*Chin. Ann. of Math.*
**20B**: 2(1999),255-260.

# DISCRETE REGULARIZATION FOR OPERATOR EQUATION OF HAMMERSTEIN'S TYPE

Nguyen BUONG*

**Abstract**

The purpose of tthis note is to study a convergence for a method in form of combination of discrete approximations with regularization for solving operator equations of Hammerstein's type in Banach spaces. For illustration, an example in the theory of nonlinear integral equations is given.

**Keywords** Discrete regularization, Operator equations of Hammerstein's type, Integral equation
**1991 MR Subject Classification** 47H30, 47A58
**Chinese Library Classification** O177.2

## §1. Introduction

In [5] we presented a method of regularization for solving the operator equation of Hammerstein's type

$$x + F_2F_1(x) = f_0, \quad f_0 \in X, \tag{1.1}$$

where both the operators $F_2 : X^* \to X$ and $F_1 : X \to X^*$ are nonlinear, hemicontinuous and monotone; $X$ is a reflexive, strictly convex Banach space having the $E$-property, i.e. weak convergence and convergence of norms for any sequence in $X$ follow it strong convergence, and $X^*$ denotes the adjoint space of $X$. Obviously, any uniformly convex Banach space has the $E$-property. From now on we suppose that $X$ and $X^*$ are uniformly convex. Further, for the sake of simplicity norms of $X$ and $X^*$ will be denoted by one symbol $\|\cdot\|$ and we write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Our method of regularization in [5] is described in the form of operator equation

$$x + F_{2\alpha}F_{1\alpha} = f_0, \tag{1.2}$$

where $F_{i\alpha} = F_i + \alpha U_i,\ i = 1, 2,\ \alpha > 0$ and $U_1 : X \to X^*$ and $U_2 : X^* \to X$ are standard dual mappings (see [19]). Equation (1.2), for all fixed $\alpha > 0$, has a unique solution $x_\alpha$, and $x_\alpha \to x_0$, a solution of (1.1), as $\alpha \to 0$.

In [5] we also considered a problem of approximating $x_\alpha$ by Galerkin method

$$x_n + F^n_{2\alpha}F^n_{1\alpha}(x_n) = f_{0n}, \quad x_n \in X_n, \tag{1.3}$$

Manuscript received December 26, 1996.

*Institute of Information Technology, National Centre for Natural Science and Technology, Hanoi 10000, Vietnam. **E-mail:** nbuong@ioit.ncst.ac.vn

where $F_{2\alpha} = P_n F_{2\alpha} P_n^*$; $F_{1\alpha}^n = P_n^* F_{1\alpha} P_n$; $f_{0n} = P_n f_0$ and $P_n$ is the linear and bounded projection from $X$ onto its subspaces $X_n$. For each $n$ and $\alpha > 0$, Equation (1.3) has a unique solution $x_{\alpha n}$ and $x_{\alpha n} \to x_\alpha$ as $n \to +\infty$. Up to now, it is still open the question about convergence of the sequence $\{x_{\alpha n}\}$ to a solution of (1.1), as both $\alpha \to 0$ and $n \to +\infty$ at once.

In the present note, we shall consider this problem in more general form, when the approximating spaces $X_n$ are not the subspaces of $X$. In Section 2 some necessary notations and facts in the theory of discrete approximations needed in the suitable sections are listed. In Section 3 we present the main results. An example for illustation is given in Section 4.

Note that, recently, the problem of approximating a solution of (1.1) is investigated by several authors because of its importance in applications (see [7–10, 12–19]).

