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MELNIKOV FUNCTIONS AND PERTURBATION OF A PLANAR HAMILTONIAN SYSTEM***

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Abstract

In this paper, Melnikov functions which appear in the study of limit cycles of a perturbed planar Hamiltonian system are studied. There are two main contributions here. The first one is related to the explicit formulae for these functions: a new method is developed to achieve the goal for the second one (Theorem A). the authors also discover a close relation between Melnikov functions and focal quantities (Theorem B). This relation is useful in both judging when a Melnikov function is identically zero and simplifying the computation of a Melnikov function (see §5). Despite these results, this paper also includes other related results, e.g. some estimations of the upper bound for the number of limit cycles in a perturbed Hamiltonian system.

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§1. Introduction

Consider the following planar Hamiltonian system

$$\dot{x} = f(x) = J \cdot DH(x), \tag{1.1}$$

where $x \in \mathbb{R}^2$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, D denotes the derivative operator, and the Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ is C^∞. Suppose that (1.1) has a family of closed orbits $\Gamma_h : \ x = q(h,t), \ \ 0 \le t \le T(h), \ h_0 < h < h_1$, satisfying $H(q(h,t)) \equiv h$. Here $T(h)$ is the period of Γ_h, and the initial values $q(h,0)$ form a curve $L = \{x = q(h,0)| \ h_0 < h < h_1\}$. We assume that $q(h,0)$ is C^∞ for $h \in (h_0,h_1)$, and $|\frac{d}{dh}q(h,0)| > a$ for some positive constant a.

Now let us consider the following one-parameter perturbation of the above system

$$\dot{x} = f(x) + \varepsilon g(x,\varepsilon), \tag{1.2}$$

where $g(x,\varepsilon)$ is C^∞ so that we have expansion $g(x,\varepsilon) = \sum\limits_{j=0}^{k} g_{j+1}(x)\varepsilon^j + O(\varepsilon^{k+1})$ for any $k \ge 1$. According to our assumption, the curve L is transversal to the vector field defined by system (1.2) when $|\varepsilon|$ is sufficiently small. Thus the Poincare map $P(h,\varepsilon)$ with respect to L is well-defined for $h \in (h_0,h_1), \ |\varepsilon| < \varepsilon_0(h)$. Let $x(t,h,\varepsilon)$ be the solution of system

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(1.2) satisfying $x(0, h, \varepsilon) = q(h, 0)$, $T = T(h, \varepsilon)$ be the minimal positive number such that $x(T, h, \varepsilon) \in L$ (note that $T(h) = T(h, 0)$). Then $P(h, \varepsilon) = H(x(T, h, \varepsilon))$, which is also uniquely determined by the equation $q(P(h, \varepsilon), 0) = x(T, h, \varepsilon)$. (Here we identify a point on L with its parameter h.) Now let us consider the successor function $\Psi(h, \varepsilon) = P(h, \varepsilon) - h$. It is obviously C^∞, so that we have the expansion

$$\Psi(h, \varepsilon) = \sum_{j=1}^{k} M_j(h)\varepsilon^j + O(\varepsilon^{k+1})$$

for any $k \geq 1$ where $M_j(h)$ is the so-called j-th Melnikov function.

It is well-known that the number of limit cycles of system (1.2) can be estimated by using the first non-identically-zero Melnikov function $M_k(h)$. Thus it is important to find explicit formulae for these functions. This goal has achieved only for $M_1(h)$. Recently Zhang Zhifen and Li Baoyi[6,7] derived a function of the form

$$\widetilde{M}_2(h) = \oint_{\Gamma_h} \Big[f \wedge g_2 + \mathrm{div}(g_1) \int_0^t f \wedge g_1 \, ds \Big] dt \tag{1.3}$$

$$= \oint_{\Gamma_h} \Big[f \wedge g_2 - (f \wedge g_1) \int_0^t \mathrm{div}(g_1) \, ds \Big] dt, \tag{1.4}$$

under the restriction $T'(h) > 0$, and obtained some interesting results on the bifurcation of system (1.2) in the case $M_1(h) \equiv 0$. Notice that the definitions of $M_2(h)$ and $\widetilde{M}_2(h)$ are different, although $\widetilde{M}_2(h)$ is called as the second Melnikov function in their papers.

Motivated by this work, we consider the following questions in the presenr paper:

(1) what is the explicit formula for $M_k(h)$, $k > 1$;

(2) how to know when $M_1(h) = \cdots = M_{k-1}(h) \equiv 0$;

(3) how about the bifurcation of system (1.2).

The first question is studied in §2 where it is successfully proved that $M_2(h) = \widetilde{M}_2(h)$ without any restriction. More exactly, we will prove the following theorem:

Theorem A. *If $M_1(\alpha) = 0$ for some fixed $\alpha \in (h_0, h_1)$, then $M_2(\alpha) = \widetilde{M}_2(\alpha)$.*

To give some answer to the second question, we establish in §3 a close relation between Melnikov functions and focal quantities. Suppose Γ_{h_0} is an elementary focus or center for small ε, whose j-th focal quantity is $v_{2j+1}(\varepsilon)$. Then we have

Theorem B. (a) *Let $M_j^*(\sqrt{h - h_0}) = M_j(h)$, then $M_j^*(s)$ is C^∞ at $s = 0$. If f and g in* (1.2) *are analytic, so is $M_j^*(s)$ at $s = 0$.*

(b) *If system* (1.2) *is analytic on the plane, then*

(1) *$M_1(h) = \cdots = M_k(h) \equiv 0$ if and only if*

$$v_{2i+1}(\varepsilon) = O(\varepsilon^{k+1}) \text{ for any } i; \tag{1.5}$$

(2) *$M_1(h) = \cdots = M_k(h) \equiv 0$ but $M_{k+1}(h) \not\equiv 0$ if and only if* (1.5) *holds. And moreover, there exists some natural number $m > 0$ such that*

$$\begin{cases} v_{2i+1}(\varepsilon) = O(\varepsilon^{k+2}), \ i = 1, 2, \cdots, m-1, \\ v_{2m+1}(\varepsilon) = b_m \varepsilon^{k+1} + O(\varepsilon^{k+2}), \ b_m \neq 0. \end{cases} \tag{1.6}$$

In this case, $M_k(h) = b_m (h - h_0)^{m+1}(1 + O(\sqrt{h - h_0}))$ as $h \to h_0$.

The third question has been solved for the bifurcation at Γ_h, $h \in (h_0, h_1)$ (see, for example, [9]). However only partial results are known for the case $h = h_0$ or h_1 (see [6,7]).

We generalized these results in §4 (Theorems 4.1 and 4.2).

The calculation in §5 is an application of Theorem B. For a typical perturbation (system (5.4)) of the general quadratic Hamiltonian system, this theorem is sufficient to determine when a Melnikov function is identically zero. Our calculation shows that the first 6 Melnikov functions are involved in such a simple case (Theorem 5.1). Thus it seems that a complete study of a perturbed planar Hamiltonian system would be very complicated.

