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ASYMPTOTIC BEHAVIOR OF HARMONIC MAPS FROM COMPLETE MANIFOLDS**

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Abstract

In this paper, the author considers a class of complete noncompact Riemannian manifolds which satisfy certain conditions on Ricci curvature and volume comparison. It is shown that any harmonic map with finite energy from such a manifold M into a normal geodesic ball in another manifold N must be asymptotically constant at the infinity of each large end of M. A related existence theorem for harmonic maps is established.

Keywords Ricci curvature, Volume comparison, Fatou's property, Harmonic map
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§1. Introduction

Let M be an m-dimensional complete noncompact Riemannian manifold, $p \in M$. P. Li and L. F. Tam introduced in [8] a volume comparison condition **(VC)** (see the definition below). They obtained some important analytic properties if M satisfies **(VC)** and the Ricci curvature condition $\mathrm{Ric}_M(x) \geq -(m-1)K/(1+r(x))^2$, where $r(x)$ is the distance from p to x, $K \geq 0$ is a constant. It is interesting to know more about the analysis on such a manifold M. Let N be a complete Riemannian manifold with the sectional curvature K_N bounded above by some constant $\bar{K} \geq 0$. $\bar{B}_q(\tau)$ denotes the geodesic ball of radius τ centered at q in N. We assume that $\bar{B}_q(\tau)$ lies inside the cut locus of q and $\tau < \pi/2\sqrt{\bar{K}}$. In this paper, we consider the harmonic map $u : M \to N$ with $u(M) \subset \bar{B}_q(\tau)$.

Let us first give the following definitions:

Definition. *Let D be a compact subset of M. We call each unbounded component of $M\backslash D$ an end of M with respect to D. An end E of M is called a large end if $\int_r^\infty \frac{t}{V_{p,E}(t)}dt < \infty$, where $V_{p,E}(r)$ denotes the volume of $B_p(r) \cap E$.*

(VC) There exists a constant $\zeta > 0$ such that for all $r > 0$ and $x \in \partial B_p(r), V_p(r) \leq \zeta V_x(r/2)$, where $V_x(r)$ denotes the volume of the geodesic ball of radius r centered at x in M.

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For an end E of M, we say that E satisfies **(VC)** if there is a constant $\zeta > 0$ such that for all $r > 0$ and $x \in \partial B_p(r) \cap E$,

$$V_{p,E}(r) \leq \zeta V_{x,E}\Big(\frac{r}{2}\Big).$$

For example, manifolds with nonnegative Ricci curvature must satisfy **(VC)**. Each end of any complete manifold with asymptotically nonnegative sectional curvature must satisfy **(VC)** (see [8] for details).

In [2], S. Y. Cheng proved a Liouville theorem for harmonic maps which asserts that if $\mathrm{Ric}_M \geq 0$ and $K_N \leq 0$, then any harmonic map $u : M \to N$ with relatively compact image is constant. H. I. Choi[3] generalized Cheng's result to the case of general target manifolds as follows: if $\mathrm{Ric}_M \geq 0$, $K_N \leq \bar{K}$, then any harmonic map $u : M \to N$ with $u(M) \subset \bar{B}_q(\tau)$ is constant.

In general, if the condition $\mathrm{Ric}_M \geq 0$ is not available, M may have more complicated geometric structure. One natural problem then is to consider Fatou's property for harmonic map u from M, that is, wether u is asymptotically constant at the infinity of ends of M. This is the main purpose of the present paper.