## §2. Discrete Approximations

Let $X_n$, for each $n$, be an $n$-dimensional Banach space, and $X_n^*$ be its adjoint space. Assume that $p_n$: $X \to X_n$ and $q_n : X^* \to X_n^*$ are the operators having the properties

$$\|p_n x\|_n \to \|x\|, \quad \|q_n x^*\|_n \to \|x^*\|, \quad \text{and } \langle q_n x^*, p_n x\rangle_n \to \langle x^*, x\rangle \text{ as } n \to \infty$$

for $x$ and $x^*$ belonging to sets which are dense in $X$ and $X^*$, respectively, where $\langle \cdot\ ,\ \cdot\rangle_n$ denotes the dual bearing between $X_n$ and $X_n^*$. The sequence $\{x_n\}$, $x_n \in X_n$, is called $d$-strongly convergent to $x \in X$ or $\{x_n\}$ converges $d$-strongly to $x \in X$ (and written by $x = s$-$\lim x_n$), if $\|p_n x - x_n\|_n \to 0$ (similarly, for $X^*$). This sequence is $d$-weakly convergent to $x$ or it converges $d$-weakly to $x$ (and written by $x = w$-$\lim x_n$), if $\langle x_n^*, x_n\rangle_n \to \langle x^*, x\rangle$, for each $d$-strongly convergent sequence $x_n^*$. It is well-known (see [11, 13, 16, 20]) that from the $d$-strong convergence it implies the $d$-weak convergence, and $\|x_n\|_n \le$ const., $\|x\| \le \liminf \|x_n\|_n$, if $\{x_n\}$ is $d$-weakly convergent. Moreover, if $X$ is reflexive, from the boundedness of $\{x_n\}$ it follows the existence of a subsequence of the sequence $\{x_n\}$ that is $d$-weakly convergent. If $X$ possesses $E$-property, then $d$-weak convergence of $\{x_n\}$ and convergence of $\{\|x_n\|_n\}$ follow $d$-strong convergence of the sequence $\{x_n\}$. The sequence of pairs $\{X_n, p_n\}$ satisfying the above properties is called the $d$-approximations of the space $X$.

Now, we list some materials of $d$-approximation for any operator $A : X \to X^*$. If from $s$-$\lim x_n = x \Rightarrow s$-$\lim A_n(x_n) = A(x)$, then the operators $A_n : X_n \to X_n^*$ are called the strong $d$-approximations of $A$. $D$-approximations of $A$ by $A_n$ is called uniformly bounded if for each $R > 0$ there is $r > 0$ such that $x_n \in X_n$, $\|x_n\|_n \le R \Rightarrow \|A_n(x_n)\|_n \le r$. The family of $d$-approximations $A_n$ is called uniformly coercive, if

$$\langle A_n(x_n), x_n\rangle_n \ge \gamma(\|x_n\|_n)\|x_n\|_n,$$

with $\gamma(t) : R^+ \to R^+$, $\gamma(t) \to +\infty$ as $t \to +\infty$.

In the subsequence, we require that $\|p_n\|$, $\|q_n\| \le T$, $\forall n$, $T > 0$, $\{X_n, p_n\}$, $\{X_n^*, q_n\}$ are the $d$-approximations of the spaces $X$ and $X_n^*$, respectively, and they possess $d$-P.P. property (see [11,13]), where the $d$-P.P. property of $X$ means that from $x = w$-$\lim x_n$ and $\|x_n\|_n \to \|x\|$ it follows $x = s$-$\lim x_n$, and $F_i^n$, $n \in \mathbf{N}$, $i = 1, 2$ are the monotone, continuous and strong $d$-approximations of $F_i$, as well as that $U_1^n : X_n \to X_n^*$ and $U_2^n : X_n^* \to X_n$ are the standard dual mappings.

## §3. Main Results

Consider the equation

$$x_n + \widetilde{F}^n_{2\alpha}F^n_{1\alpha}(x_n) = p_n f_0, \quad \widetilde{F}^n_{i\alpha} = F^n_i + \alpha U^n_i. \tag{3.1}$$

**Theorem 3.1.** *For each $n$, $\alpha > 0$ and $f_0 \in x$, Equation (3.1) has a unique solution $x^n_\alpha$. The sequence $\{x^n_\alpha\}$, for each $\alpha > 0$, convrges $d$-strongly to $x_\alpha$.*

