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HOPF-JACOBSON RADICAL FOR COMODULE ALGEBRAS***

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Abstract

Let H be a Hopf algebra over a field k (not necessarily finite dimensional). In this paper the Hopf-Jacobson radical $J^H(A)$ of right H-comodule algebra A (not necessarily with identity) is studied. The relationships between $J^H(A)$ and the Jacobson radical of the smash product $A\#H^{*\mathrm{rat}}$ are discussed. The density theorem is given for left (A,H)-Hopf simple module.

Keywords Comodule algebra, Hopf-Jacobson radical, Density theorem
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§1. Introduction

Fisher[7] discussed the Hopf-Jacobson radical $\mathbf{J}_H(A)$ of the H-module algebra A, where H is an irreducible Hopf algebra. In [5], Cai generalized Fisher's result. He proved that $\mathbf{J}_H(A)\#H \subseteq \mathbf{J}(A\#H)$, where H is an arbitrary Hopf algebra, A an H-module algebra. A version of the Chevalley-Jacobson density theorem for H-module algebra was also proved by Cai[5].

Dually, Liu Guilong[8] defined and studied the Hopf-Jacobson radical of H-comodule algebra A. In this paper, we study further the Hopf-Jacobson radical of the H-comodule algebra A, and we give a version of the Chevalley-Jacobson density theorem for H-comodule algebra.

In Section 2, we discuss the right H-comodule algebra A (not necessarily with identity 1), and the smash product $A\#H^{*\mathrm{rat}}$. We first show that $H^{*\mathrm{rat}}$ is an essential left (right) ideal of H^*. Next, we show that $\widetilde{A}\#H^{*\mathrm{rat}}$ is an essential right ideal of $\widetilde{A}\#H^*$, where A is a right H-comodule algebra. Finally, we show that M is a left (A,H)-Hopf simple module if and only if it is a left $A\#H^{*\mathrm{rat}}$-simple module.

In Section 3, we first study the Hopf-Jacobson radical of the right H-comodule algebra A. We discuss the relationships between $\mathbf{J}(A^{\mathrm{co}H})$ and $\mathbf{J}(A\#H^{*\mathrm{rat}})$, between $\mathbf{J}^H(A)$ and $\mathbf{J}(A\#H^{*\mathrm{rat}})$ and between $\mathbf{J}^H(A)$ and $\mathbf{J}(A^{\mathrm{co}H})$ respectively, where $\mathbf{J}(R)$ denotes the Jacobson radical of algebra R, $\mathbf{J}^H(A)$ is the Hopf-Jacobson radical of the right H-comodule algebra A. Next, we show that if $A\#H^{*\mathrm{rat}}$ is a primitive algebra then $A\#H^*$ is a primitive algebra.

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In Section 4, we first show that $\mathrm{Hom}_A(M,N)$ is a left H-module, and we point out that $\mathrm{HOM}_A(M,N) = [\mathrm{Hom}_A(M,N)]^{\mathrm{rat}}$, where the definition of $\mathrm{HOM}_A(M,N)$ was given by K. H. Ulbrich[11]. Next, we give a version of the Chevally-Jacobson density theorem for right H-comodule algebras.

Throughout this paper, H denotes a Hopf algebra over a field k (not necessarily finite dimensional), with comultiplication $\triangle$, counit ε, and antipode $\mathbf{S}$. H^* denotes the dual algebra of H, and ε is an identity of H^*. $\int^l$ (resp. $\int^r$) denotes the left (resp. right) integral space. In this paper, unless otherwise stated, we use the notation from [10], $\otimes$ means $\otimes_k$, Hom means Hom_k and we assume that $\int^l \neq 0$ ($\int^r \neq 0$). In this case, $H^{*\mathrm{rat}}$ is dense in H^* by [1, p.433], and $\mathbf{S}$ is bijective by [1, Proposition 2]. Let $\overline{\mathbf{S}}$ denote the inverse of $\mathbf{S}$.