We have the following

Theorem 1.1. *Let M be an m-dimensional complete noncompact manifold satisfying* **(VC)** *and*

$$\mathrm{Ric}_M(x) \geq -(m-1)K/(1+r(x))^2,$$

$E_1, \cdots, E_l$ be large ends of M, N be a complete manifold with $K_N \leq \bar{K}$ ($\bar{K} \geq 0$) and $u : M \to N$ be a harmonic map with $u(M) \subset \bar{B}_q(\tau), \tau < \pi/2\sqrt{\bar{K}}$. If u has finite energy, then there exist $p_1, \cdots, p_l \in \bar{B}_q(\tau)$ such that

$$\lim_{x \in E_A \to \infty} u(x) = p_A, \quad A = 1, \cdots, l.$$

We would like to point out that similar problems were studied by other authors. Y. H. Yang[12] proved Fatou's property for bounded harmonic maps with finite energy from M under the assumption that M, with only large ends, has nonnegative sectional curvature outside a compact subset and two conditions on Green's function on M. Our result concerns more general M while the conditions on Green's function are not needed. Moreover, our method also works for the case in [12]. Recently, based on a result in [10], S. Y. Cheng, L. F. Tam and T. Y. H. Wan[4] asserted that if M satisfies the condition **(A)** that every harmonic function with finite Dirichlet integral on M is bounded and another condition **(D)**, that is, for some bounded domain D of M, every bounded harmonic function on M is asymptotically constant near the infinity of each unbounded component of $M\backslash\bar{D}$, then for any harmonic map with finite energy from M to a Cartan-Hadamard manifold N holds Fatou's property. Note that the target manifold N in Theorem 1.1 may be any complete manifold, instead of Cartan-Hadamard manifolds. Moreover, our result gives a direct link between the geometric conditions and the asymptotic properties of harmonic maps.

It is natural to establish the existence theorem for the above considered harmonic maps. We have the following

Theorem 1.2. *Let M be an m-dimensional complete noncompact manifold satisfying*

$$\mathrm{Ric}_M(x) \geq -(m-1)K/(1+r(x))^2,$$

N and $\bar{B}_q(\tau)$ be as in Theorem 1.1. *Suppose that M has only finitely many ends $E_A, A = 1, \cdots, l$ with each E_A being large and satisfying* **(VC)**. *Then for any $p_A \in \bar{B}_q(\tau), A = 1, \cdots, l$, there exists a unique harmonic map $u : M \to N$ with finite energy such that*

$$\lim_{x \in E_A \to \infty} u(x) = p_A, \quad A = 1, \cdots, l.$$

§2. Preliminaries

For two Riemannian manifolds (M, μ) and (N, g), we consider a smooth map $u : M \to N$. Choose local orthonormal frames $\{e_\alpha, \alpha = 1, \cdots, m\}$ in M, then the energy density of u can be defined as $e(u) = \frac{1}{2} \sum\limits_{\alpha=1}^{m} \langle u_* e_\alpha, u_* e_\alpha \rangle_g$. The energy of u is $E(u) = \int_M e(u)$. u is called a harmonic map if it is a critical point of the energy functional E.

We first establish some differential inequalities about u.

Lemma 2.1. *Let $M, N, u, \bar{B}_q(\tau)$ be as in Theorem* 1.1. *Then there exists a constant $\beta(m, K, \bar{K}, \tau) > 0$ such that $\triangle e(u) \geq -\beta e(u)$.*

Proof. Choose local orthonormal frames $\{e_\alpha, \alpha = 1, \cdots, m\}$ in M. Denote by $B(u)$ the second fundamental form of u. $\mathrm{Ric}_M(\cdot)$ and $R_N(\cdot, \cdot)$ denote the Ricci curvature of M and the curvature operator of N respectively.

We have (see [5], also see [11] for our notations)

$$\begin{aligned} \triangle e(u) = |B(u)|^2 - \sum_{\alpha,\beta=1}^{m} \langle R_N(u_* e_\alpha, u_* e_\beta) u_* e_\alpha, u_* e_\beta \rangle \\ + \sum_{\alpha=1}^{m} \langle u_* Ric_M e_\alpha, u_* e_\alpha \rangle. \end{aligned} \tag{2.1}$$

The Ricci curvature condition of M implies $\mathrm{Ric}_M(x) \geq -(m-1)K$. Substituting this and the assumption on K_N into (2.1), we obtain

$$\begin{aligned} \triangle e(u) &\geq -4\bar{K} e^2(u) - 2(m-1)K e(u) \\ &= -e(u)[4\bar{K} e(u) + 2(m-1)K]. \end{aligned}$$