Notations. Throughout this paper, the right-hand side of the notation $\stackrel{\text{def.}}{=}$ are new symbols introduced. The wedge product $\wedge$ is defined by the equality $(a,b)\wedge(c,d) = ad - bc$. And $O(*)$ denotes the asymptotic estimation $|O(*)| < A \cdot |*|$ in certain limit process which will not be explicitly written out. The reader can easily find the limit process by requiring $* \to 0$. For example, the limit process for $O(\varepsilon)$ is $\varepsilon \to 0$.

§2. Formulae for Melnikov Functions

In this section we give a method to derive the formula for $M_j(h)$, and then prove Theorem A. To begin with, we first introduce the following change of variables

$$x = q\Big(h, \frac{T(h)}{2\pi}\theta\Big) \stackrel{\text{def.}}{=} G(h,\theta), \tag{2.1}$$

and prove the following lemma.

Lemma 2.1. *System* (1.2) *is changed by* (2.1) *into the following* 2π*-periodic system*

$$\frac{dh}{d\theta} = \frac{\varepsilon T(h) f\wedge g}{2\pi\big(1-\varepsilon\frac{\partial G}{\partial h}\wedge g\big)} \stackrel{\text{def.}}{=} \sum_{i=1}^{k} H_i(h,\theta)\varepsilon^i + O(\varepsilon^{k+1}), \tag{2.2}$$

where

$$\begin{aligned} H_1(h,\theta) &= \frac{T(h)}{2\pi} f(G)\wedge g_1(G),\\ H_2(h,\theta) &= \frac{T(h)}{2\pi}\Big[f(G)\wedge g_2(G) + \Big(\frac{\partial G}{\partial h}\wedge g_1\Big)(f\wedge g_1)\Big]. \end{aligned} \tag{2.3}$$

Proof. It is easy to see that

$$H(G(h,\theta)) \equiv h, \quad \frac{\partial G}{\partial\theta} = \frac{T(h)}{2\pi} f(G). \tag{2.4}$$

From (2.1) and (2.4), we have

$$DH(G)\cdot\frac{\partial G}{\partial h} = f(G)\wedge\frac{\partial G}{\partial h} = 1, \tag{2.5}$$

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial G}{\partial h}\cdot\frac{dh}{dt} + \frac{\partial G}{\partial\theta}\cdot\frac{d\theta}{dt}\\ &= f(G) + \varepsilon g(G,\varepsilon). \end{aligned} \tag{2.6}$$

Thus, by (2.5) and (2.6), we obtain

$$\frac{dh}{dt} = \varepsilon f(G)\wedge g(G,\varepsilon), \tag{2.7}$$

$$\frac{T(h)}{2\pi}\cdot\frac{d\theta}{dt} = 1 - \varepsilon\frac{\partial G}{\partial h}\wedge g(G,\varepsilon). \tag{2.8}$$

Now (2.2) follows from (2.7) and (2.8), whereas (2.3) is obtained by direct calculation. The proof is finished.

Suppose

$$h(\theta,\alpha,\varepsilon)=\sum_{i=0}^{k}h_i(\theta)\varepsilon^i+O(\varepsilon^{k+1}) \tag{2.9}$$

is the solution of (2.2) satisfying $h(0,\alpha,\varepsilon)=\alpha$. We find that

$$\sum_{j=0}^{k}h_j'(\theta)\varepsilon^j+O(\varepsilon^{k+1})=\sum_{j=1}^{k}H_j\Big(\sum_{i=0}^{k}h_i(\theta)\varepsilon^i,\theta\Big)\varepsilon^j+O(\varepsilon^{k+1})$$
$$\overset{\text{def.}}{=}\sum_{j=1}^{k}S_j(\theta,h_0(\theta),\cdots,h_{j-1}(\theta))\varepsilon^j+O(\varepsilon^{k+1}),$$

where

$$\begin{aligned}S_1\big(\theta,h_0(\theta)\big)&=H_1\big(h_0(\theta),\theta\big),\\ S_2(\theta,h_0(\theta),h_1(\theta))&=H_2(h_0(\theta),\theta)+\frac{\partial H_1}{\partial h}(h_0(\theta),\theta)h_1(\theta),\end{aligned} \tag{2.10}$$

and, in general, S_j is a C^∞ function in j variables. By comparing the coefficients of ε^j in the above equality, we obtain

$$h_0'(\theta)=0,\quad h_j'(\theta)=S_j\Big(\theta,h_0(\theta),\cdots,h_{j-1}(\theta)\Big),\quad j=1,2,\cdots. \tag{2.11}$$

Since $h(0,\alpha,\varepsilon)\equiv\alpha$, we have $h_0(0)=\alpha,\ h_j(0)=0,\ j=1,2,\cdots$. Thus

$$h_0(\theta)\equiv\alpha,\quad h_j(\theta)=\int_0^\theta S_j(\theta,\alpha,h_1(\theta),\cdots,h_{j-1}(\theta))\,d\theta,\quad j=1,2,\cdots. \tag{2.12}$$

From (2.9) we obtain

$$h(2\pi,\alpha,\varepsilon)-\alpha=\sum_{j=1}^{k}h_j(2\pi)\varepsilon^j+O(\varepsilon^{k+1}). \tag{2.13}$$

Now we can prove the following lemma, which (and (2.12)) provides a method to derive explicit formulae for Melnikov functions.

Lemma 2.2. *$M_j(\alpha)=h_j(2\pi),\ j=1,2,\cdots$. In particular,*

$$\begin{aligned}M_1(\alpha)&=\int_0^{2\pi}H_1(\alpha,\theta)\,d\theta=\oint_{\Gamma_\alpha}f(x)\wedge g_1(x)\,dx,\\ M_2(\alpha)&=\int_0^{2\pi}\Big[H_2(\alpha,\theta)+\frac{\partial H_1}{\partial h}(\alpha,\theta)h_1(\theta)\Big]\,d\theta.\end{aligned} \tag{2.14}$$

Proof. It suffices to prove $P(\alpha,\varepsilon)=h(2\pi,\alpha,\varepsilon)$. Notice that (2.8) has a solution $\theta=\theta(t,\alpha,\varepsilon)$ with $\theta(0,\alpha,\varepsilon)=0$ when h is replaced by $h(\theta,\alpha,\varepsilon)$ in this equation. Whereas $x(t,\alpha,\varepsilon)=G(h(\theta(t,\alpha,\varepsilon),\alpha,\varepsilon),\theta(t,\alpha,\varepsilon))$ is a solution of (1.2) satisfying $x(0,\alpha,\varepsilon)=q(\alpha,0)$. By the definition of $T(\alpha,\varepsilon)$ (see §1), we must have

$$\begin{aligned}\theta(T(\alpha,\varepsilon),\alpha,\varepsilon)&=2\pi,\\ x(T(\alpha,\varepsilon),\alpha,\varepsilon)&=G(h[\theta(T(\alpha,\varepsilon),\alpha,\varepsilon),\alpha,\varepsilon],\theta(T(\alpha,\varepsilon),\alpha,\varepsilon))\\ &=G(h(2\pi,\alpha,\varepsilon),2\pi)=q(h(2\pi,\alpha,\varepsilon),0).\end{aligned}$$

Thus $h(2\pi,\alpha,\varepsilon)=H(q(h(2\pi,\alpha,\varepsilon),0))=P(\alpha,\varepsilon)$. This ends the proof.