**Proof.** Let $Z_n = X_n \times X^*_n$, $Z = X \times X^*$ be Banach spaces with the norm $\|z_n\|_n = (\|x_n\|^2_n + \|x^*_n\|^2_n)^{1/2}$ and $\|z\| = (\|x\|^2 + \|x^*\|^2)^{1/2}$, where $z_n = [x_n, x^*_n]$, $x_n \in X_n$, $x^*_n \in X^*_n$ and $z = [x, x^*]$, $x \in X$, $x^* \in X^*$. In the Banach space $Z_n$ we consider the equation

$$\mathcal{A}^n_\alpha(z_n) \equiv \mathcal{A}^n(z_n) + \alpha J^n(z_n) = \bar{f}_0, \tag{3.2}$$

where

$$\mathcal{A}^n(z_n) = [F^n_1(x_n), F^n_2(x^*_n)] + [-x^*_n, x_n],$$
$$J^n(z_n) = [U^n_1(x_n), U^n_2(x^*_n)], \quad \bar{f}_0 = [\theta^*_n, p_n f_0],$$

and $\theta^*_n$ denotes the zero element of $X^*_n$. It is easy to verify that $\mathcal{A}^n$ is monotone and continuous, and $J^n$ is standard dual mapping from $Z_n$ onto $Z^*_n$ (see [5]). It is also easy to verify that $\mathcal{A}^n_\alpha$ are uniformly coercive. Hence, Equation (3.2) possesses a unique solution $z^n_\alpha = [x^n_\alpha, \widetilde{F}^n_{1\alpha}(x^n_\alpha)]$ (see [3, 17]). Therefore, $x^n_\alpha$ is the unique solution of (3.1). Further, in order to prove that the sequence $\{x^n_\alpha\}$ converges $d$-strongly to $x_\alpha$ we consider the equation

$$\mathcal{A}_\alpha(z) \equiv \mathcal{A}(z) + \alpha J(z) = \bar{f}_0, \tag{3.3}$$

where

$$\mathcal{A}(z) = [F_1(x), F_2(x^*)] + [-x^*, x],$$
$$J(z) = [U_1(x), U_2(x^*)], \quad \bar{f}_0 = [\theta^*, f_0],$$

and $\theta^*$ denotes the zero element of $X^*$. Equation (3.3), for each $\alpha > 0$, has a unique solution $z_\alpha = [x_\alpha, F_{1\alpha}(x_\alpha)]$ (see [3, 17]).

First, we prove that $\{x^n_\alpha\}$ is bouneded. Indeed, since $\langle \mathcal{A}^n(z^n_\alpha) + \alpha J^n(z^n_\alpha), z^n_\alpha\rangle_n = \langle \bar{f}^n_0, z^n_\alpha\rangle_n$,

$$\|z^n_\alpha\|_n \le (\|p_n \bar{f}_0\|_n + \|\mathcal{A}^n(\theta_n)\|_n)/\alpha.$$

Hence, the sequence $\{z^n_\alpha\}$ is bounded. Let $\tilde{z}_\alpha = w\text{-}\lim z^n_\alpha$. We shall prove that $\tilde{z}_\alpha$ is a solution of (3.3), i.e. $\tilde{z}_\alpha = z_\alpha$. Because of monotone property of $\mathcal{A}^n_\alpha$ we have

$$\langle \mathcal{A}^n_\alpha(z_n) - \mathcal{A}^n_\alpha(z^n_\alpha), z_n - z^n_\alpha\rangle_n \ge 0, \quad \forall z_n \in Z_n, \tag{3.4}$$

$$z = s\text{-}\lim z_n.$$

After passing $n \to \infty$ in this inequality, form $d$-approximative property of $F^n_i$ and (3.4), it implies that $\langle \mathcal{A}_\alpha(z) - \bar{f}_0, z - \tilde{z}_\alpha\rangle \ge 0$, $\forall z \in Z$.

By Minty's Lemma (see [19]), we have $\tilde{z}_\alpha$ is a solution of (3.3), since Equation (3.3) has only one solution $\tilde{z}_\alpha = z_\alpha$ and the entire sequence $\{z^n_\alpha\}$ converges $d$-weakly to $z_\alpha$.