Because $\mathrm{Ric}_M(x) \geq -(m-1)K, K_N \leq \bar{K}, u$ and $\bar{B}_q(\tau)$ are as above, a well-known result of H. I. Choi[3] implies that there exists a constant $C_1(m, \tau, \bar{K}) > 0$ such that $e(u) \leq C_1 K$. Therefore

$$\triangle e(u) \geq -e(u)[4C_1 K\bar{K} + 2(m-1)K] = -\beta e(u),$$

where $\beta = 4C_1 K\bar{K} + 2(m-1)K = \beta(m, K, \bar{K}, \tau)$.

Let $\{y^i, i = 1, \cdots, n\}$ be the normal coordinates in the geodesic ball $\bar{B}_q(\tau)$ in N under which u has components $u^1, \cdots, u^n$. Choose local coordinates $\{x_\alpha, \alpha = 1, \cdots, m\}$ in M. Set $u^i_\alpha = \frac{\partial u^i}{\partial x_\alpha}, i = 1, \cdots, n, \alpha = 1, \cdots, m$. Let $\{\mu_{\alpha\beta}, \alpha, \beta = 1, \cdots, m\}$ and $\{g_{ij}, i, j = 1, \cdots, n\}$ be the metric tensors of M and N respectively.

Lemma 2.2. *Let $u : M \to N$ be a harmonic map with $u(M) \subset \bar{B}_q(\tau)$. Then there exist* *constants $C, C' > 0$ depending only on the geometry of $\bar{B}_q(\tau)$ such that*

$$e(u) \le C' \sum_{j=1}^{n} |\nabla u^j|^2,$$

$$|\nabla u^i|^2 \le Ce(u), \quad 1 \le i \le n,$$

where ∇u^i denotes the gradient of the function u^i on M.

Proof. It is easy to see that

$$|\nabla u^i|^2 = \sum_{\alpha,\beta=1}^{m} u^i_\alpha u^i_\beta \mu^{\alpha\beta},$$

$$e(u) = \frac{1}{2} \sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{n} u^i_\alpha u^j_\beta \mu^{\alpha\beta} g_{ij}.$$

Set

$$A = (A_{ij}), \quad A_{ij} = \sum_{\alpha,\beta=1}^{m} u^i_\alpha u^j_\beta \mu^{\alpha\beta}, \quad G = (g_{ij}), \qquad U = (u^i_\alpha)$$
$$\Gamma = (\mu^{\alpha\beta}), \quad i, j = 1, \cdots, n; \qquad \alpha, \beta = 1, \cdots, m.$$

It is clear that

$$e(u) = \frac{1}{2}\mathrm{Trace}(AG), \tag{2.2}$$

$$|\nabla u^i|^2 = A_{ii}, \quad 1 \le i \le n. \tag{2.3}$$

Because Γ is positive definite, there is an $m \times m$ nondegenerate matrix J such that $\Gamma = JJ'$, where J' denotes the transpose of J. Hence, $A = U\Gamma U' = (UJ)(UJ)'$ which means that A is semi-positive definite.

On the other hand, since G is positive definite, there is an orthonormal matrix P such that

$$PGP^{-1} = PGP' = [\lambda^1, \cdots, \lambda^n] \text{ (diagonal matrix)},$$

where $\lambda^1, \cdots, \lambda^n > 0$ are the eigenvalues of G in $\bar{B}_q(\tau)$. We have

$$\begin{aligned}\mathrm{Trace}(AG) &= \mathrm{Trace}(PAP^{-1}[\lambda^1, \cdots, \lambda^n]) \\ &= \mathrm{Trace}(\tilde{A}[\lambda^1, \cdots, \lambda^n]) \\ &= \lambda^1 \tilde{A}_{11} + \cdots + \lambda^n \tilde{A}_{nn},\end{aligned}$$

in which $\tilde{A} = PAP^{-1} = PAP'$ is also semi-positive definite. It is clear that there exist constants $C, C' > 0$ depending only on the geometry of $\bar{B}_q(\tau)$ such that $2C' > \lambda^1, \cdots, \lambda^n > \frac{2}{C}$. Thus

