Chin. Ann. of Math.
20B: 2(1999),223-232.

NONLINEAR STABILITY OF RAREFACTION WAVES FOR A RATE-TYPE VISCOELASTIC SYSTEM***

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Abstract

The authors study a 3×3 rate-type viscoelastic system, which is a relaxation approximation to a 2×2 quasi-linear hyperbolic system, including the well-known p-system. It is shown that the rarefaction waves are nonlinear asymptotically stable in this relaxation approximation.

Keywords Nonlinear stability, Rarefaction waves, Relaxation approximation
1991 MR Subject Classification 35L60, 35F25
Chinese Library Classification O175.27

§1. Introduction

In this paper, we study the following rate-type viscoelastic system, i.e.,

$$\begin{cases} v_t - u_x = 0, \qquad u_t + p_x = 0, \\ (p+Ev)_t = \dfrac{p_R(v) - p}{\tau}, \end{cases} \tag{1.1}$$

where v and $(-p)$ denote strain and stress, u is related to the particle velocity, E is a positive constant called the dynamic Young's modulus, $\tau > 0$ is a relaxation time.

This system was proposed in [16] to introduce a relaxation approximation to the following system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_R(v)_x = 0. \end{cases} \tag{1.2}$$

Since the system (1.2) can be obtained from (1.1) by an expansion procedure as the first order, it is natural to expect that the solution of (1.1) converges to that of (1.2) as $\tau \to 0$. However, the zero limit convergence has not been established yet, although some numerical experiments on (1.1) have been made[14] and certain effort on the L^2-estimates for the difference $|p - p_R(v)|$ of (1.1) have been done[2].

A tightly related problem is the nonlinear stability of waves for this relaxation approximation. As far as shock waves of (1.2) are concerned, the stability results have been proved in [4, 8]. In the present paper, we investigate the asymptotic stability of rarefaction waves for this relaxation approximation. For any given suitably weak rarefaction waves for the reduced system (1.2), we consider an initial value, which is a small perturbation of the rarefaction wave, and prove that the solution of this initial value problem for (1.1) exists globally and

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***Project supported by the National Natural Science Foundation of China.

converges, in L^∞-norm, to the rarefaction wave, as $t \to +\infty$. Namely the rarefaction wave is a global attractor for (1.1), or the stability of rarefaction wave is obtained.

To approximate a hyperbolic system of conservation laws, the viscosity method has been usually used, we refer to [3, 10, 13, 17] and the references therein. A different approximation method is to introduce some relaxation mechanism. Compared with viscosity, the dissipation of relaxation is weaker. This makes differences between these approximations.

For relaxation approximation, the stability of elementary waves has been proved in [7] when the corresponding equilibrium equation is scalar. Here, the corresponding equilibrium system (1.2) is a 2×2 system, more difficulties occur certainly. Different from the stability of shock profile, which is compressible and is the exact solution of (1.1), the rarefaction waves are expansive and can not solve (1.1) exactly.

As far as the multi-dimensional case is concerned, we refer [9] and [12] in which the stability for planar rarefaction waves and shock profiles are obtained respectively for a relaxation model where the corresponding equilibrium equations is scalar.

The organization for this paper is as follows. In section 2, we give the rarefaction wave solutions of the Riemman problem for (1.2) and their smooth approximations which are named expansion waves, for which important properties have been established[13,17]. In section 3 and section 4, we will prove the stabilities of the rarefaction waves constructed in section 2. The energy method is used to get some key estimates for two different cases.

§2. Preliminaries

Consider the following Riemann problem

$$\begin{cases} v_t - u_x = 0, \\ u_t + (p_R(v))_x = 0, \end{cases} \tag{2.1}$$

$$(v(x,0), u(x,0)) = (v_0^r(x), u_0^r(x)), \tag{2.2}$$

where

$$(v_0^r(x), u_0^r(x)) = \begin{cases} (v_-, u_-), & x < 0, \\ (v_+, u_+), & x > 0, \end{cases}$$

with (v_-, u_-) and (v_+, u_+) being two constant states.

We give the following hypotheses: for some constants c_1 and d_1 such that $-\infty < c_1 < v_-, v_+ < d_1 < +\infty$, it holds

(H$_1$) $p_R'(v) < -a_1 < 0$, (H$_2$) $p_R''(v) > a_2 > 0$,

with some positive constants a_1 and a_2,

(H$_3$) $|p_R'(v)| < E$, (H$_4$) $p_R(v)$, p_R', p_R'', p_R''' are bounded,

where $v \in [c_1, d_1]$.