On the other hand, from (3.2) and (3.4) we also have

$$\alpha(\|z^n_\alpha\|_n - \|z'^n_\alpha\|_n)^2 \le \alpha\langle -J^n(z'^n_\alpha), z^n_\alpha - z'^n_\alpha\rangle_n + \langle \bar{f}_0 - \mathcal{A}^n(z^n_\alpha), z'^n_\alpha\rangle_n,$$

where $z'^n_\alpha \in Z_n : z_\alpha = s\text{-}\lim z'^n_\alpha$. From the last inequality it implies that $\|z^n_\alpha\|_n \to \|z_\alpha\|$. Then, $x_\alpha = w\text{-}\lim x^n_\alpha$ and $\|x^n_\alpha\|_n \to \|x_\alpha\|$ (see [13]). Since the space $X$ possesses the $E$-property, the sequence $x^n_\alpha \to x_\alpha$ as $n \to +\infty$.

As mentioned above that $x_\alpha \to x_0$ as $\alpha \to 0$. Therefore, it is natural to ask when $x_\alpha^n$ converges $d$-strongly to $x_0$ as $\alpha \to 0$ and $n \to +\infty$. The following theorem answers this question.

**Theorem 3.2.** *Assume that all the above condition concerning the space $X$ and the operators $F_i,\ i = 1, 2$, are satisfied and*

$$\|F_1^n(p_n x) - q_n F_1(x)\|_n \le r_n \tilde{f}_0(x),$$
$$\|F_2^n(q_n x^*) - p_n F_2(x^*)\|_n \le r_n \tilde{f}(x^*), \quad x^* \in R(F_1),$$

*where $R(F_1)$ denotes the range of $F_1, \tilde{f}_0(x)$ and $\tilde{f}(x^*)$ are the functionals on $X$ and $X^*$ respectively. If $\{F_i^n\}$ are uniformly bounded and $\limsup\limits_{\alpha,1/n\to 0} \frac{r_n}{\alpha} = 0$, then the sequence $\{x_\alpha^n\}$ converges $d$-strongly to $x_0$.*

**Proof.** Since, if $z_0 \in \tilde{S}_0 = S_0 \times F_1(S_0)$, where $S_0$ denotes the set of solutions of (1.1),

$$\begin{aligned} 0 &= \langle \mathcal{A}_\alpha^n(z_\alpha^n) - \bar{f}_0, z_\alpha^n - \mathcal{P}_n z_0\rangle_n \\ &= \langle \mathcal{A}^n(z_\alpha^n) - \mathcal{A}^n(\mathcal{P}_n z_0), z_\alpha^n - \mathcal{P}_n z_0\rangle_n \\ &\quad + \langle \mathcal{A}^n(\mathcal{P}_n z_0) - \mathcal{Q}_n\mathcal{A}(z_0), z_\alpha^n - \mathcal{P}_n z_0\rangle_n + \alpha\langle J^n(z_\alpha^n), z_\alpha^n - \mathcal{P}_n z_0\rangle_n \\ &\ge -r_n(\tilde{f}_0(x_0) + \tilde{f}(x_0^*))\|z_\alpha^n - \mathcal{P}_n z_0\|_n + \alpha\langle J^n(z_\alpha^n), z_\alpha^n - \mathcal{P}_n z_0\rangle_n, \end{aligned}$$

where $\mathcal{Q}_n = [q_n, p_n]$ and $\mathcal{P}_n = [p_n, q_n]$, then

$$\langle J^n(z_\alpha^n), z_\alpha^n - \mathcal{P}_n z_0\rangle_n \le \frac{r_n(\tilde{f}_0(x_0) + \tilde{f}(x_0^*))}{\alpha}\|z_\alpha^n - \mathcal{P}_n z_0\|_n, \tag{3.5}$$

hence, the sequence $\{z_\alpha^n\}$ is bounded. Let $\bar{z} = w\text{-}\lim z_\alpha^n$. We prove that $\bar{z}$ is a solution of $\mathcal{A}(z) = \bar{f}_0$. Indeed, for any $z \in Z : z = s\text{-}\lim \mathcal{P}_n z$,