Now we can prove Theorem A. Let α be a fixed value of h.

Proof of Theorem A. We obtain from (2.3) and (2.13) that

$$\begin{aligned} M_2(\alpha) &= \frac{T(\alpha)}{2\pi}\int_0^{2\pi} f(G)\wedge g_2(G)\,d\theta \\ &\quad + \frac{T(\alpha)}{2\pi}\int_0^{2\pi}\Big(\frac{\partial G}{\partial h}\wedge g_1(G)\Big)\,(f(G)\wedge g_1(G))\,d\theta \\ &\quad + \int_0^{2\pi}\frac{\partial}{\partial h}\Big[\frac{T(h)}{2\pi}f(G)\wedge g_1(G)\Big]h_1(\theta)\,d\theta \\ &\overset{\text{def.}}{=} M_{21}(\alpha)+M_{22}(\alpha)+M_{23}(\alpha). \end{aligned} \tag{2.15}$$

Since $M_1(\alpha)=h_1(2\pi)=0$, we have

$$\begin{aligned} \int_0^{2\pi}\frac{T'(\alpha)}{2\pi}f(G)\wedge g_1(G)h_1(\theta)\,d\theta &= \frac{T'(\alpha)}{T(\alpha)}\int_0^{2\pi}h_1'(\theta)h_1(\theta)\,d\theta \\ &= \frac{T'(\alpha)}{2T(\alpha)}M_1^2(\alpha)=0. \end{aligned}$$

Therefore,

$$\begin{aligned} M_{23}(\alpha) &= \int_0^{2\pi}\frac{\partial}{\partial h}\Big[\frac{T(h)}{2\pi}f(G)\wedge g(G)\Big]h_1(\theta)\,d\theta \\ &= \int_0^{2\pi}\frac{T(\alpha)}{2\pi}\Big[\Big(Df(G)\frac{\partial G}{\partial h}\Big)\wedge g_1(G) \\ &\quad + f(G)\wedge\Big(Dg_1(G)\frac{\partial G}{\partial h}\Big)\Big]h_1(\theta)\,d\theta. \end{aligned} \tag{2.16}$$

Integrating $M_{22}(\alpha)$ by parts, we obtain

$$\begin{aligned} M_{22}(\alpha) &= \int_0^{2\pi}\frac{T(\alpha)}{2\pi}(f(G)\wedge g_1(G))\Big(\frac{\partial G}{\partial h}\wedge g_1(G)\Big)\,d\theta \\ &= \int_0^{2\pi}h_1'(\theta)\frac{\partial G}{\partial h}\wedge g_1(G)\,d\theta \\ &= \int_0^{2\pi}h_1(\theta)\frac{\partial}{\partial\theta}\Big(\frac{\partial G}{\partial h}\wedge g_1(G)\Big)\,d\theta. \end{aligned} \tag{2.17}$$

Notice that at $h=\alpha$,

$$\begin{aligned} &\frac{\partial}{\partial\theta}\Big(\frac{\partial G}{\partial h}\wedge g_1(G)\Big) = \frac{\partial^2 G}{\partial\theta\partial h}\wedge g_1(G)+\frac{\partial G}{\partial h}\wedge\frac{\partial}{\partial\theta}g_1(G) \\ &= \frac{\partial}{\partial h}\Big[\frac{T(h)}{2\pi}f(G)\Big]\wedge g_1(G)+\frac{\partial G}{\partial h}\wedge\Big[Dg_1(G)\frac{\partial G}{\partial\theta}\Big] \\ &= \Big[\frac{T'(\alpha)}{2\pi}f(G)+\frac{T(\alpha)}{2\pi}Df(G)\frac{\partial G}{\partial h}\Big]\wedge g_1(G)+\frac{T(\alpha)}{2\pi}\frac{\partial G}{\partial h}\wedge(Dg_1(G)\cdot f(G)) \\ &= \frac{T'(\alpha)}{T(\alpha)}h_1'(\theta)+\frac{T(\alpha)}{2\pi}\Big[\Big(Df(G)\frac{\partial G}{\partial h}\Big)\wedge g_1(G)+\frac{\partial G}{\partial h}\wedge(Dg_1(G)f(G))\Big]. \end{aligned} \tag{2.18}$$

Combining (2.16), (2.17) and (2.18), we have

$$M_{22}(\alpha)+M_{23}(\alpha)=\int_0^{2\pi}\frac{T(\alpha)}{2\pi}\Big[f(G)\wedge\Big(Dg_1(G)\frac{\partial G}{\partial h}\Big)-\frac{\partial G}{\partial h}\wedge(Dg_1(G)f(G))\Big]h_1(\theta)\,d\theta.$$

It can be directly verified that

$$f(G)\wedge\Big(Dg_1(G)\frac{\partial G}{\partial h}\Big)-\frac{\partial G}{\partial h}\wedge(Dg_1(G)f(G))=\operatorname{div}(g_1(G))DH(G)\cdot\frac{\partial G}{\partial h},$$

where $\operatorname{div}\big(g_1(G)\big)$ is the divergence of the vector function $g_1(G)$. Using (2.5), we obtain

$$\begin{aligned} M_{22}(\alpha)+M_{23}(\alpha) &= \int_0^{2\pi}\frac{T(\alpha)}{2\pi}\operatorname{div}(g_1(G))h_1(\theta)\,d\theta \\ &= \int_0^{T(\alpha)}\operatorname{div}(g_1(q(\alpha,t)))h_1\left(\frac{2\pi t}{T(\alpha)}\right)\,dt \\ &= \int_0^{T(\alpha)}\Big[\operatorname{div}(g_1(q(\alpha,t)))\int_0^t f(q(\alpha,s))\wedge g_1(q(\alpha,s))\,ds\Big]\,dt \\ &= \oint_{\Gamma_\alpha}\Big[\operatorname{div}(g_1)\int_0^t f(x)\wedge g_1(x)\,ds\Big]\,dt. \end{aligned} \tag{2.19}$$

Combining (2.15) and (2.19) we obtain (1.3), whereas (1.4) comes from (1.3) by integrating by parts. The proof of the theorem is finished.

Corollary 2.1. *$M_2(h)\equiv\tilde M_2(h)$ if $M_1(h)\equiv 0$ for $h\in(h_0,h_1)$.*

§3. Properties of Melnikov Functions

The calculation in the last section shows that $M_1(h)$ and $M_2(h)$ are independent of the choice of the curve L. In fact we have the following general fact.

Theorem 3.1. *Let $L_i, i=1,2,$ be two curves parametrized by $H(x)=h$. If they are transversal to the vector field defined by system (1.2), then we can define the maps $P_i(h)$ and $\Psi_i(h)$, $i=1,2$, as done for the curve L in §1. Suppose*

$$\Psi_1(h)=M_k^{(1)}(h)\varepsilon^k(1+O(\varepsilon)),\quad M_k^{(1)}(h)\not\equiv 0.$$

Then we have

$$\Psi_2(h)=M_k^{(1)}(h)\varepsilon^k(1+O(\varepsilon)).$$

Proof. Define $\pi: L_2\to L_1$ such that $\pi(h)\in L_1$ is the first point at which L_1 intersects the positive semi-trajectory starting from $h\in L_2$. It is easy to see that

$$\pi(h)=h(1+O(\varepsilon)),\quad P_2(h)=\pi^{-1}\circ P_1\circ\pi(h).$$

Using the Lagrange mean value theorem, we obtain

$$\begin{aligned} \Psi_2(h) &= P_2(h)-h=\pi^{-1}\circ P_1\circ\pi(h)-\pi^{-1}\circ\pi(h) \\ &= (\pi^{-1})'(\xi)(P_1\circ\pi(h)-\pi(h))=(\pi^{-1})'(\xi)\Psi_1(\pi(h)) \\ &= (1+O(\varepsilon))\Psi_1(h(1+O(\varepsilon)))=M_k^{(1)}(h)\varepsilon^k(1+O(\varepsilon)). \end{aligned}$$

This is the conclusion of the theorem.