(H$_3$) is so-called subcharacteristic condition[7].

It is easy to see that, under (H$_1$)–(H$_2$), (2.1) is strictly hyperbolic and genuinely nonlinear, with eigenvalues

$$\lambda_1 = -(-p_R'(v))^{\frac{1}{2}} < 0 < (-p_R'(v))^{\frac{1}{2}} = \lambda_2. \tag{2.3}$$

It is well known[1] that, to connect (v_-, u_-) to (v_+, u_+) by centred rarefaction waves, we have the following cases:

(1) Single Mode Case. (v_+, u_+) is on the k-rarefaction wave curve $R_k(v_-, u_-)$ ($k = 1$, or 2). In this case, the rarefaction wave solution (v^r, u^r) is a k-centred rarefaction wave connecting (v_-, u_-) to (v_+, u_+) .

(2) Two Modes Case. We can find a unique state on the 1-rarefaction wave curve $R_1(v_-, u_-)$, i.e., $(\overline{v}, \overline{u}) \in R_1(v_-, u_-)$, such that (v_+, u_+) is on the 2-rarefaction wave curve $R_2(\overline{v}, \overline{u})$. In this case, we denote the 1-rarefaction wave connecting (v_-, u_-) to $(\overline{v}, \overline{u})$ as (v_1^r, u_1^r) and the 2-rarefaction wave connecting $(\overline{v}, \overline{u})$ to (v_+, u_+) as (v_2^r, u_2^r), then the corresponding rarefaction waves solution can be defined as

$$\begin{cases} \tilde{v}^r(x,t) = v_1^r(x,t) + v_2^r(x,t) - \overline{v}, \\ \tilde{u}^r(x,t) = u_1^r(x,t) + u_2^r(x,t) - \overline{u}. \end{cases} \tag{2.4}$$

For the single mode case, (v^r, u^r) is completely determined by the following relations[6,15]

$$\begin{cases} u^r(x,t) - u_- = -\displaystyle\int_{v_-}^{v^r(x,t)} \lambda_k(v)dv, \\ \lambda_k(v^r)(x,t) = w^r(x,t), \end{cases} \tag{2.5}$$

where $w^r(x,t)$ is the solution of the following problem

$$\begin{cases} w_t^r + (\dfrac{w^{r2}}{2})_x = 0, \\ w^r(x,0) = w_0^r(x) \end{cases} \tag{2.6}$$

with

$$w_0^r(x) = \begin{cases} \lambda_k(v_-) \text{ for } x < 0, \\ \lambda_k(v_+) \text{ for } x > 0. \end{cases}$$

Now we define (V, U) by the following relations

$$\begin{cases} U(x,t) - u_- = -\displaystyle\int_{v_-}^{V(x,t)} \lambda_k(v)dv, \\ \lambda_k(V)(x,t) = W(x,t), \end{cases} \tag{2.7}$$

where $W(x,t)$ is the solution of the following problem

$$\begin{cases} W_t + \Big(\dfrac{W^2}{2}\Big)_x = 0, \\ W(x,0) = W_0(x) \end{cases} \tag{2.8}$$

with $W_0(x) = \frac{1}{2}(\lambda_k(v_-) + \lambda_k(v_+)) + \frac{1}{2}[(\lambda_k(v_+) - \lambda_k(v_-))]\tanh x$. By the characteristic method, the following lemmas can be easily verified[17].

Lamma 2.1. *Under* (H_1)–(H_2)*, there exists a smooth function* $(V(x,t), U(x,t))$*, which is the smooth approximation of* (v^r, u^r) *in the following sense:*

(1)

$$\begin{cases} V_t - U_x = 0, \\ U_t + (p_R(V))_x = 0, \end{cases}$$

(2) $$\lim_{t\to+\infty} \sup_{x\in\mathbf{R}^1} \{|v^r(x,t) - V(x,t)| + |u^r(x,t) - U(x,t)|\} = 0.$$

Lemma 2.2. *The smooth function* $(V(x,t), U(x,t))$ *in Lemma* 2.1 *has the following properties:*

(1) $\frac{\partial V}{\partial t} > 0,\ \forall x \in \mathbf{R}^1,\ t \geq 0;$

(2) $\forall p \in [1, +\infty]$, $\exists c_p > 0$*, s.t.* $\forall t \geq 0$,

$$\|(V_x, U_x)\|_{L^p} \leq c_p \delta^{\frac{1}{p}} (1+t)^{-1+\frac{1}{p}}, \quad \|(V_x, U_x)\|_{L^\infty} \leq c_\infty \delta;$$