$$\mathrm{Trace}(AG) > \frac{2}{C} \sum_{j=1}^{n} \tilde{A}_{jj} = \frac{2}{C} \sum_{j=1}^{n} A_{jj}$$

and $\mathrm{Trace}(AG) < 2C' \sum_{j=1}^{n} A_{jj}$ because $\tilde{A}_{jj}, A_{jj} \ge 0, j = 1, \cdots, n$. Therefore

$$e(u) < C' \sum_{j=1}^{n} A_{jj} = C' \sum_{j=1}^{n} |\nabla u^j|^2,$$

$$|\nabla u^i|^2 \le \sum_{j=1}^{n} A_{jj} < Ce(u), \quad 1 \le i \le n.$$

§3. Proof of Theorems

Lemma 3.1. *Let M be an m-dimensional complete noncompact manifold satisfying* **(VC)** *and*

$$\mathrm{Ric}_M(x) \geq -(m-1)K/(1+r(x))^2,$$

$u : M \to N$ be a harmonic map with $u(M) \subset \bar{B}_q(\tau)$, and E be an end of M. E_{r_0} denotes the unbounded component of $E\backslash \bar{B}_p(r_0)$ for $r_0 > 0$. Then there is a constant $C_4(m, \zeta, K, \bar{K}, \tau, \bar{B}_q(\tau)) > 0$ such that for all $r \geq 2r_0, x \in E_{r_0} \cap (B_p(2r)\backslash \bar{B}_p(r))$,

$$\mathrm{OSC}_{E_{r_0}\cap B_x(\frac{r}{4})} u^i \leq C_4 \Big(\int_r^{2r} \frac{t}{V_{p,E}(t)} dt \Big)^{\frac{1}{2}} \Big(\int_{E_{r_0}\cap (B_p(3r)\backslash B_p(\frac{r}{2}))} e(u) \Big)^{\frac{1}{2}}.$$

Proof. For any $y \in E_{r_0} \cap B_x(\frac{r}{4})$, let $\mu : [0, l] \to M$ be a normal minimal geodesic with length l and $\mu(0) = x, \mu(l) = y$. It is clear that $l \leq \frac{r}{4}$ and, for any $t \in [0, l]$,

$$B_{\mu(t)}\Big(\frac{1}{12} r(\mu(t))\Big) \subset E_{r_0} \cap \Big(B_p(3r)\backslash B_p\Big(\frac{r}{2}\Big)\Big), \quad B_{\mu(t)}\Big(\frac{1}{6} r(\mu(t))\Big) \subset M\backslash B_p\Big(\frac{3}{8} r\Big). \tag{3.1}$$

From Lemma 2.1, $\triangle e(u) \geq -\beta e(u)$. In $B_{\mu(t)}(\frac{1}{6} r(\mu(t)))$, we apply Lemma 1.6 in [8] to $f = e(u)$ by setting $R = r(\mu(t))/12, x = \mu(t)$ and $K_0 = C_5/(1+r)^2$, where $C_5(m, K) > 0$ is some constant. We have

$$e(u)(\mu(t)) \leq \frac{C_6}{V_{\mu(t)}(r(\mu(t))/12)} \int_{B_{\mu(t)}(r(\mu(t))/12)} e(u), \tag{3.2}$$

where $C_6(m, K, \bar{K}, \tau) > 0$ is some constant.

From the condition **(VC)** and Lemma 1.3 in [8], we can conclude that

$$V_p(2r) \leq \frac{\zeta}{C_7} V_{\mu(t)}(r(\mu(t))/12),$$

where $C_7(m, K, \zeta) > 0$ is a constant. Hence, there is a constant $C_8(m, K, \zeta, \bar{K}, \tau,) > 0$ such that

$$e(u)(\mu(t)) \leq \frac{C_8}{V_p(2r)} \int_{E_{r_0}\cap (B_p(3r)\backslash B_p(\frac{r}{2}))} e(u). \tag{3.3}$$