Remark 3.1. Only the first non-identically-zero Melnikov function is important. In fact, we will show in §4 that the number of limit cycles of system (1.2) can be estimated from above by this function. Thus Theorem 3.1 tells us that when computing $M_j(h)$, we can choose any particular L without changing its value.

Our next aim is to prove Theorem B. Without loss of generality, we assume

$$f(0)=g(0,\varepsilon)=0,\quad H(x)=x_1^2+x_2^2+O(|x|^3),\quad h_0=0, \tag{3.1}$$

so that system (1.2) has an elementary center or a focus at the origin when $|\varepsilon|\ll 1$. To study $M_j(h)$ near $h=0$, we choose $L=\{(x_1,0)|\ 0<x_1\ll 1\}$ (see Remark 3.1). Consider

the polar coordinate transformation $x = ru(\theta)$, $u(\theta) = (\cos\theta, \sin\theta)$. Notice that

$$f + \varepsilon g = \frac{dx}{dt} = u(\theta)\frac{dr}{dt} + ru'(\theta)\frac{d\theta}{dt}, \quad u(\theta)\cdot u'(\theta) = 0.$$

So we have

$$\frac{dr}{d\theta} = \frac{r[f(ru(\theta)) + \varepsilon g(ru(\theta),\varepsilon)]\cdot u(\theta)}{[f(ru(\theta)) + \varepsilon g(ru(\theta),\varepsilon)]\cdot u'(\theta)} \overset{\text{def.}}{=} \sum_{i=0}^{k} R_j(\theta, r)\varepsilon^j + O(r\varepsilon^{k+1}), \tag{3.2}$$

where R_j is 2π-periodic in θ and $R_j(\theta, 0) \equiv 0$. Let

$$r(\theta, \beta, \varepsilon) \overset{\text{def.}}{=} \sum_{j=0}^{k} r_j(\theta, \beta)\varepsilon^j + O(\beta\varepsilon^{k+1})$$

be a solution of equation (3.2) satisfying $r(0, \beta, \varepsilon) \equiv \beta$. It is easy to see that $r_j(\theta, \beta)$ depends only on $R_0, \cdots, R_j, r_0(\theta, \beta), \cdots, r_{j-1}(\theta, \beta)$. Set

$$W_j(\beta) = r_j(2\pi, \beta), \quad j = 1, 2, \cdots.$$

Since $r(2\pi, \beta, 0) = \beta$, we have the following successor function for equation (3.2)

$$\overline{\Psi}(\beta, \varepsilon) = r(2\pi, \beta, \varepsilon) - \beta = \sum_{i=1}^{k} W_j(\beta)\varepsilon^j + O(\beta\varepsilon^{k+1}). \tag{3.3}$$

It is related to the function $\Psi(\beta, \varepsilon)$ by the following equalities

$$h = H(\beta, 0) = \beta^2 + \cdots, \tag{3.4}$$

$$\Psi(h, \varepsilon) = H(\beta + \overline{\Psi}(\beta, \varepsilon), 0) - H(\beta, 0), \tag{3.5}$$

where $\cdots$ denotes the higher order terms. From (3.5) and (3.3), we have

$$\begin{aligned}\Psi(h, \varepsilon) &= H\Big(\beta + \sum_{j=1}^{k} W_j(\beta)\varepsilon^j + O(\beta\varepsilon^{k+1}), 0\Big) - H(\beta, 0)\\ &\overset{\text{def.}}{=} \frac{\partial H}{\partial x_1}(\beta, 0)W_1(\beta)\varepsilon + \sum_{j=2}^{k}\Big[N_j(\beta, W_1, \cdots, W_{j-1})\\ &\quad + \frac{\partial H}{\partial x_1}(\beta, 0)W_j(\beta)\Big]\varepsilon^j + O(\beta\varepsilon^{k+1}),\end{aligned}$$

where $N_2(\beta, W_1) = \frac{1}{2}\frac{\partial^2 H}{\partial x_1^2}(\beta, 0)W_1^2(\beta)$, and generally

$$N_j = F\Big(\frac{\partial^2 H}{\partial x_1^2}(\beta, 0), \cdots, \frac{\partial^j H}{\partial x_1^j}(\beta, 0), W_1, \cdots, W_{j-1}\Big)$$

for some polynomial F with constant coefficients. Furthermore

$$N_j(\beta, 0, \cdots, 0) = 0, \quad j = 2, 3, \cdots. \tag{3.6}$$

By the definition of Melnikov functions, we obtain

$$\begin{cases} M_1(h) = \dfrac{\partial H}{\partial x_1}(\beta, 0)W_1(\beta), \\ M_2(h) = \dfrac{1}{2}\dfrac{\partial^2 H}{\partial x_1^2}(\beta, 0)W_1^2(\beta) + \dfrac{\partial H}{\partial x_1}(\beta, 0)W_2(\beta), \\ M_j(h) = N_j\big(\beta, W_1(\beta), \cdots, W_{j-1}(\beta)\big) + \dfrac{\partial H}{\partial x_1}(\beta, 0)W_j(\beta), \quad j = 3, \cdots. \end{cases} \tag{3.7}$$

Proof of Theorem B. By the implicit function theorem, we have a function H^* such that $h = H(H^*(\sqrt{h}), 0)$. It is easy to see that $H^*(s)$ is C^∞ at $s = 0$; if system (1.2) is analytic, then so is $H^*(s)$. Now part (a) in Theorem B follows easily from the relation $\beta = H^*(\sqrt{h})$ (see (3.4)). Further we can easily obtain from (3.6) and (3.7) the conclusion that $M_j(h) \equiv 0,\ j = 1, \ldots, k,\ 0 < h \ll 1$, for some natural number $k \geq 1$ if and only if

$$W_j(\beta) \equiv 0 \ \text{ for } \ j = 1, \cdots, k, \quad |\beta| \ll 1. \tag{3.8}$$

Notice that we have the following well-known expansion

$$\overline{\Psi}(\beta, \varepsilon) = \sum v_{2i+1}(\varepsilon)\beta^{2i+1} f_i(\beta, \varepsilon),$$

where $f_i(0, \varepsilon) = 1$ for all i. Thus (1.5) is equivalent to the equality $\overline{\Psi}(\beta, \varepsilon) = O(\varepsilon^{k+1})$. If, furthermore, there exists an integer $m > 0$ such that (1.6) is satisfied, then we have

$$\overline{\Psi}(\beta, \varepsilon) = b_m \beta^{2m+1} \varepsilon^{k+1} + O(\varepsilon^{k+2} + \beta^{2m+2}).$$

By (3.3), (3.4) and (3.7), we obtain part (b). The proof is finished.

