisfies $-\frac{1}{\rho_0^2} \le K \le 0$ which will be negatively small if we let ρ_0 be fixed sufficiently large.

Now we come to our main problem. Since we are working locally, we shall fix a point $p \in M^2$ and work in a small neighborhood, still denoted by M^2, of p. Without loss of generality, we assume $M^2 = \Omega$ to be an open domain of R^2 containing the origin $O = (0,0)$ and $p = O$, the origin. If $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ over Ω, then we want to find three smooth functions $\rho(u,v)$, $\theta(u,v)$, $\phi(u,v)$ such that

$$d\rho^2 + f^2(\rho)(d\theta^2 + \mathrm{ch}^2\theta d\phi^2) = Edu^2 + 2Fdudv + Gdv^2 \tag{2.7}$$

in a neighborhood, still denoted by Ω, of p.

For the model N_{ρ_0} (2.7) becomes

$$d\rho^2 + (\rho_0+\rho)^2(d\theta^2 + \mathrm{ch}^2\theta d\phi^2) = Edu^2 + 2Fdudv + Gdv^2. \tag{2.8}$$

Now we apply Lemma 2.1 to derive the equivalent 2-order Darboux equation of (2.8) as follows.

Let $e^{\omega} = \rho_0 + \rho$, then (2.8) is an equivalent that the metric

$$(Edu^2 + 2Fdudv + Gdv^2)e^{-2w} - dw^2 = d\theta^2 + \mathrm{ch}^2\theta d\phi^2$$

has Gauss curvature -1. Thus we have

$$-1 = \frac{1}{1-|\widetilde{\nabla} w|^2}\Big[\widetilde{K} - \frac{\det(\widetilde{\nabla}_{ij} w)}{\det(\widetilde{g})(1-|\widetilde{\nabla} w|^2)}\Big] \tag{2.9}$$

from Lemma 2.1, where $\widetilde{\nabla}$, $\widetilde{\nabla}_{ij}$ and $\widetilde{K}$ denote the gradient operator, covariant derivative in i,j direction and the Gauss curvature of the metric $\widetilde{g} = (Edu^2 + 2Fdudv + Gdv^2)e^{-2w}$ respectively. Let

$$\gamma_{ij}^k = -\delta_{ik}\frac{\partial w}{\partial u_j} - \delta_{jk}\frac{\partial w}{\partial u_i} + g_{ij}g^{kl}\frac{\partial w}{\partial u_l}, \tag{2.10}$$

where (g_{ij}) denotes the metric tensor of g (i.e., $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$) and (g^{kl}) its inverse matrix, and $(u_1, u_2) = (u, v)$ the coordinate system around $p \in M^2$. From (2.9) we have

$$\begin{aligned}
&\det(\nabla_{ij} w + \gamma_{ij}^k w_k) - (e^{-2w}(EG-F^2) - Ew_2^2 - Gw_1^2 + 2Fw_1w_2)\Delta_g w \\
&= K(e^{-2w}(EG-F^2) - Ew_2^2 - Gw_1^2 + 2Fw_1w_2) \\
&\quad + \frac{1}{EG-F^2}(e^{-2w}(EG-F^2) - Ew_2^2 - Gw_1^2 + 2Fw_1w_2)^2.
\end{aligned} \tag{2.11}$$

This is the Darboux equation corresponding to the ambient space N_{ρ_0}. We shall prove that (2.11) always has a local solution for any given metric g with the aid of the above equation (2.8) in the next section.

§3. Local Solvability

The following statements can be obtained by standard methods in the theory of nonlinear elliptic or hyperbolic equations, which may be found in many detailed books of PDEs (see, for example, [1, 2, 3]).