For $u^i, 1 \leq i \leq n$, we have $|\nabla u^i|^2 \leq C e(u)$ from Lemma 2.2. Combing this with (3.3) leads to the following

$$|\nabla u^i|(\mu(t)) \leq \frac{C_9}{\sqrt{V_p(2r)}} \Big(\int_{E_{r_0}\cap (B_p(3r)\backslash B_p(\frac{r}{2}))} e(u) \Big)^{\frac{1}{2}} \tag{3.4}$$

in which $C_9(m, K, \zeta, \bar{K}, \tau, \bar{B}_q(\tau)) > 0$ is some constant. Therefore

$$\begin{aligned} |u^i(x) - u^i(y)| &\leq \int_0^l |\nabla u^i|(\mu(t)) dt \\ &\leq l \frac{C_9}{\sqrt{V_p(2r)}} \Big(\int_{E_{r_0}\cap (B_p(3r)\backslash B_p(\frac{r}{2}))} e(u) \Big)^{\frac{1}{2}} \\ &\leq \frac{C_9}{4} \Big(\frac{r}{\sqrt{V_p(2r)}} \Big) \Big(\int_{E_{r_0}\cap (B_p(3r)\backslash B_p(\frac{r}{2}))} e(u) \Big)^{\frac{1}{2}} \\ &\leq \frac{C_4}{2} \Big(\int_r^{2r} \frac{t}{V_p(t)} dt \Big)^{\frac{1}{2}} \Big(\int_{E_{r_0}\cap (B_p(3r)\backslash B_p(\frac{r}{2}))} e(u) \Big)^{\frac{1}{2}}, \end{aligned}$$

where $C_4 = \frac{1}{2} C_9$.

Noticing that $V_{p,E}(t) \le V_p(t)$, we obtain

$$\mathrm{OSC}_{E_{r_0}\cap B_x(\frac{r}{4})} u^i \le C_4\Big(\int_r^{2r} \frac{t}{V_{p,E}(t)}dt\Big)^{\frac12}\Big(\int_{E_{r_0}\cap(B_p(3r)\backslash B_p(\frac{r}{2}))} e(u)\Big)^{\frac12}, \quad 1\le i\le n.$$

We introduce the so-called ball covering lemma in [8]:

Lemma 3.2. *Let M be as in Theorem 1.1. Then for all $r>0$, $0<\alpha\le 1/4, B_p(2r)\backslash \bar{B}_p(r)$ can be covered by s geodesic balls of radius αr with centers in $B_p(2r)\backslash\bar{B}_p(r)$, and s can be bounded above by a constant depending only on $m,\zeta,K,$ and α.*

We can now establish the following property about the behavior of the harmonic map u near the infinity of each large end of M.

Proposition 3.1. *Let $M,N,u:M\to N,u(M)\subset\bar{B}_q(\tau)$ be as in Theorem 1.1, and E be a large end of M. $E_{\frac{r}{2}}$ denotes the unbounded component of $E\backslash\bar{B}_p(\frac{r}{2})$ for $r>0$. If x,y can be jointed by a curve γ in $E\backslash\bar{B}_p(r)$, then*

$$|u^k(x)-u^k(y)| \le C_{10}\Big(\int_r^{\infty}\frac{t}{V_{p,E}(t)}dt\Big)^{\frac12}\Big(\int_{E_{\frac{r}{2}}} e(u)\Big)^{\frac12}, \quad 1\le k\le n,$$

where $C_{10}(m,K,\zeta,\bar{K},\tau,\bar{B}_q(\tau))>0$ is some constant.