Remarks 3.2. (1) Due to Theorem B(a), it seems to the authors that $M_j(h)$ may not be smooth at $h = 0$ even in the case that system (1.2) is analytic.

(2) Generally speaking, the Abelian integral $M_1(h)$ can be asymptotically expanded as $\sum a_{k,\alpha} h^\alpha (\ln h)^k$ at a polycycle (i.e. at $h = h_0$ or $h = h_1$ in our notation) (see, for example, [1]). Theorem B(a) provides a specific expansion in the particular case that Γ_{h_0} is a center. As another such example, one can prove that at a polycycle which is composed by hyperbolic singular points, we have the following expansion

$$M_1(h) = \sum_{i=0}^{\infty} c_i \cdot (h_1 - h)^i + \ln(h_1 - h) \cdot \sum_{i=1}^{\infty} d_i \cdot (h_1 - h)^i$$

if the system (1.2) is analytic (see [10] for a proof).

(3) It is interesting to compare Theorem B(b) with a result (see [4, Theorem 1]) proved by Il'yashenko which says that when the integral $M_1(h)$ (see §1) is evaluated in the complex domain, i.e. integrating paths are replaced by the closed curves (of real dimension 1) on the complex manifolds Γ_h, and is identically zero, then system (1.2) is Hamiltonian if g is independent of ε.

(4) Theorem B(b) suggests a method which is useful in the computation of $M_j(h)$. For example, to calculate $M_k(h)$ for the polynomial system

$$\dot{x}_1 = \alpha(\varepsilon)x_1 + x_2 + \sum_{i+j=2}^{n} a_{ij}(\varepsilon)x_1^i x_2^j, \quad \dot{x}_2 = -x_1 + \alpha(\varepsilon)x_2 + \sum_{i+j=2}^{n} b_{ij}(\varepsilon)x_1^i x_2^j \tag{3.9}$$

under the condition $M_1(h) = \cdots = M_{k-1}(h) \equiv 0$, we first compute

$$v_{2i+1}^{(k)} = \frac{d^k v_{2i+1}}{d\varepsilon^k}(0), \quad i = 0, 1, \cdots, N,$$

where $v_1(\varepsilon) = \alpha(\varepsilon),\ v_3(\varepsilon), \cdots,\ v_{2N+1}(\varepsilon)$ are all the focal quantities of (3.9). Then we can take $a_{ij}^{(s)}(0)$ (or $\alpha^{(s)}(0), b_{ij}^{(s)}(0)$) to be zero if it does not appear in $v_{2l+1}^{(k)},\ \forall\ 1 \leq l \leq N$. This method will be used in §5 to simplify $M_1(h)$.

§4. Bifurcations

In this section we are interested in the upper bound for the number of limit cycles in system (1.2) near $\Gamma_\alpha, \alpha \in [h_0, h_1]$, with the assumption

$$M_j(h) \equiv 0 \text{ for } j = 1, \cdots, k-1, \quad \text{and} \quad M_k(h) \not\equiv 0 \tag{4.1}$$

for some fixed integer $k \geq 1$. It is well-known that the best estimation has been obtained for the case $\alpha \in (h_0, h_1)$. So the problem is left only for the case $\alpha = h_0$ or h_1 which will be studied here. To start with, we first collect an easy result which is not totally new.

Theorem 4.1. *Suppose for some* $k \geq 1, \ m \geq 1$,

$$M_j(h) \equiv 0, \ j = 1, \cdots, k-1, \quad M_k(h) = a_m h^{m+1} + O(h^{m+\frac{3}{2}}), \ a_m \neq 0 \tag{4.2}$$

for $0 = h_0 < h \ll 1$ (*here we assume that* (3.1) *is satisfied*). *Then for sufficiently small* ε, *system* (1.2) *has at most* m *limit cycles near the origin for sufficiently small* $|\varepsilon| > 0$.

This result appeared in [6] in the case $k = 1, 2$. But the authors failed to find it in the general form in the literature, so we gave it a proof in [10].

The rest of this section is devoted to the limit cycle bifurcation at Γ_{h_1}, which is assumed here to be a homoclinic loop with a unique hyperbolic singular point. For the sake of simplicity, we suppose, without loss of generality, that the segment $L = \{x_2 = 0, x_1 \geq 0\}$ is transversal to Γ_{h_1} at $q(h_1, 0) \stackrel{\text{def.}}{=} Q_0$. One feature of this particular case is that $\frac{\partial H}{\partial x_1}(Q_0) > 0$. This fact will be used below. Now for $0 < |\varepsilon| \ll 1$, the stable manifold of the unique saddle will intersect L at a point $Q(\varepsilon) \stackrel{\text{def.}}{=} (a(\varepsilon), 0)$ near Q_0. Obviously, $Q(0) = Q_0 = (a(0), 0)$.

For sufficiently small $u > 0$, the positive semi-trajectory of system (1.2) passing through the initial point $Q_1 = (a(\varepsilon) - u, 0)$ will intersect L at several points, the first of which is denoted by $Q_2 = \big(a(\varepsilon) - \overline{P}(u, \varepsilon), 0\big)$. Clearly, for sufficiently small $u \geq 0, \ |\varepsilon| \geq 0$, the function $\overline{P}(u, \varepsilon)$ is well-defined. Since $\overline{P}(u, 0) = u$, we can write

$$\overline{P}(u, \varepsilon) - u \stackrel{\text{def.}}{=} \varepsilon \overline{M}_1(u) + \varepsilon^2 \overline{M}_2(u) + \cdots + \varepsilon^k \overline{M}_k(u) + O(\varepsilon^{k+1}). \tag{4.3}$$

Using the results in [8], we know that

$$\overline{M}_1(u) = m_0^{(1)} + m_1^{(1)} u \ln u + m_2^{(1)} u + m_3^{(1)} u^2 \ln u + \cdots,$$

$$\overline{M}_k(u) = m_0^{(k)} + m_1^{(k)} u \ln u + m_2^{(k)} u + m_3^{(k)} u^2 \ln u + \cdots, \tag{4.4}$$

if $\overline{M}_j(u) \equiv 0, \ j = 1, \ldots, k-1$. Here $m_i^{(j)}$ is a real constant for any $i \geq 0, j \geq 1$.