Lemma 3.1. (a) *Consider the following Dirichlet problem with a parameter ε:*

$$\begin{cases} \sum_{ij}^{2} a_{ij} \frac{\partial^2 w}{\partial u_i \partial u_j} = \varepsilon h(\varepsilon, u, w, \partial w, \partial^2 w), \quad in\ B = \{u_1^2 + u_2^2 < 1\}, \\ w|_{\partial B} = 0, \end{cases} \tag{3.1}$$

where $(a_{ij})_{2\times 2}$ is a uniformly positive definite matrix over B (i.e., strictly elliptic) and $h(\cdot)$ is a smooth function with respect to all the appearing variables $(\varepsilon, u, w, \partial w, \partial^2 w)$, $\partial = \frac{\partial}{\partial u_i}$, $\partial^2 = \frac{\partial^2}{\partial u_i \partial u_j}$ $(1 \le i, j \le 2)$. Then there exists a positive $\varepsilon_0 > 0$ such that problem (3.1) has a solution $w \in C^\infty(B) \cap C^0(\overline{B})$ with w bounded uniformly in $\bar{B}$ for any ε satisfying $-\varepsilon_0 \le \varepsilon \le \varepsilon_0$.

(b) *Consider the following Cauchy problem in $B = \{|\ |u_1| \le \frac{1}{\sqrt{2}}, |u_2| < \frac{1}{\sqrt{2}}\}$ with a parameter ε:*

$$\begin{cases} \sum_{ij}^{2} a_{ij} \frac{\partial^2 w}{\partial u_i \partial u_j} = \varepsilon h(\varepsilon, u, w, \partial w, \partial^2 w), \qquad in\ \ B, \\ w|_{u_1=0} = 0, \quad \frac{\partial w}{\partial u_1}|_{u_1=0} = 0, \quad |u_2| \le \frac{1}{\sqrt{2}}, \end{cases} \tag{3.2}$$

where $(a_{ij})_{2\times 2}$ is a non-degenerate matrix with signature $\{-1, +1\}$ over B (i.e., strictly hyperbolic). Then there exists a positive $\varepsilon_0 > 0$ such that problem (3.2) has a uniformly bounded solution w in a neighborhood, which is the determinate region of the equation (3.2) with respect to the interval $[-\frac{\sqrt{2}}{2}, +\frac{\sqrt{2}}{2}]$, of the origin O, for any ε satisfying $-\varepsilon_0 \le \varepsilon \le \varepsilon_0$.

Our aim is to show that (2.8) or (2.11) always has a local solution for any given (Ω, ds^2). Without loss of generality, we choose the normal coordinate system $(u, v) = (u_1, u_2)$ of $ds^2 = E du_1^2 + 2F du_1 du_2 + G du_2^2$ around a fixed point, say the origin, $O \in \Omega$, i.e., $E(0) = 1$, $F(0) = 0$, $G(0) = 1$ and $\Gamma_{ij}^k(0) = 0$ $(1 \le i, j, k \le 2)$.

In the following, we consider the local solvability of a slightly general equation of (2.11)

$$\det(\nabla_{ij}\omega - \gamma_{ij}^k \omega_k) - a(u, w, \partial w)\Delta\omega = b(u, w, \partial w), \tag{3.3}$$

where $u = (u_1, u_2) \in \Omega \subset R^2$, $a(u, w, \partial u)$, $b(u, w, \partial w)$ are given smooth functions with $b(0) \triangleq b(0, c_0, 0) \ne -a^2(0) \triangleq a^2(0, c_0, 0)$ and $a(0) \ne 0$ for some chosen constant c_0.

The first step for solving (3.3) is to find an approximate solution $w = w_0$ which satisfies (3.3) at the origin $(0, 0)$. For this purpose, we choose $w_0 = c_0 + \frac{a_1 u_1^2 + a_2 u_2^2}{2}$, where the constants a_1, a_2 will be fixed later. Substituting $w = w_0$ into (3.3) and taking the value at $(0, 0)$ we have

$$a_1 \cdot a_2 - a(0)(a_1 + a_2) = b(0).$$

Since $a(0) \ne 0$, we may choose $a_2 = -\frac{b(0)}{a(0)}$ and $a_1 = 0$. Hence $w_0 = c_0 - \frac{b(0)}{2a(0)} u_2^2$ is the desired approximate solution at the origin $(0, 0)$.