(3) *for* $j \geq 2$, $\forall p \in [1, +\infty)$, $\exists c_{p,j} > 0$, *s.t.*, $\forall t \geq 0$,

$$\Big\| \frac{\partial^j}{\partial x^j}(V, U) \Big\|_{L^p} \leq c_{p,j}\delta(1+t)^{-1},$$

(4) $\exists c > 0$, *s.t.* $|V_t| \leq c|V_x|$, $|U_t| \leq c|U_x|$, *where* δ *is the strength of the wave, namely,* $\delta \equiv |v_+ - v_-| + |u_+ - u_-|$.

For two modes case, $(v_i^r(x,t), u_i^r(x,t))$ $(i = 1,\ 2)$ can be determined by the similar way as (2.5)-(2.6). Similar to (2.7)-(2.8), we define $(\tilde{V}_i, \tilde{U}_i)(i = 1,\ 2)$ by the following relations respectively,

$$\begin{cases} \tilde{U}_1(x,t) - u_- = -\displaystyle\int_{v_-}^{\tilde{V}_1(x,t)} \lambda_1(v)dv, \\ \lambda_1(\tilde{V}_1)(x,t) = \tilde{W}_1(x,t), \end{cases} \tag{2.9}$$

$$\begin{cases} \tilde{U}_2(x,t) - \overline{u} = -\displaystyle\int_{\overline{v}}^{\tilde{V}_2(x,t)} \lambda_2(v)dv, \\ \lambda_2(\tilde{V}_2)(x,t) = \tilde{W}_2(x,t), \end{cases} \tag{2.10}$$

where $\tilde{W}_i(x,t)(i = 1,\ 2)$ is the solution of the following problem

$$\begin{cases} (\tilde{W}_i)_t + \Big(\dfrac{\tilde{W}_i^2}{2}\Big)_x = 0, \\ \tilde{W}_i(x,0) = \tilde{W}_0^i(x) \end{cases} \tag{2.11}$$

with

$$W_0^1(x) = \frac{1}{2}(\lambda_1(\overline{v}) + \lambda_1(v_-)) + \frac{1}{2}[(\lambda_1(\overline{v}) - \lambda_1(v_-))]\tanh x$$

or

$$W_0^2(x) = \frac{1}{2}(\lambda_2(v_+) + \lambda_2(\overline{v})) + \frac{1}{2}[(\lambda_2(v_+) - \lambda_2(\overline{v}))]\tanh x,$$

respectively.

Moreover, it can be verified that

Lemma 2.3. *Under* (H_1)–(H_2), *there exist smooth functions* $(\tilde{V}_i(x,t), \tilde{U}_i(x,t))$ $(i = 1,\ 2)$, *satisfying*

(1)

$$\begin{cases} (\tilde{V}_i)_t - (\tilde{U}_i)_x = 0, \\ (\tilde{U}_i)_t + (p_R(\tilde{V}_i))_x = 0, \end{cases}$$

(2) $\displaystyle\lim_{t\to+\infty} \sup_{x\in\mathbf{R}^1} \{|v_i^r(x,t) - \tilde{V}_i(x,t)| + |u_i^r(x,t) - \tilde{U}_i(x,t)|\} = 0.$

Now we set

$$(\tilde{V}, \tilde{U}) = (\tilde{V}_1 + \tilde{V}_2 - \overline{v}, \tilde{U}_1 + \tilde{U}_2 - \overline{u}). \tag{2.12}$$

It is easy to know that, there exists a positive constant α such that

$$\tilde{V} = \begin{cases} \tilde{V}_1 + F_1(x,t) \text{ on } \Omega_1, \\ \tilde{V}_2 + F_1(x,t) \text{ on } \Omega_2; \end{cases} \tag{2.13}$$

$$\tilde{U} = \begin{cases} \tilde{U}_1 + F_2(x,t) \text{ on } \Omega_1, \\ \tilde{U}_2 + F_2(x,t) \text{ on } \Omega_2, \end{cases} \tag{2.14}$$

where $F_i(x,t) = O(1)\delta\exp[-\alpha(t+|x|)]$, $\Omega_1 = \{(x,t)|x \le 0, t \ge 0\}$, $\Omega_2 = \{(x,t)|x \ge 0, t \ge 0\}$. Then we conclude by Lemma 2.3 that $(\tilde V, \tilde U)$ satisfies (2.1) approximately with an exponential error, i.e.,