Let

$$H(a(\varepsilon) - u, 0) \stackrel{\text{def.}}{=} h^*(u, \varepsilon), \quad 0 < u \ll 1. \tag{4.5}$$

Then h^* is C^∞ and $h^*(0, 0) = h_1$. Since

$$\frac{\partial h^*}{\partial u}(0, 0) = -\frac{\partial H}{\partial x_1}(a(0), 0) \stackrel{\text{def.}}{=} b < 0,$$

$h = h^*(u, \varepsilon)$ has a C^∞ inverse

$$u = u^*(h, \varepsilon) = b^{-1} \cdot (h - h_1) + O(|\varepsilon| + |h - h_1|^2). \tag{4.6}$$

By applying the implicit function theorem to (4.6), one can easily obtain the solution $h = \bar{h}(\varepsilon) = h_1 + O(\varepsilon)$ of the equation $u^*(h, \varepsilon) = 0$. Evidently this solution is C^∞ and exists for sufficiently small $|\varepsilon| > 0$. Furthermore we have

$$u^*(h, \varepsilon) \geq 0 \Longleftrightarrow h \leq \bar{h}(\varepsilon). \tag{4.7}$$

Since $u^*(h^*(u,\varepsilon),\varepsilon) \equiv u$ for small $u, |\varepsilon| > 0$, we have

$$\begin{aligned}\overline{P}(u,\varepsilon) - u &= u^*(h^*(\overline{P}(u^*(h,\varepsilon),\varepsilon),\varepsilon),\varepsilon) - u^*(h,\varepsilon)\\ &= \frac{\partial u^*}{\partial h}(\tilde{h},\varepsilon)[h^*(\overline{P}(u^*(h,\varepsilon),\varepsilon),\varepsilon) - h],\end{aligned} \tag{4.8}$$

where $\tilde{h}$ lies between h and $h^*(\overline{P}(u^*(h,\varepsilon),\varepsilon),\varepsilon)$, u and h are related by (4.6). From (4.5), we have

$$h^*(\overline{P}(u^*(h,\varepsilon),\varepsilon),\varepsilon) = H(a(\varepsilon) - \overline{P}(u^*(h,\varepsilon),\varepsilon),\varepsilon,0).$$

Because $\overline{P}$ and the Poincare map $P(h,\varepsilon)$ are all defined by the same orbit of system (1.2), we can easily find a relation between them, which will be formulated below.

Consider the positive semi-trajectory of (1.2) starting at $(a(\varepsilon) - u^*(h,\varepsilon), 0) \in L$. Then the first point at which it intersects L is $(a(\varepsilon) - \overline{P}(u^*(h,\varepsilon),\varepsilon), 0) \in L$. According to the definition of P, the h-value at this point will be $P(h,\varepsilon)$, since the h-value of the initial point is

$$H(a(\varepsilon) - u^*(h,\varepsilon), 0) = h^*(u^*(h,\varepsilon),\varepsilon) = h.$$

Thus we have

$$P(h,\varepsilon) = H(a(\varepsilon) - \overline{P}(u^*(h,\varepsilon),\varepsilon),\varepsilon,0) = h^*(\overline{P}(u^*(h,\varepsilon),\varepsilon),\varepsilon).$$

Furthermore, their corresponding successor functions are also closely related. Let u and h be the two parameterizations of L which are related by (4.6). Then the relation

$$\overline{P}(u,\varepsilon) - u = b^{-1}[1 + O(|\varepsilon| + |h - h_1|)](P(h,\varepsilon) - h)$$

follows easily from (4.8). Equivalently,

$$P(h,\varepsilon) - h = b[1 + O(|\varepsilon| + |h - h_1|)][\overline{P}(u^*(h,\varepsilon),\varepsilon) - u^*(h,\varepsilon)] \tag{4.9}$$

for $h \le \bar{h}(\varepsilon)$.

By (4.3), (4.9) and the definition of Melnikov funtions, we obtain

(1) $M_1(h) = b\overline{M}_1(u^*(h,0))F_1(h), \quad F_1 \in C^\infty, \quad F_1(h_1) = 1$;

(2) $M_k(h) = b\overline{M}_k(u^*(h,0))F_k(h), \quad F_k \in C^\infty, \quad F_k(h_1) = 1$, if $M_j(h) = \overline{M}_j(u) \equiv 0, \ j = 1, \cdots, k-1$.

Therefore from (4.4) and (4.6), we have the following expansion

$$\begin{aligned}M_k(h) &= P_0^{(k)} + P_1^{(k)}(h_1 - h)\ln(h_1 - h) + P_2^{(k)}(h_1 - h)\\ &\quad + P_3^{(k)}(h_1 - h)^2\ln(h_1 - h) + \cdots\end{aligned} \tag{4.10}$$

if $M_j(h) \equiv 0$ for $j = 1, \cdots, k-1$. The coefficients $P_j^{(k)}, \ j \ge 0, k \ge 1$, are real constants and have the following property

$$P_j^{(k)} = 0, \ j = 0,1,\cdots,n-1, \quad P_n^{(k)} \ne 0 \iff m_j^{(k)} = 0, \ j = 0,1,\cdots,n-1, \quad m_n^{(k)} \ne 0.$$

Now the following theorem holds through a discussion similar to that in [8,3].

Theorem 4.2. *Suppose $M_j(h) \equiv 0, \ \forall 1 \le j \le k-1$, but $M_k(h) \not\equiv 0$ for some $k \ge 1$. Then $M_k(h)$ can be expanded as (4.10) at $h = h_1$. Further, if $P_j^{(k)} = 0$ for $j = 1, \ldots, n-1$, but $P_n^{(k)} \ne 0$, then system (1.2) has at most n limit cycles near Γ_{h_1} for small $|\varepsilon| > 0$.*

§5. Melnikov Functions for Quadratic Systems

In this section we shall consider Melnikov functions for quadratic systems. This seems to be the simplest nontrivial case, but there are still (the first) 6 Melnikov functions to be calculated before the perturbed system gets to be integrable (Theorem 5.1). The calculation is done by using Theorem B(b). Before going to general quadratic systems, let us consider the following example which has been studied in [5] recently.

Example 5.1. Any quadratic Hamiltonian system with an invariant line can be written as

$$\begin{aligned} \dot{x} &= -y + lx^2 + ny^2, \\ \dot{y} &= x(1 - 2ly). \end{aligned} \tag{5.1}$$

Its quadratic perturbation can be written as

$$\begin{aligned} \dot{x} &= -y + \varepsilon\bar{\delta}x + (l + \varepsilon\bar{l})x^2 + \varepsilon\bar{m}xy + (n + \varepsilon\bar{n})y^2, \\ \dot{y} &= x(1 + \varepsilon\bar{a}x + (\varepsilon\bar{b} - 2l)y), \end{aligned} \tag{5.2}$$

where ε is a small parameter. By Theorem 12.3[9,§12] we have

$$\begin{cases} \overline{W}_1 = \varepsilon[\bar{m}(l + n) + \varepsilon(\bar{m}(\bar{l} + \bar{n}) - \bar{a}(\bar{b} + 2\bar{l}))], \\ \overline{W}_2 = \varepsilon^3\bar{m}\bar{a}(5\bar{a} - \bar{m})[(l + n + \varepsilon(\bar{l} + \bar{n}))^2(n - 2l + \varepsilon(\bar{n} + \bar{b})) \\ \qquad - (\varepsilon\bar{a})^2(n + \varepsilon(\bar{b} + 2\bar{l} + \bar{n}))], \\ \overline{W}_3 = \varepsilon^3\bar{m}\bar{a}^2[2\bar{a}^2\varepsilon^2 + (n + \bar{n}\varepsilon)(l + 2n + \varepsilon(\bar{l} + 2\bar{n}))] \cdot \\ \qquad \cdot [(l + n + \varepsilon(\bar{l} + \bar{n}))^2(n - 2l + \varepsilon(\bar{n} + \bar{b})) - (\varepsilon\bar{a})^2(n + \varepsilon(\bar{b} + 2\bar{l} + \bar{n}))], \end{cases} \tag{5.3}$$

where $\overline{W}_1, \overline{W}_2, \overline{W}_3$ are equivalent to the focal quantities v_3, v_5, v_7 respectively. Without changing $M_1(h)$, we can take $\bar{a} = \bar{n} = \bar{l} = \bar{b} = 0$ (see Remark 3.2(4)). Thus

$$M_1(h) = \int_{\Gamma_h} (\bar{\delta}x + \bar{m}xy)\, dy.$$

If one is only interested in the generic (i.e. $M_1(h) \not\equiv 0$) bifurcation, it can be assumed that $\bar{a} = \bar{n} = \bar{l} = \bar{b} = 0$ in system (5.2), which turns out to be a quadratic system with an invariant line. By a well-known result[9,Theorem 15.4], this kind of quadratic systems have at most one limit cycle. Thus we have proved one of the main results in [5]: generically system (5.2) generates at most one limit cycle.