$$\begin{aligned} \langle \mathcal{A}^n(\mathcal{P}_n z) - \mathcal{Q}_n\bar{f}_0, \mathcal{P}_n z - z_\alpha^n\rangle_n &\ge \langle \mathcal{A}^n(z_\alpha^n) - \mathcal{Q}_n\bar{f}_0, \mathcal{P}_n z - z_\alpha^n\rangle_n \\ &= \langle \mathcal{A}_\alpha^n(z_\alpha^n) - \mathcal{Q}_n\bar{f}_0, \mathcal{P}_n z - z_\alpha^n\rangle_n - \alpha\langle J^n(z_\alpha^n), \mathcal{P}_n z - z_\alpha^n\rangle_n \\ &\ge \alpha\langle J^n(\mathcal{P}_n z), z_\alpha^n - \mathcal{P}_n z\rangle_n. \end{aligned}$$

From this inequality, after passing $\alpha$ and $1/n$ to zero, we obtain

$$\langle \mathcal{A}(z) - \bar{f}_0, z - \bar{z}\rangle \ge 0, \quad \forall z \in Z.$$

By virtue of Minty's Lemma (see [19]), $\mathcal{A}(\bar{z}) = \bar{f}_0$. Now, from (3.5) it follows that

$$\langle J^n(\mathcal{P}_n z_0), z_\alpha^n - \mathcal{P}_n(z_0)\rangle_n \le \frac{r_n(\tilde{f}_0(x_0) + \tilde{f}(x_0^*))}{\alpha}\|z_\alpha^n - \mathcal{P}_n z_0\|_n.$$

By tending $\alpha, 1/n \to 0$ in the last inequality, it implies that

$$\langle J(z_0), , \bar{z} - z_0\rangle \le 0, \quad \forall z_0 \in \overline{S}_0.$$

By the similar argument, as in [13], we have $\|\bar{z}\| \le \|z_0\|,\ \forall z_0 \in \overline{S}_0$. The element $\bar{z}$ of $\overline{S}_0$ with the last property is defined uniquely. Therefore, all the sequence $\{z_\alpha^n\}$ converges $d$-weakly to $\bar{z}$.

Since $\langle J^n(z_\alpha^n), z_\alpha^n - \mathcal{P}_n(\bar{z})\rangle_n \ge \langle J^n(\mathcal{P}_n\bar{z}), z_\alpha^n - \mathcal{P}_n\bar{z}\rangle_n \to 0, \liminf\langle J^n(z_\alpha^n), z_\alpha^n - \mathcal{P}_n\bar{z}\rangle_n \ge 0$. On the other hand, from (3.5) we also have

$$\limsup\langle J^n(z_\alpha^n), z_\alpha^n - \mathcal{P}_n\bar{z}\rangle_n \le 0.$$

These two conclusions give the result $\lim\limits_{\alpha,1/n\to 0}\langle J^n(z_\alpha^n), z_\alpha^n - \mathcal{P}_n\bar{z}\rangle_n = 0$. As

$$(\|z_\alpha^n\|_n - \|\mathcal{P}_n\bar{z}\|)^2 \le \langle J^n(z_\alpha^n) - J^n(\mathcal{P}_n\bar{z}), z_\alpha^n - \mathcal{P}_n\bar{z}\rangle_n,$$

then $\|z_\alpha^n\|_n \to \|\bar{z}\|$, when $\alpha, 1/n \to 0$. In the similar way, as in [5], $\|x_\alpha^n\|_n \to \|\bar{x}\|$ and $\bar{x} = w$-$\lim x_\alpha^n$. Consequently, the $E$-property of the space $X$ gives the conclusion that $\bar{x} = s$-$\lim x_\alpha^n$.