$$\begin{cases} \tilde V_t - \tilde U_x = 0, \\ \tilde U_t + (p_R(\tilde V))_x = (G(\tilde V))_x, \end{cases} \tag{2.15}$$

where

$$G(\tilde V) = p_R(\tilde V) - p_R(\tilde V_1) - p_R(\tilde V_2) + p_R(\overline{v}), \tag{2.16}$$

$$\frac{\partial^j}{\partial x^j}(G(\tilde V)) = O(1)\delta\exp[-\alpha(t+|x|)]. \tag{2.17}$$

Noticing that the solutions of (2.11) are monotonely increasing, it is easy to show that[17]:

$$\frac{\partial \tilde V}{\partial t} > 0. \tag{2.18}$$

Furthermore, by the results for the single mode case, we have the following lemma.

Lemma 2.4. *The smooth function* $(\tilde V(x,t), \tilde U(x,t))$ *has the following properties:*

(1) $\frac{\partial \tilde V}{\partial t} > 0,\ \forall x \in \mathbf{R}^1,\ t \ge 0$;

(2) $\forall p \in [1,+\infty]$, $\exists c_p > 0$, *s.t.* $\forall t \ge 0$

$$\|(\tilde V_x, \tilde U_x)\|_{L^p} \le c_p \delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}, \quad \|(\tilde V_x, \tilde U_x)\|_{L^\infty} \le c_\infty \delta;$$

(3) *for* $j \ge 2$, $\forall p \in [1,+\infty)$, $\exists c_{p,j} > 0$, *s.t.*, $\forall t \ge 0$,

$$\Big\|\frac{\partial^j}{\partial x^j}(\tilde V, \tilde U)\Big\|_{L^p} \le c_{p,j}\delta(1+t)^{-1},$$

(4) $\exists c > 0$, *s.t.* $|\tilde V_t| \le c|\tilde V_x|,\ |\tilde U_t| \le c|\tilde U_x|$;

(5) *for any positive integer* j *and* $\forall p \in [1,+\infty]$, $\exists c'_{p,j} > 0$, *and* $\alpha > 0$, *independent of* t, *satisfying* $\|\frac{\partial^j}{\partial x^j}G(\tilde V)\|_{L^p} \le c'_{p,j}\delta\exp(-\alpha t),\ \forall t \ge 0$.

All the above discussions can be found in [13, 17], the readers are refered there for the detail.

We will use energy method with the help of Lemmas 2.1–2.4 to estabilish the stability results for the single mode case in Section 3 and the two modes case in Section 4.

§3. Stability Analysis—Single Mode Case

Since we are interested in the large time behavior for fixed τ, we may assume $\tau = 1$ in system (1.1), without loss of generality, i.e., we will consider

$$\begin{cases} v_t - u_x = 0, \qquad u_t + p_x = 0, \\ (p + Ev)_t = p_R(v) - p, \end{cases} \tag{3.1}$$

with initial data

$$(v(x,0), u(x,0), p(x,0)) = (v_0(x), u_0(x), p_0(x)). \tag{3.2}$$

Suppose the Riemann data (v_-, u_-) and (v_+, u_+) in (2.2) can be connected by a k-centred rarefaction wave (v^r, u^r), constructed in Section 2, where $k = 1$, or 2, fixed. We denote $p^r(x,t) = p_R(v^r)$, and $p_0^r(x) = p_R(v_0^r(x))$.

The purpose in this section is to show that, if the rarefaction wave is weak (i.e. δ is small), then (v^r, u^r, p^r) is a global attractor for (3.1). More precisely, it says that

Theorem 3.1. *Under* (H_1)–(H_4), *suppose* (v_-, u_-) *and* (v_+, u_+) *can be connected by* (v^r, u^r), *then there exist positive constants* δ_0 *and* ε_0, *such that if* $\delta < \delta_0$ *and*

$$\|(v_0 - V(x,0), u_0 - U(x,0), p_0 - P(x,0))\|_{H^1} \le \varepsilon_0,$$

then the problem (3.1)–(3.2) *has a unique smooth global solution* (v, u, p)*, which tends to* (v^r, u^r, p^r) *uniformly in* x *as* $t \to +\infty$*, where* (V, U) *is defined in Section* 2*, and* $P = p_R(V)(x, t)$*.*