Now let us consider the general quadratic system in the following form

$$\begin{aligned} \dot{x} &= \lambda_1 x - y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2, \end{aligned} \tag{5.4}$$

where $\lambda_j = \lambda_{j_0} + \varepsilon\delta_j,\ j = 1, \cdots, 6$. We assume (5.4) is Hamiltonian in the case $\varepsilon = 0$, i.e.

$$\lambda_{10} = \lambda_{40} = \lambda_{50} = 0. \tag{5.5}$$

Rewrite (5.4) in the form

$$\begin{aligned} \dot{x} &= -\frac{\partial H}{\partial y} + \varepsilon h_1(x, y), \\ \dot{y} &= \frac{\partial H}{\partial x} + \varepsilon h_2(x, y), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned}H(x,y)&=\frac{1}{2}(x^2+y^2)+\frac{1}{3}\lambda_{20}x^3+\lambda_{30}x^2y-\lambda_{20}xy^2-\frac{1}{3}\lambda_{60}y^3,\\h_1(x,y)&=\delta_1x-\delta_3x^2+(2\delta_2+\delta_5)xy+\delta_6y^2,\\h_2(x,y)&=\delta_1y+\delta_2x^2+(2\delta_3+\delta_4)xy-\delta_2y^2.\end{aligned}$$

From [9, Lemma 9.2], we have for (5.6) that

$$r(2\pi,\beta,\varepsilon)-\beta=\sum_{i=0}^{3}v_{2i+1}(\varepsilon)f_{2i+1}(\beta,\varepsilon)$$

with $f_{2i+1}(0,\varepsilon)=1$, and

$$\begin{cases}v_1(\varepsilon)=e^{2\pi\lambda_1}-1=2\pi\lambda_1+O(\lambda_1^2),\\v_3(\varepsilon)=-\dfrac{\pi}{4}\lambda_5(\lambda_3-\lambda_6),\\v_5(\varepsilon)=\dfrac{\pi}{24}\lambda_2\lambda_4(\lambda_3-\lambda_6)(\lambda_4+5\lambda_3-5\lambda_6),\\v_7(\varepsilon)=-\dfrac{5}{32}\pi\lambda_2\lambda_4(\lambda_3-\lambda_6)^2(\lambda_3\lambda_6-2\lambda_6^2-\lambda_2^2).\end{cases}\tag{5.7}$$

Direct computation shows that

$$\begin{aligned}v_1(\varepsilon)&=2\pi\delta_1\varepsilon+O(\varepsilon^2\delta_1^2),\\\frac{4}{\pi}v_3(\varepsilon)&=-\delta_5(\lambda_{30}-\lambda_{60})\varepsilon-\delta_5(\delta_3-\delta_6)\varepsilon^2,\\\frac{24}{\pi}v_5(\varepsilon)&=5\delta_4\lambda_{20}(\lambda_{30}-\lambda_{60})^2\varepsilon+\delta_4(\lambda_{30}-\lambda_{60})[\lambda_{20}(\delta_4+10\delta_3-10\delta_6)+5\delta_2(\lambda_{30}-\lambda_{60})]\varepsilon^2\\&\quad+\delta_4[\lambda_{20}(\delta_3-\delta_6)(\delta_4+5\delta_3-5\delta_6)+\delta_2(\lambda_{30}-\lambda_{60})(\delta_4+10\delta_3-10\delta_6)]\varepsilon^3\\&\quad+\delta_4\delta_2(\delta_3-\delta_6)(\delta_4+5\delta_3-5\delta_6)\varepsilon^4,\\\frac{32}{5\pi}v_7(\varepsilon)&=-\delta_4\lambda_{20}(\lambda_{30}-\lambda_{60})^2(\lambda_{30}\lambda_{60}-2\lambda_{60}^2-\lambda_{20}^2)\varepsilon\\&\quad-\delta_4[\lambda_{20}(\lambda_{30}-\lambda_{60})^2(\delta_3\lambda_{60}+\lambda_{30}\delta_6-4\lambda_{60}\delta_6-2\delta_2\lambda_{20})\\&\quad+(\lambda_{30}-\lambda_{60})(\delta_2(\lambda_{30}-\lambda_{60})+2\lambda_{20}(\delta_3-\delta_6))(\lambda_{30}\lambda_{60}-2\lambda_{60}^2-\lambda_{20}^2)]\varepsilon^2\\&\quad-\delta_4[\lambda_{20}(\lambda_{30}-\lambda_{60})^2(\delta_3\delta_6-2\delta_6^2-\delta_2^2)+(\lambda_{30}-\lambda_{60})\\&\quad\cdot[\delta_2(\lambda_{30}-\lambda_{60})+2\lambda_{20}(\delta_3-\delta_6)](\delta_3\lambda_{60}+\lambda_{30}\delta_6-4\lambda_{60}\delta_6-2\delta_2\lambda_{20})\\&\quad+(\delta_3-\delta_6)[\lambda_{20}(\delta_3-\delta_6)+2\delta_2(\lambda_{30}-\lambda_{60})](\lambda_{30}\lambda_{60}-2\lambda_{60}^2-\lambda_{20}^2)]\varepsilon^3\\&\quad-\delta_4[(\lambda_{30}-\lambda_{60})(\delta_2(\lambda_{30}-\lambda_{60})+2\lambda_{20}(\delta_3-\delta_6))(\delta_3\delta_6-2\delta_6^2-\delta_2^2)\\&\quad+(\delta_3-\delta_6)(\lambda_{20}(\delta_3-\delta_6)+2\delta_2(\lambda_{30}-\lambda_{60}))(\delta_3\lambda_{60}+\lambda_{30}\delta_6-4\lambda_{60}\delta_6-2\delta_2\lambda_{20})\\&\quad+\delta_2(\delta_3-\delta_6)^2(\lambda_{30}\lambda_{60}-2\lambda_{60}^2-\lambda_{20}^2)]\varepsilon^4\\&\quad-\delta_4[(\delta_3-\delta_6)(\lambda_{20}(\delta_3-\delta_6)+2\delta_2(\lambda_{30}-\lambda_{60}))(\delta_3\delta_6-2\delta_6^2-\delta_2^2)\\&\quad+\delta_2(\delta_3-\delta_6)^2(\delta_3\lambda_{60}+\lambda_{30}\delta_6-4\lambda_{60}\delta_6-2\delta_2\lambda_{20})]\varepsilon^5\\&\quad-\delta_4\delta_2(\delta_3-\delta_6)^2(\delta_3\delta_6-2\delta_6^2-\delta_2^2)\varepsilon^6.\end{aligned}$$

Now the following conclusion follows from Theorem B(b) by using direct computation.