## §4. Example

Consider the nonlinear integral equation of Hammerstein's type

$$\varphi(t) + \int_a^b k(t,s) f(\varphi(s)) ds = f_0(t), \quad t \in [a,b], \tag{4.1}$$

where the functions $f_0 \in L_{p_1}[a,b]$, $k$ and $f$ that are continuous on $[a,b] \times [a,b]$ and $R$, are given and satisfy the conditions

$$k(t,s) \geq 0, \quad |f(t)| \leq a_0 + b_0|t|^{p_1-1}, \quad a_0 + b_0 > 0, \quad a_0, b_0 \geq 0,$$
$$f(t_1) \leq f(t_2), \quad \text{iff } t_1 \leq t_2.$$

Then the operators $F_i,\ i = 1,2$, defined by

$$(F_2\xi)(t) = \int_a^b k(t,s)\xi(s)ds, \quad \xi \in L_{p_2}[a,b], \quad p_1^{-1} + p_2^{-1} = 1,$$
$$(F_1\varphi)(t) = f(\varphi(t)), \quad \varphi \in L_{p_1}[a,b]$$

are monotone and continuous with $X = L_{p_1}[a,b]$ and $X^* = L_{p_2}[a,b]$. The standard dual mapping of $L_{p_i}[a,b]$ has the form (see [19])

$$(U_i(\varphi))(t) = \|\varphi\|_{L_{p_i}[a,b]}^{2-p_i} |\varphi(t)|^{p_i-2}\varphi(t).$$

Let $p_n$ be defined by

$$p_n:\ \varphi(t) \to \Big(\frac{1}{h_1^n}\int_{t_0^n}^{t_1^n} \varphi(t)dt, \cdots, \frac{1}{h_n^n}\int_{t_{n-1}^n}^{t_n^n} \varphi(t)dt\Big),$$

where $t_0^n = a < t_1^n,\ t_2^n < \cdots < t_n^n = b$ is a partition of $[a,b]$ with $h_i^n = t_i^n - t_{i-1}^n$, $\lim \max h_i^n = 0$, as $n$ tends to $+\infty$. Therefore, $\|p_n\| \leq 1$. As $X_n$, we use the space $l_{p_1}^n$ with the dual mapping $U_1^n(x) = \|x\|_{l_{p_1}^n}^{2-p_1} z,\ z = (|x_1|^{p_1-2}x_1, \cdots, |x_n|^{p_1-2}x_n),\ x = (x_1, \cdots, x_n)$. The approximating operators $F_i^n, i = 1,2$ are defined by

$$(F_2^n\xi^n)_i = \sum_{j=1}^n h_j^n k_{ij}^n \xi^n, \quad i = 1, \cdots, n, \quad \xi^n \in l_{p_2}^n,$$
$$k_{ij} = k(t_i^n, t_j^n), \quad \text{if } j = 2, 3 \cdots, n-1; \quad k_{ij}^n = \frac{1}{2}k(t_i^n, t_j^n), \quad j = 1, n,$$
$$(F_1^n(\varphi^n))_i = f(\varphi_i^n), \quad i = 1, 2, \cdots, n, \quad \varphi \in l_{p_1}^n.$$

Obviously, $F_i^n,\ i = 1,2$, are discret $d$-approximations of $F_i$ and uniformly bounded (see [16, 20]). We have

$$(F_2^n(q_n\xi))_i - (p_n(F_2\xi)(t))_i = \sum_{j=1}^n h_j^n k_{ij}^n \xi(t_j^n) - \frac{1}{h_i^n}\int_{t_{i-1}^n}^{t_i^n} \Big(\int_a^b k(t,s)\xi(s)ds\Big)dt$$
$$= \sum_{j=1}^n h_j^n k_{ij}^n \xi(t_j^n) - \int_a^b k(t_i^n, s)\xi(s)ds = O(h^2),$$

if $t_i^n - t_{i-1}^n = h = (b-a)/(n-1)$. On the other hand,

$$(F_1^n(p_n\varphi) - q_nF_1(\varphi))_i = f\Big(\frac{1}{h_i^n}\int_{t_{i-1}^n}^{t_i^n} \varphi(t)dt\Big) - \frac{1}{h_i^n}\int_{t_{i-1}^n}^{t_i^n} f(\varphi(t))dt.$$