To prove this theorem, we introduce

$$(\phi, \psi, w) = (v, u, p) - (V, U, P). \tag{3.3}$$

Then (3.1), (3.3) and Lemma 2.1 give

$$\begin{cases} \phi_t - \psi_x = 0, \qquad \psi_t + w_x = 0, \\ w_t + E\psi_x + w + (p_R(V) - p_R(V + \phi)) + (E + p_R'(V))U_x = 0, \\ (\phi, \psi, w)(x, 0) = (v, u, p)(x, 0) - (V, U, P)(x, 0). \end{cases} \tag{3.4}$$

We denote

$$L_1 \equiv \phi_t - \psi_x = 0. \tag{3.5}$$

Combining $(3.4)_2$ and $(3.4)_3$, one can easily get

$$L_2 \equiv \psi_{tt} - E\phi_{xx} + \phi_t - A(V, \phi)_x - B(V, U_x)_x = 0 \tag{3.6}$$

with

$$A(V, \phi) = p_R(V) - p_R(V + \phi), \tag{3.7}$$

$$B(V, U_x) = [E + p_R'(V)]U_x. \tag{3.8}$$

It is clear that (3.5)–(3.6) give a closed system for (ϕ, ψ). We consider (3.5)–(3.6) with initial data

$$\begin{cases} \phi(x, 0) = \phi_0(x) = v_0(x) - V(x, 0), \\ \psi(x, 0) = \psi_0(x) = u_0(x) - U(x, 0), \\ \psi_t(x, 0) = \psi_1(x) = p_0'(x) - P'(x, 0). \end{cases} \tag{3.9}$$

By virtue of Lemma 2.1, it is easy to know that one only needs to show Theorem 3.2 in order to prove Theorem 3.1.

Theorem 3.2. *Under* (H_1)–(H_4)*, suppose* (v_-, u_-) *and* (v_+, u_+) *can be connected by* (v^r, u^r)*, then there exist positive constants* δ_0 *and* ε_0*, such that if* $\delta < \delta_0$ *and*

$$\|(\phi_0, \psi_0, w)\|_{H^1} \le \varepsilon_0,$$

then the problem (3.4) *has a unique smooth global solution* (ϕ, ψ, w)*, which tends to* $(0, 0, 0)$ *uniformly in* x *as* $t \to +\infty$*.*

We will solve the Cauchy problem (3.4) in the space $X(0, T) = \{(\phi, \psi, w) \in C^0(0, T; H^1)\}$, for some $T > 0$. Firstly, we proceed the a priori estimate for the solution of (3.5)–(3.9), then the bounds on w can be derived from (3.4). In the following, we always assume a priorily that $(\phi, \psi, w) \in X(0, T)$ is the solution of (3.4) for some $T > 0$.

Let

$$\gamma^2 := \sup_{0 \le t \le T} (\|(\phi, \psi, w)\|_1^2(t)). \tag{3.10}$$

To prove Theorem 3.2, we need the following a priori estimates.

Lemma 3.3. *Suppose the conditions in Theorem* 3.2 *are satisfied,* $\delta < \delta_0$*, and* $\gamma \le \varepsilon_0$*, then it holds*

$$\sup_{0 \le t \le T} \|(\phi, \psi, w)(t)\|_1^2 + \int_0^T \|(\phi_x, \psi_x, w_x)(t)\|^2 dt \le K(\|(\phi, \psi, w)(\cdot, 0)\|_1^2 + \delta_0)$$

for $(\phi, \psi, w) \in X(0, T)$*, where* $K > 1$ *is a positive constant which does not depend on* T*.*

To prove this lemma, we establish the following Lemmas 3.4–3.5 next.

By Sobolev embedding theorem, $H^{m+1} \hookrightarrow C^m$, $m \geq 0$. Thus if $\gamma \leq \varepsilon_0$, then

$$\|(\phi, \psi, w)\|_{C^0} \leq C\varepsilon_0.$$

From these facts, we know that there are constants $-\infty < c < d < +\infty$ such that $c > c_1$ and $d < d_1$, and $v \in [c, d]$.