Next we let $u_i = \varepsilon^2 x_i$ $(i = 1, 2)$ and try to find $z = z(x)$ such that $w = w_0 + \varepsilon^5 z$ satisfies

$$\Phi(z) \triangleq \widetilde{\Phi}(w_0 + \varepsilon^5 z) = \det(\nabla_{ij} w - \gamma_{ij}^k w_k) - a(u, w, \partial w)\Delta w - b(u, w, \partial w) = 0. \tag{3.4}$$

Lemma 3.2. *$\Phi(z)|_{z=0} = \varepsilon^2 f(x, \varepsilon)$ for some smooth function f.*

Proof. By a direct calculation.

The linearized operator $L_z v$ of $\Phi(z)$ is

$$L_z v = \lim_{t\to 0} \frac{\Phi(z + tv) - \Phi(z)}{t} = \sum_{i,j=1}^{2} \Phi^{ij}(z)\nabla_{ij}(\varepsilon^5 v) - a(z)\Delta(\varepsilon^5 v) + \text{lower order terms}, \tag{3.5}$$

where $a(z) = a(\varepsilon^2 x, w_0 + \varepsilon^5 z, \partial(w_0 + \varepsilon^5 z))$ and

$$(\Phi^{ij}(z))_{2\times 2} = \begin{pmatrix} -\frac{b(0)}{a(0)} + \varepsilon^2 a_{11}(\varepsilon, x, z, Dz, D^2 z) + \varepsilon D_{22} z & \varepsilon^2 a_{12}(\varepsilon, x, z, Dz, D^2 z) + \varepsilon D_{12} z \\ \varepsilon^2 a_{12}(\varepsilon, x, z, Dz, D^2 z) + \varepsilon D_{12} z & \varepsilon^2 a_{22}(\varepsilon, x, z, Dz, D^2 z) + \varepsilon D_{11} z \end{pmatrix},$$

where $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, and $a_{ij}(\varepsilon, x, z, Dz, D^2 z)$ $(1 \le i, j \le 2)$ denote some corresponding smooth functions with respect to ε, x, z, 1-order derivative of z (i.e., Dz) and 2-order derivative of z (i.e., $D^2 z$). Thus (3.5) can be reexpressed as follows

$$\begin{aligned} L_z v &= \varepsilon \Phi^{ij}(z) D_{ij} v - \varepsilon a(z)(D_{11} + D_{22}) v + \varepsilon^2 \text{ lower order terms} \\ &= \varepsilon A^{ij}(z) D_{ij} v + \varepsilon^2 \text{ lower order terms}, \end{aligned} \tag{3.6}$$

where

$$A = (A^{ij}(z)) = \begin{pmatrix} \Phi^{11}(z) - a(z) & \Phi^{12}(z) \\ \Phi^{12}(z) & \Phi^{22}(z) - a(z) \end{pmatrix}. \tag{3.7}$$

Now equation (3.4) reads

$$0 = \Phi(z) = \Phi(w_0 + \varepsilon^5 z) = \widetilde{\Phi}(w_0) + L_0(\varepsilon^5 z) + \varepsilon^2 Q(z, z, \varepsilon) \tag{3.8}$$

for some smooth Q. Substituting (3.5) into (3.7), we have

$$0 = \widetilde{\Phi}(w_0) + \Big[\varepsilon \sum A^{ij}(0) D_{ij} z + \varepsilon^2 \text{ lower order terms}\Big] + \varepsilon^2 Q(z, z, \varepsilon). \tag{3.9}$$

Noticing that $\widetilde{\Phi}(w_0) = \Phi(z)|_{z=0} = \varepsilon^2 f(x, \varepsilon)$ from Lemma 3.2, we finally get the following equation by cancelling ε on both sides,

$$A^{ij}(0) D_{ij} z = \varepsilon h(\varepsilon, x, z, Dz, D^2 z), \tag{3.10}$$

where $h(\varepsilon, x, z, Dz, D^2 z)$ is the algebra sum of the terms with coefficient ε^2 in (3.9).