Proof. We can find an integer $J>0$ large enough, so that $\gamma\subset E_{\frac{r}{2}}\cap(B_p(2^Jr)\backslash\bar{B}_p(r))$. From Lamma 3.2, for $1\le j\le J$, there are $x_1^j,\cdots,x_{s_j}^j\in B_p(2^jr)\backslash\ \bar{B}_p(2^{j-1}r)$ such that

$$B_p(2^jr)\backslash\bar{B}_p(2^{j-1}r)\subset\bigcup_{i=1}^{s_j}B_i^j,$$

where $B_i^j=B_{x_i^j}(2^{j-1}r/4)$. Furthermore, s_j can be bounded above by a constant depending only on m,ζ and K. If $B_i^j\cap E_{\frac{r}{2}}\ne\emptyset$, then $B_i^j\subset E_{\frac{r}{2}}$. So we may assume that $\{B_i^j\}$ cover $E_{\frac{r}{2}}\cap(B_p(2^jr)\backslash B_p(2^{j-1}r))$ and $B_i^j\subset E_{\frac{r}{2}}$. Clearly,

$$B_i^j\subset E_{\frac{r}{2}}\cap(B_p(3\cdot 2^{j-1}r)\backslash B_p(2^{j-2}r)) \quad\text{and } \gamma\subset\bigcup_{i,j}B_i^j.$$

Hence

$$|u^k(x)-u^k(y)|\le\sum_{j=1}^J\sum_{i=1}^{s_j}\mathrm{OSC}_{B_i^j}u^k.$$

Denote $E_{\frac{r}{2}}\cap B_p(3\cdot 2^{i-1}r)\backslash B_p(2^{i-2}r))$ by $E_{r,i}$. Lemma 3.1 implies that

$$\begin{aligned}|u^k(x)-u^k(y)| &\le C_{11}\sum_{i=1}^{\infty}\Big(\int_{2^{i-1}r}^{2^ir}\frac{t}{V_{p,E}(t)}dt\Big)^{\frac12}\Big(\int_{E_{r,i}}e(u)\Big)^{\frac12}\\ &\le C_{11}\Big(\sum_{i=1}^{\infty}\int_{2^{i-1}r}^{2^ir}\frac{t}{V_{p,E}(t)}dt\Big)^{\frac12}\Big(\sum_{i=1}^{\infty}\int_{E_{r,i}}e(u)\Big)^{\frac12}\\ &\le C_{10}\Big(\int_r^{\infty}\frac{t}{V_{p,E}(t)}dt\Big)^{\frac12}\Big(\int_{E_{\frac{r}{2}}}e(u)\Big)^{\frac12},\end{aligned}$$

where $C_{10},C_{11}>0$ are constants depending only on $m,K,\zeta,\bar{K},\tau,$ and $\bar{B}_q(\tau)$.

Proof of Theorem 1.1. For each large end E_A, Proposition 3.1 implies that $u^k(1\le k\le n)$ is asymptotically a constant a_A^k at the infinity of E_A. Clearly, a_A^k is finite because $\sum\limits_{i=1}^n(u^i)^2<\tau^2<+\infty$. Set $p_A=(a_A^1,\cdots,a_A^n)\in\bar{B}_q(\tau)$, then $u(x)\to p_A$ as $x\in E_A\to\infty$, for

$A = 1, \cdots, l$.

Corollary 3.1. *Let M, N, u be as in Theorem* 1.1. *Soppose that M has only one end which is large. If u has finite energy, then u must be constant.*

Remark 3.1. The domain manifold M may have signed Ricci curvature and large volume growth. From this point of view, our result can be considered as a kind of generalization of the previous Liouville theorems.

Proof of Theorem 1.2. We use the argument in [1] and [12]. Since

$$\mathrm{Ric}_M(x) \geq -(m-1)K/(1+r(x))^2,$$

and each large end E_A $(1 \leq A \leq l)$ satisfies **(VC)**, from Theorem 1.9 in [8], E_A is non-parabolic end, namely, E_A admits a nonconstant positive Green's function. Then from [9], there exist harmonic functions f_A, $A = 1, \cdots, l$ such that

$$\begin{aligned} &0 < f_A < 1, \\ &\lim_{x \in E_A \to \infty} f_A(x) = 1, \quad \lim_{x \in E_B \to \infty} f_A(x) = 0 \quad (B \neq A), \\ &\int_M |\nabla f_A|^2 < +\infty. \end{aligned} \tag{3.5}$$