Lemma 3.4. *Suppose the conditions in Lemma* 3.3 *are satisfied,* $\delta \leq \delta_0$*, and* $\gamma \leq \varepsilon_0$ *for some suitably small* δ_0 *and* ε_0*, then we have*

$$\|(\phi, \psi, \psi_t, \psi_x)(t)\|^2 + \int_0^t \|(\psi_x, \psi_t, V_t^{\frac{1}{2}}\phi)(\tau)\|^2 d\tau \leq C(\|(\phi, \psi)(\cdot, 0)\|_1^2 + \delta_0). \tag{3.11}$$

Proof. We consider the equality

$$AL_1 + (\mu\psi_t + \psi)L_2 = 0 \tag{3.12}$$

with a positive constant $\mu = \frac{E_1+E}{2E_1}$, and $E_1 = \sup\limits_{v\in[c,d]} |p'_R(v)|$. By Taylor's formula, it follows

$$A = -p'_R(V)\phi + g(V, \phi)\phi^2, \tag{3.13}$$

where $g(V, \phi)$ is a smooth function. Therefore, (3.12) can be reduced into

$$(G_4 + G_5) + \sum_{l=6}^{10} G_l + G_{11x} = 0, \tag{3.14}$$

where

$$\begin{aligned}
&G_4 = \tfrac{1}{2}\psi^2 + \tfrac{\mu}{2}\psi_t^2 + \psi\psi_t, && G_5 = \tfrac{1}{2}\Big(D - \tfrac{1}{3}g\phi\Big)\phi^2 + \mu D\phi\psi_x + \tfrac{1}{2}\mu\psi_x^2,\\
&G_6 = (E + \mu(p'_R(V) - g_\phi\phi^2 - 2g\phi))\psi_x^2, && G_7 = (\mu - 1)\psi_t^2,\\
&G_8 = \tfrac{1}{2}\Big(p''_R(V) + \tfrac{1}{3}g_V\phi\Big)V_t\phi^2, && G_9 = \mu(p''_R(V) - g_V\phi)V_t\phi\psi_x,\\
&G_{10} = \tfrac{1}{3}g_\phi\phi^3\psi_x - B_x(\psi + \mu\psi_t), && G_{11} = -(A\psi + E\psi\psi_x + \mu E\psi_t\psi_x + \mu A\psi_t).
\end{aligned} \tag{3.15}$$

Due to (H_1)–(H_3), and the smallness of δ_0 and ε_0, it holds that $E > E_1 + c_1\varepsilon_0 > D - \frac{1}{3}g\phi > c_2 > 0$, and $E > E_1 + c_1\varepsilon_0 > D > c_2 > 0$, for some positive constants c_1 and c_2. Thus, there are positive constants $c_i (i = 3, \cdots, 9)$ such that

$$\begin{gathered}
c_3(\psi^2 + \psi_t^2) \leq G_4 \leq c_4(\psi^2 + \psi_t^2), \quad c_5(\phi^2 + \psi_x^2) \leq G_5 \leq c_6(\phi^2 + \psi_x^2),\\
G_6 + G_7 \geq c_7(\psi_x^2 + \psi_t^2), \quad G_8 \geq c_8 V_t\phi^2, \quad |G_9| \leq \frac{1}{2}c_8 V_t\phi^2 + c_9\delta_0\psi_x^2.
\end{gathered} \tag{3.16}$$

Now we integrate (3.14) over $[0, t] \times (-\infty, +\infty)$. Integrating by parts, we arrive at

$$\begin{aligned}
&\|(\phi, \psi, \psi_t, \psi_x)\|^2(t) + \int_0^t \|(\psi_t, \psi_x, V_t\phi^2)\|^2(\tau)\ d\tau\\
&\leq C\|(\phi, \psi, w)(\cdot, 0)\|_1^2 + \Big|\int_0^t \int_{-\infty}^{+\infty} G_{10}\ d\tau\Big|.
\end{aligned} \tag{3.17}$$

We estimate each term in G_{10} next.

Using Young's inequality and Lemma 2.2, it can be shown that

$$\begin{aligned}
\int_0^t \int_{-\infty}^{+\infty} |B_x\psi_t|\ d\tau &\leq \alpha_1 \int_0^t \int_{-\infty}^{+\infty} \psi_t^2\ d\tau + C(\alpha_1)\int_0^t \int_{-\infty}^{+\infty} B_x^2\ d\tau\\
&\leq \alpha_1 \int_0^t \|\psi_t\|^2(\tau)\ d\tau + C(\alpha_1)\delta_0^2,
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&\Big|\int_0^t\int_{-\infty}^{+\infty}B_x\psi\ d\tau\Big|\le C\int_0^t\|\psi\|^{\frac12}\|\psi_x\|^{\frac12}\|B_x\|_{L^1}\ d\tau\\ \le C&\int_0^t(\|\psi\|^2\|\psi_x\|^2+\|B_x\|_{L^1}^{\frac43})d\tau\le C(\delta_0^{\frac43}+\varepsilon_0^2\int_0^t\|\psi_x\|^2d\tau),\end{aligned}\tag{3.19}$$