If $f$ has a bounded derivative, then

$$f\Big(\frac{1}{h}\int_{t_{i-1}^n}^{t_i^n}\varphi(\tau)d\tau\Big)-\frac{1}{h}\int_{t_{i-1}^n}^{t_i^n}f(\varphi(t))dt$$

$$=f\Big(\frac{1}{h}\int_{t_{i-1}^n}^{t_i^n}\varphi(\tau)d\tau\Big)-f(\varphi(t))|_{t\in(t_{i-1}^n,t_i^n)}+O(h^2)$$

$$=f'(\varphi(t))\Big(\frac{1}{h}\int_{t_{i-1}^n}^{t_i^n}\varphi(\tau)d\tau-\varphi(t)\Big)+O(h^2)=O(h^2).$$

Hence, $\alpha$ can be chosen such that $\alpha=h^\mu,\ 0<\mu<2$.

## References


[1] Amann, H., Ein Existenz und Eindeutigkeitssatz fur die Hammersteinshe Gleichung in Banachraumen, *Math. Z.,* **111**(1969), 175–190.

[2] Amann, H., Uber die naherungsweise Losung nichlinearer Integralgleichungen, *Numer. Math.,* **19**(1972), 29–45.

[3] Brezis, H. & Browder, F., Nonlinear integral equations and systems of Hammerstein's type, *Adv. Math.,* **10**(1975), 115–144.

[4] Browder, F., Nonlinear functional analysis and nonlinear equations of Hammerstein and Uryshon type, in Contribution to Nonlinear Functional Analysis, E, H. Zarantonello ed., Academic Press, New York, 1971, 425–500.

[5] Buong, N., On solutions of the equations of Hammerstein type in Banach spaces, *J. of Math. Computation* & *Math. Physics, USSR,* **25:**8(1985), 1256–1280 (in Russian).

[6] Dolph, C. & Minty, G., On nonlinear integral equations of Hammerstein type, in Nonlinear Integral Equations, P. Anselon ed., Wis. Press, Nadison, 1964, 99–154.

[7] Kaneko, H., Noren, R. & Xu, Y., Numerical solution for weakly singular Hammerstien equations and their superconvergence, *J. Int. Eq. Appl.,* **4**(1992), 391–407.

[8] Kaneko, H. & Xu, Y., Degenerate kernel method for Hammerstein equations, *Math. Comp.,* **56**(1991), 141–148.

[9] Kannan, R. & Salehi, H., Random nonlinear equations and monotonic nonlinearities, *J. Math. Anal. Appl.,* **78**(1980), 488–496.

[10] Kumar, S., Superconvergence of a collocation-type method for Hammerstein equations, *IMA J. of Numerical Analysis,* **7**(1987), 313–325.

[11] Liskovets, O. A., Regularization for problems involving monotone operators under discrete approximations for the spaces and operators, *J. of Math. Computation* & *Math. Physics USSR,* **27:**1(1987), 3–15 (in Russian).

[12] Panchal, C. D., Existence theorems for equations of Hammerstein type, *The Quart. J. of Math.,* **35**(1984), 311–320.

[13] Pankov. A. A., Discrete approximation of convex sets and convergence of solution of variational inequalities, *Math. Nachr.,* **91**(1979), 7–12.

[14] Pascali, D. & Sburlan, S., Nonlinear mappings of monotone type, Bucur. Roumania, 1978.

[15] Petryshyn, W. V. & Fitzpatrick, R. M., New existence theorems for nonlinear equation of Hammerstein type, *Trans. AMS,* **160**(1971), 39–63.

[16] Stummel, F., Discrete convergence of mappings, in Numer. Analysis, Proc. Conf. Numer. Analysis, Dublin, 1972, London-New York, 1973, 285–310.

[17] Tartar, L., Topics in nonlinear analysis, Publ. D'orsey, 1978.

[18] Vaclav, D., Monotone operators and applications in control and network theory, Ams. Oxf. New York, Elsevier, 1979.

[19] Vainberg, M. M., Variational method and method of monotone operators, Moscow, Nauka, 1972 (in Russion).

[20] Vassin, V. V., Discrete approximation and stability in extremal problems, *J. Comput. Math. and Math. Physics,* **22**(1982), 824–839.