Theorem 5.1. *Assume in system (5.4) that $\lambda_j=\lambda_{j0}+\varepsilon\delta_j$ with δ_j being a constant independent of ε. Then the first non-identically-zero Melnikov function can be $M_k(h)$ for any $k=1,2,3,4,6$. If $M_1(h)=\cdots=M_6(h)\equiv0$, then $M_k(h)\equiv0$ for any $1\le k<\infty$.*

Indeed, we have the following details.

(1) $M_1(h) = a_0^{(1)} h(1 + O(\sqrt{h}))$ with $a_0^{(1)} \neq 0 \Longleftrightarrow \delta_1 \neq 0$;

(2) $M_1(h) = a_1^{(1)} h^2(1 + O(\sqrt{h}))$ with $a_1^{(1)} \neq 0 \Longleftrightarrow \delta_1 = 0, \delta_5(\lambda_{30} - \lambda_{60}) \neq 0$;

(3) $M_1(h) = a_2^{(1)} h^3(1 + O(\sqrt{h}))$ with $a_2^{(1)} \neq 0 \Longleftrightarrow \delta_1 = \delta_5 = 0,\ \lambda_{20}\delta_4(\lambda_{30} - \lambda_{60}) \neq 0$;

(4) $M_1(h) \equiv 0$ if and only if

$$\delta_1 = \delta_5(\lambda_{30} - \lambda_{60}) = \lambda_{20}\delta_4(\lambda_{30} - \lambda_{60}) = 0; \tag{5.8}$$

(5) $M_1(h) \equiv 0, M_2(h) = a_1^{(2)} h^2(1 + O(\sqrt{h}))$ with $a_1^{(2)} \neq 0$ if and only if

$$\delta_1 = \lambda_{30} - \lambda_{60} = 0, \quad \delta_5(\delta_3 - \delta_6) \neq 0;$$

(6) $M_1(h) \equiv 0, M_2(h) = a_2^{(2)} h^3(1 + O(\sqrt{h}))$ with $a_2^{(2)} \neq 0$ if and only if

$$\delta_1 = \delta_5 = \lambda_{20} = 0, \quad \delta_2\delta_4(\lambda_{30} - \lambda_{60}) \neq 0;$$

(7) $M_1(h) = M_2(h) \equiv 0 \Longleftrightarrow$ both (5.8) and the following hold

$$\delta_5(\delta_3 - \delta_6) = \delta_2\delta_4(\lambda_{30} - \lambda_{60}) = 0; \tag{5.9}$$

(8) $M_1(h) = M_2(h) \equiv 0,\ M_3(h) = a_2^{(3)} h^3(1 + O(\sqrt{h})),\ a_2^{(3)} \neq 0 \Longleftrightarrow$ (5.8) and (5.9) hold, and $\delta_4\lambda_{20}(\delta_3 - \delta_6)(\delta_4 + 5\delta_3 - 5\delta_6) \neq 0$;

(9) $M_1(h) = M_2(h) \equiv 0,\ M_3(h) = a_3^{(3)} h^4(1 + O(\sqrt{h})),\ a_3^{(3)} \neq 0$ if and only if

$$\delta_1 = \delta_5 = \delta_4 + 5\delta_3 - 5\delta_6 = 0, \quad \delta_4\lambda_{20}(\delta_3 - \delta_6) \neq 0;$$

(10) $M_1(h) = M_2(h) = M_3(h) \equiv 0$ if and only if

$$\delta_1 = \lambda_{30} - \lambda_{60} = \delta_5(\delta_3 - \delta_6) = \delta_4\lambda_{20}(\delta_3 - \delta_6), \text{ or} \tag{5.10}$$

$$\delta_1 = \delta_5 = \delta_4\lambda_{20} = \delta_2\delta_4 = 0; \tag{5.11}$$

(11) $M_j(h) \equiv 0,\ j = 1, 2, 3,\ M_4(h) = a_2^{(4)} h^3(1 + O(\sqrt{h}))$ with $a_2^{(4)} \neq 0$ if and only if

$$\delta_1 = \delta_5 = \lambda_{20} = \lambda_{30} - \lambda_{60} = 0, \quad \delta_2\delta_4(\delta_3 - \delta_6)(\delta_4 + 5\delta_3 - 5\delta_6) \neq 0;$$

(12) $M_j(h) \equiv 0,\ j = 1, 2, 3,\ M_4(h) = a_3^{(4)} h^4(1 + O(\sqrt{h}))$ with $a_3^{(4)} \neq 0 \Longleftrightarrow$ (5.10) holds, and $\delta_4 + 5\delta_3 - 5\delta_6 = 0,\ \delta_2\delta_4(\delta_3 - \delta_6)\lambda_{60} \neq 0$;

(13) $M_j(h) \equiv 0, j = 1, 2, 3, 4 \Longleftrightarrow$ either (5.11) holds, or

$$\delta_1 = \delta_3 - \delta_6 = \lambda_{30} - \lambda_{60} = 0, \text{ or} \tag{5.12}$$

$$\delta_1 = \delta_5 = \delta_4\lambda_{20} = \delta_2\delta_4\lambda_{30} = \lambda_{30} - \lambda_{60} = \delta_2\delta_4(\delta_4 + 5\delta_3 - 5\delta_6) = 0; \tag{5.13}$$

(14) $M_j(h) \equiv 0, j = 1, 2, 3, 4 \Longrightarrow M_5(h) \equiv 0$;

(15) $M_j(h) \equiv 0, j = 1. \cdots, 5,\ M_6(h) = a_3^{(6)} h^4(1 + O(\sqrt{h})),\ a_3^{(6)} \neq 0$ if and only if (5.13) and $\delta_2\delta_4(\delta_3 - \delta_6)(\delta_3\delta_6 - 2\delta_6^2 - 2\delta_2^2) \neq 0$ hold;

(16) $M_j(h) \equiv 0$ for all $j \geq 1 \Longleftrightarrow$ (5.11)–(5.13) and $\delta_2\delta_4(\delta_3 - \delta_6)(\delta_3\delta_6 - 2\delta_6^2 - 2\delta_2^2) = 0$ hold.

(17) Moreover, in the cases (1)–(3), the Abelian integral $M_1(h)$ can be simplified as

$$M_1(h) = \int_{\Gamma_h} h_1 dy - h_2 dx = \int_{\Gamma_h} (\delta_1 x + \delta_5 xy) dy - \delta_4 xy dx. \tag{5.14}$$

If $\lambda_{30} = \lambda_{60}$, we can take $\delta_4 = \delta_5 = 0$ in (5.15); if $\lambda_{20} = 0$, we can take $\delta_4 = 0$.

Notice that these calculations can be done even in the case that δ_i depends on ε. Evidently it will be more complicated.

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