where we have used the Sobolev inequality. We also note that

$$\Big|\int_0^t\int_{-\infty}^{+\infty}\psi_x\phi^3d\tau\Big|\le\alpha_2\int_0^t\|\psi_x\|^2(\tau)d\tau+C(\alpha_2)\int_0^t\int_{-\infty}^{+\infty}\phi^6d\tau,\tag{3.20}$$

$$\int_0^t\int_{-\infty}^{+\infty}(\phi^6)d\tau\le\int_0^t\|\phi\|^4\|\phi_x\|^2dxd\tau\le\varepsilon_0^4\int_0^t\|\phi_x\|^2(\tau)\ d\tau.\tag{3.21}$$

Thus, for suitably small α_1 and α_2, (3.17)–(3.21) imply that

$$\begin{aligned}&\|(\psi,\psi_t,\phi,\psi_x)(t)\|^2+\int_0^t\|(\psi_t,\psi_x,V^{\frac12}\phi)(\tau)\|^2d\tau\\ &\le C\varepsilon_0^4\int_0^t\|\phi_x(\tau)\|^2d\tau+C\delta_0^{\frac43}+C\|(\phi,\psi,w)(\cdot,0)\|_1^2.\end{aligned}\tag{3.22}$$

To bound ϕ_x, we investigate the following equation

$$\begin{aligned}0&=(E\phi_x-\psi_t)\partial_xL_1-\phi_xL_2\\ &=(\frac12E\phi_x^2-\psi_t\phi_x-\frac12\psi_x^2)_t+(\psi_t\psi_x)_x+A_x\phi_x+\phi_x\psi_t+B_x\phi_x,\end{aligned}\tag{3.23}$$

where

$$A_x\phi_x=(-p_R'(V)+g_\phi\phi^2+2g\phi)\phi_x^2+g_VV_x\phi^2\phi_x-p_R''(V)V_x\phi\phi_x.\tag{3.24}$$

Then the Cauchy inequality and (3.22) yield

$$\|\phi_x\|^2(t)+\int_0^t\|\phi_x\|^2(\tau)d\tau\le C\delta_0^{\frac43}+C\|(\phi,\psi,w)(\cdot,0)\|_1^2.\tag{3.25}$$

(3.22) and (3.25) imply the Lemma 3.4.

The next aim is to deduce the estimate on w by the equations (3.4) and the above results. In fact, we can show the following lemma.

Lemma 3.5. *Under the conditions cited in Lemma* 3.3, *we have*

$$\|(w,w_x,w_t)(t)\|^2+\int_0^t\|(w_x,w_t)(\tau)\|^2d\tau\le C(\|(\phi,\psi,w)(\cdot,0)\|_1^2+\delta_0).\tag{3.26}$$

Proof. By (3.4), we see that $w_x=-\psi_t$, thus the estimate of w_x in (3.22) comes from Lemma 3.4 directly.

Turn to w and w_t next. We know from (3.4) that

$$L_3\equiv w_{tt}-Ew_{xx}+w_t+A_t+B_t=0.\tag{3.27}$$

Thus

$$\begin{aligned}w_tL_3&=w_tw_{tt}-Ew_tw_{xx}+w_t^2+A_tw_t+B_tw_t\\ &=\Big(\frac12w_t^2+\frac12Ew_x^2\Big)_t-E(w_xw_t)_x+w_t^2-A_tw_t-B_tw_t=0.\end{aligned}\tag{3.28}$$

Integrating (3.28) over $[0,t]\times(-\infty,+\infty)$, integrating by parts and using the Cauchy inequality with the estimate in Lemma 3.4, we have

$$\|w_t\|^2(t)+\int_0^t\|w_t\|^2(\tau)d\tau\le C(\|(\phi,\psi,w)(0)\|_1^2+\delta_0).\tag{3.29}$$

At last, we can get the estimate on w by taking L^2-norm in the third equation of (3.4) directly.