In normal coordinates in $\bar{B}_q(\tau)$, we assume $p_A = (p_A^1, \cdots, p_A^n), A = 1, \cdots, l$. Let

$$h(x) = \Big(\sum_{A=1}^{l} p_A^1 f_A(x), \cdots, \sum_{A=1}^{l} p_A^n f_A(x) \Big),$$

$$v(x) = \frac{1}{2} \sum_{A=1}^{l} \sum_{i=1}^{n} (p_A^i)^2 f_A,$$

then $\lim\limits_{x \in E_A \to \infty} h(x) = p_A$ and $\lim\limits_{x \to \infty} (v(x) - \frac{1}{2}|h(x)|^2) = 0$. Denote $B_k := B_p(R_k)$, $k = 1, 2, \cdots$, where $R_k \to +\infty$ $(k \to +\infty)$. From results in [6], there exist harmonic maps $u_k : B_k \to \bar{B}_q(\tau)$ with $u_k|_{\partial B_k} = h|_{\partial B_k}$, $k = 1, 2 \cdots$. Furthermore, we may assume that $\{u_k\}$ converges uniformly to a harmonic map u on compact subsets of M. Let v_k be a harmonic function on B_k such that $v_k|_{\partial B_k} = \frac{1}{2}|h(x)|^2|_{\partial B_k}$. By Lemma 3.1 in [1], we have

$$[\rho(u_k(x), h(x))]^2 \leq C_{11} \Big(v_k(x) - \frac{1}{2}|h(x)|^2 \Big) \tag{3.6}$$

for all $x \in B_k, k = 1, 2, \cdots$, and for some constant $C_{12} > 0$ depending only on the geometry of $\bar{B}_q(\tau)$. Since v and v_k are harmonic functions, we have $v_k(x) < v(x)$ on B_k, $k = 1, 2, \cdots$. From this and (3.6), we see easily that

$$[\rho(u(x), h(x))]^2 \leq C_{11} \Big(v(x) - \frac{1}{2}|h(x)|^2 \Big) \tag{3.7}$$

for all $x \in M$. Hence $\lim\limits_{x \in E_A \to \infty} u(x) = p_A$, $A = 1, \cdots, l$.

To show the uniqueness, let u_1 and u_2 be two harmonic maps from M into $\bar{B}_q(\tau)$ with the same values at the infinity of M. From Theorem A in [7], the function

$$F(x) = \frac{Q(\rho(u_1(x), u_2(x)))}{\cos\sqrt{\bar{K}}\rho(q, u_1(x)) \cos\sqrt{\bar{K}}\rho(q, u_2(x))}$$

satisfies the maximum principle in B_k:

$$\sup_{B_k} F(x) \leq \sup_{\partial B_k} F(x), \tag{3.8}$$

where

$$Q(t) = \begin{cases} (1-\cos\sqrt{\bar{K}}t)/\bar{K}, & \text{if} \quad \bar{K} > 0, \\ t^2/2, & \text{if} \quad \bar{K} = 0. \end{cases}$$

By letting $k \to +\infty$ in (3.8), we have $u_1 \equiv u_2$ on M.

As for the energy of u, we first observe that

$$\int_{B_k} e(u_k) \le \int_{B_k} e(h), \quad k = 1, 2, \cdots .$$

Given any $R > 0$, we have $R_k > R$ for k large enough. So $\int_{B_p(R)} e(u_k) \le \int_{B_k} e(h)$. Letting $k \to \infty$ leads to $\int_{B_p(R)} e(u) \le \int_M e(h)$. But f_A's have finite Dirichlet integrals, from the construction of h and Lemma 2.2, we can conclude that $\int_M e(h) < \infty$, therefore, $\int_M e(u) < \infty$. This completes the proof.

Remark 3.2. From Theorem 1.1. and Theorem 1.2, for certain M, one can establish a one-to-one correspondence between the harmonic maps with $u(M) \subset \bar{B}_q(\tau)$ and the n-multiple points $(p_1, \cdots, p_l)$ with $p_A \in \bar{B}_q(\tau), A = 1, \cdots, l$.

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