Next, we will solve equation (3.10) over the disc domain $B = \{x \mid x_1^2 + x_2^2 < 1\}$ in x plane.

Theorem 3.1. *There exists $\varepsilon_0 > 0$ such that equation* (3.10) *has a smooth solution $z = z(x)$ in a neighborhood of the origin for $|\varepsilon| \le \varepsilon_0$.*

Proof. A simple calculation from the above discussion shows that

$$\begin{aligned} A^{11}(0) &= \Phi^{11}(0) - a(0) = -\left[\frac{b(0) + a^2(0)}{a(0)}\right] + \varepsilon^2 a_{11}(\varepsilon, x), \\ A^{12}(0) &= \Phi^{12}(0) = \varepsilon^2 a_{12}(\varepsilon, x), \\ A^{22}(0) &= \Phi^{22}(0) - a(0) = -a(0) + \varepsilon^2 a_{22}(\varepsilon, x) \end{aligned}$$

for some smooth functions $a_{11}(\varepsilon, x)$, $a_{12}(\varepsilon, x)$, $a_{22}(\varepsilon, x)$. From our assumptions, we know that $\frac{b(0)+a^2(0)}{a(0)} \ne 0$ and $a(0) \ne 0$. Thus the discussions are divided into two cases:

(i) $\left[\frac{b(0)+a^2(0)}{a(0)}\right] a(0) > 0$. In this case, it is easy to see that there exists a small $\varepsilon_1 > 0$ such that the left-hand side of equation (3.10) is strictly elliptic over the disc domain B for any ε with $|\varepsilon| \le \varepsilon_1$. Therefore we consider the following Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2} A^{ij}(0) \frac{\partial^2}{\partial x_i \partial x_j} z = \varepsilon h(\varepsilon, x, z, Dz, D^2 z) \ \text{ in } B = \{x \mid x_1^2 + x_2^2 < 1\}, \\ z|_{\partial B} = 0, \end{cases} \tag{3.11}$$

Lemma 3.1 (a) implies that there exists $\varepsilon_2 > 0$ such that (3.11) has a uniformly bounded smooth solution $z = z(x) \in C^\infty(B)$ for any ε with $|\varepsilon| \le \varepsilon_2$. Hence if we choose $\varepsilon_0 =$

$\min\{\varepsilon_1, \varepsilon_2\}$, then equation (3.10) has a uniformly bounded smooth solution $z = z(x) \in C^\infty(B)$ for any ε with $|\varepsilon| \le \varepsilon_0$.

(ii) $\frac{b(0)+a^2(0)}{a(0)} \cdot a(0) < 0$. In this case, it is easy to see that there exists a small $\varepsilon_1 > 0$ such that left-hand side of equation (3.10) is strictly hyperbolic over the disc domain B for any ε with $|\varepsilon| \le \varepsilon_1$. Therefore we consider the following Cauchy problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2} A^{ij}(0)\frac{\partial^2}{\partial x_i \partial x_j} z = \varepsilon h(\varepsilon, x, Dz, D^2 z) \\ \qquad \text{in } B = \big\{(x_1, x_2) \,\big|\, |x_1| \le \dfrac{1}{\sqrt{2}}, |x_2| < \dfrac{1}{\sqrt{2}}\big\}, \\ z|_{x_1=0} = 0, \quad \dfrac{\partial}{\partial x_1} z|_{x_1=0} = 0, \quad |x_2| \le \dfrac{1}{\sqrt{2}}. \end{cases} \tag{3.12}$$