Lemmas 3.4–3.5 imply Lemma 3.3. Namely, suppose $(\phi, \psi, w) \in X(0,T)$ is the smooth solution of (3.4) for some $T>0$, then it holds that

$$\|(\phi,\psi,w)\|_{H^1}^2 + \int_0^T \|(\phi_x,\phi_t,\psi_x,\psi_t,w_x,w_t)(\tau)\|^2\,d\tau \le C.$$

Since the local (in time) existence and uniqueness of the solution for initial value problem (3.4) can be obtained by standard procedure, it follows from Lemma 3.3 and a standard continuity argument[5] that the problem (3.4) has a unique global (in time) solution $(\phi,\psi,w) \in X(0,+\infty)$, satisfying, for any $t \ge 0$, that

$$\|(\phi,\psi,w)\|_{H^1}^2 + \int_0^t \|(\phi_x,\phi_t,\psi_x,\psi_t,w_t,w_x)(\tau)\|^2\,d\tau \le C. \tag{3.30}$$

Due to (3.30), it is known that

$$\int_0^t \Big|\frac{d}{dt}\int_{-\infty}^{+\infty} \phi_x^2(x,\tau)\,dx\Big|d\tau \le C.$$

Thus

$$\int_0^{+\infty} \Big(\|\phi_x(t)\|^2 + \Big|\frac{d}{dt}\|\phi_x(t)\|^2\Big|\Big)dt < +\infty.$$

It follows then that $\lim\limits_{t\to+\infty}\|\phi_x(t)\|^2 = 0$. So the Sobolev's inequality implies

$$\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}|\phi(x,t)| \le \lim_{t\to+\infty}(\|\phi(t)\|\|\phi_x(t)\|)^{\frac12} \le C\lim_{t\to+\infty}\|\phi_x(t)\|^{\frac12} = 0.$$

Similarly, it can be proved that

$$\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}|(\psi(x,t),w(x,t))| = 0.$$

This completes the proof of Theorem 3.2.

§4. Stability Analysis—Two Modes Case

Consider the following problem:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (p+Ev)_t = p_R(v) - p, \end{cases} \tag{4.1}$$

with initial data

$$(v(x,0),u(x,0),p(x,0)) = (v_0(x),u_0(x),p_0(x)). \tag{4.2}$$

We assume the Riemann data (v_-,u_-) and (v_+,u_+) can be connected by a 1-centred rarefaction wave and a 2-centred rarefaction wave successively. We will use the same symbols as defined in Section 2 for the two modes case. We denote $\tilde p^r(x,t) = p_R(v^r(x,t))$, and $\tilde p_0^r(x) = p_R(v_0^r(x))$.

We will show in this section that, if the rarefaction wave is weak (i.e. δ is small), then $(\tilde v^r, \tilde u^r, \tilde p^r)$ is a global attractor for (4.1). Namely,

Theorem 4.1. *Under* (H_1)–(H_4), *suppose* (v_-,u_-) *and* (v_+,u_+) *can be connected by* $(\tilde v^r, \tilde u^r)$, *then there exist positive constants* δ_0 *and* ε_0, *such that if* $\delta<\delta_0$ *and*

$$\|(v_0-\tilde V(x,0), u_0-\tilde U(x,0), p_0-\tilde P(x,0))\|_{H^1} \le \varepsilon_0$$

then the problem (4.1)–(4.2) *has a unique smooth global solution* (v, u, p), *which tends to* $(\tilde{v}^r, \tilde{u}^r, \tilde{p}^r)$ *uniformly in* x *as* $t \to +\infty$. *Here* $(\tilde{V}, \tilde{U})$ *is defined in Section* 2, *and* $\tilde{P} = p_R(\tilde{V})$.

This theorem can be proved by the similar way as in Section 3. Since we have not used the structure of rarefaction waves in the proofs for the estimates in Section 3, we are able to get the estimates required here by the same arguments as used in the proofs of Lemma 3.3 with only few modifications. The difference here is caused by the exponential small term $G(\tilde{V})_x + G(\tilde{V})_{xt}$ (see (2.16)–(2.17) for the discussion on $G(\tilde{V})$), which can be easily estimated. For instance,

$$\begin{aligned} &\Big|\int_0^t \int_{-\infty}^{+\infty} [G(\tilde{V})_{xt} + G(\tilde{V})_x]\tilde{\psi}\ dxd\tau\Big| \\ &\le C\gamma \int_0^t \int_{-\infty}^{+\infty} |G(\tilde{V})_{xt} + G(\tilde{V})_x|\ dxd\tau \\ &\le C\gamma \int_0^t \delta \exp(-\alpha\tau)\ d\tau \le C\delta\gamma. \end{aligned}$$

We omit the details.

Acknowledgement. The authors would like to thank Dr. Tao Luo for helpful discussions.

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