Lemma 3.1 (b) implies that there exists $\varepsilon_2 > 0$ such that (3.12) has a smooth solution $z = z(x) \in C^\infty(\Lambda)$ for any ε with $|\varepsilon| \le \varepsilon_2$, where Λ is the determinate region (in x plane) of the hyperbolic equation (3.12) with respect to the interval $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Obviously Λ is an open neighborhood of the origin in x plane. Hence if we choose $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, the equation (3.10) has a uniformly bounded smooth solution $z = z(x)$ with $z(0) = 0$ in the neighborhood Λ of the origin for any ε with $|\varepsilon| \le \varepsilon_0$. The proof of Theorem 3.1 is completed.

Remark 3.1. Since the solution $z = z(x)$ constructed in Theorem 3.1 is bounded uniformly over the neighborhood of the origin, it is easy to see that the corresponding solution $w = w_0 + \varepsilon^5 z$ of (3.3) has

$$e^w > e^{c_0} - 1$$

in some neighborhood of the origin. In other words, if we set $\rho_0 = e^{c_0} - 1$, then the ρ, which satisfies

$$\rho_0 + \rho = e^w,$$

has $\rho > 0$ in a neighborhood of the origin.

For the proof of the Main Theorem, we need a useful fact as follows, which is observed directly from the 1-order embedding equation (2.8).

Lemma 3.3. *Let (Ω, ds^2) be a surface. If $(\Omega, c^2 ds^2)$ can be locally isometrically embedded into the model space $N_{c\rho_0}$ around a fixed point O for some positive constants c, then (Ω, ds^2) can be locally isometrically embedded into the model space N_{ρ_0} around O.*

Proof. By the assumption in Lemma 3.3, let (ρ, θ, ϕ) be the local realization of $(\Omega, c^2 ds^2)$ into $N_{c\rho_0}$ around O. Then for the corresponding 1-order isometric embedding equation (2.8) for $f(\rho) = c\rho_0 + \rho$, it is direct to see from the equation that $(\rho/c, \theta, \phi)$ is in fact a local solution of (2.8) for $f(\rho) = \rho_0 + \rho$ which is the corresponding 1-order isometric embedding equation of the surface (Ω, ds^2) into the ambient space N_{ρ_0}.

Now we are in a position to give a complete proof of the Main Theorem.

Proof of Main Theorem. For the fixed model space $N_{\rho_0} (\rho_0 > 0)$, we consider the isometric embedding of a surface (Ω, ds^2) into it around the origin O. By Lemma 3.3 we only solve the same question of the surface $(\Omega, c^2 ds^2)$ into $N_{c\rho_0}$ around O for some sufficiently large constant $c > 0$, where c will be specialized later. Notice that the Gauss curvature of the metric $c^2 ds^2$ at the origin is $K(0)/c^2$, where $K(x)$ is the Gauss curvature of the metric

ds^2. In this situation, one can rewrite the corresponding equation (2.11) (under the normal coordinate system) in the form of equation (3.3) with

$$a(0) = e^{-2c_0} \neq 0,$$
$$b(0) = K(0)/c^2 e^{-2c_0} + e^{-4c_0},$$

where the constant c_0 is chosen as $c_0 = \ln(c\rho_0 + 1)$. For applying Theorem 3.1, we must verify that the condition $b(0) + a(0)^2 \neq 0$ in Theorem 3.1 is valid in this case. In fact it is easy to see that

$$b(0) + a(0)^2 = \frac{1}{(c\rho_0 + 1)^2}\Big(K(0)/c^2 + \frac{2}{(c\rho_0 + 1)^2}\Big).$$

Hence for any given $K(0)$, we can choose a positive c such that

$$K(0)/c^2 + 2/(c\rho_0 - 1)^2 \neq 0.$$

Thus $(\Omega, c^2 ds^2)$ can be locally isometrically embedded into $N_{c\rho_0}$ from Theorem 3.1 and so does the (Ω, ds^2) into N_{ρ_0} by Lemma 3.3.

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