Let A be a right H-comodule algebra over the field k, $\rho_A : A \longrightarrow A \bigotimes H$ the comodule structure map. Define $A^{\mathrm{co}H} = \{ a \in A \mid \rho_A(a) = a \otimes 1 \}$, the subalgebra of coinvariant elements. If A is a k-algebra with 1, then $1 \in A^{\mathrm{co}H}$. In fact, since A is a right H-comodoule, A is a left H^*-module. For any $a, b \in A$, $h^* \in H^*$, we have

$$\begin{aligned}
(h^* \rightarrow a)b &= \sum a_{(0)} b \langle h^*, a_{(1)} \rangle \\
&= \sum a_{(0)} b_{(0)} \langle h^*, a_{(1)} b_{(1)} \mathbf{S} b_{(2)} \rangle = \sum (\mathbf{S} b_{(1)} \rightharpoonup h^*) \rightarrow a b_{(0)}, \\
(h^* \rightarrow 1)b &= \sum (\mathbf{S} b_{(1)} \rightharpoonup h^*) \rightarrow b_{(0)} = \sum b_{(0)} \langle h^*, b_{(1)} \mathbf{S} b_{(2)} \rangle \\
&= \sum b_{(0)} \varepsilon(b_{(1)}) \langle h^*, 1 \rangle = b \langle h^*, 1 \rangle.
\end{aligned}$$

Thus $\rho_A(1) = 1 \otimes 1$, $1 \in A^{\mathrm{co}H}$. In this case, we say that A is a right H-comodule algebra with 1. If not, let $\widetilde{A} = A \bigoplus k$ (direct sum of vector spaces), $\widetilde{A}$ becomes an algebra with 1 by the usual way. Define

$$\rho_{\widetilde{A}} : \widetilde{A} \longrightarrow \widetilde{A} \bigotimes H, \quad \rho_{\widetilde{A}}(a + n) = \rho_A(a) + n \otimes 1,$$

then $\widetilde{A}$ becomes a right H-comodule algebra with 1. We write $\widetilde{A} = A$ for right H-comodule algebra A with 1. In this paper, unless otherwise stated, A will denote a right H-comodule algebra, and we will not require that A has an identity 1.

Let M be a left A-module, right H-comodule. The module action will be denoted by $a \cdot m$ for $a \in A$, $m \in M$ and the H-comodule structure map is given by $\rho_M : M \longrightarrow M \bigotimes H$. We write $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$ where the $m_{(0)}$'s lie in M while the $m_{(1)}$'s lie in H. M is called a left (A,H)-Hopf module if

$$\rho_M(a \cdot m) = \sum (a \cdot m)_{(0)} \otimes (a \cdot m)_{(1)} = \sum (a_{(0)} \cdot m_{(0)}) \otimes (a_{(1)} m_{(1)}).$$

M is called a right (A,H)-Hopf module, if M is a right A-module, a right H-comodule and

$$\rho_M(m \cdot a) = \sum (m \cdot a)_{(0)} \otimes (m \cdot a)_{(1)} = \sum (m_{(0)} \cdot a_{(0)}) \otimes (m_{(1)} a_{(1)}).$$

§2. A, $H^{*\mathrm{rat}}$ and $A\#H^{*\mathrm{rat}}$

In [2], Beattie defined the smash product $A\#H^*$ for the right H-comodule algebra A with 1. As a matter of fact, the smash product $A\#H^*$ may be defined for any right H-comodule algebra A. As a vector space, $A\#H^*$ is $A \bigotimes H$. Elements $a \otimes h^*$ will be written as $a\#h^*$, $a \in A$, $h^* \in H^*$. The multiplication is defined by

$$(a\#h^*)(b\#g^*) = \sum a b_{(0)} \#(h^* \leftharpoonup b_{(1)}) g^*, \quad a, b \in A, \quad h^*, g^* \in H^*.$$

Clearly $A\#H^* \subseteq \widetilde{A}\#H$. We note that the map $\gamma : A \longrightarrow A\#H^*$ given by $\gamma : a \mapsto a\#\varepsilon$ is injective, thus we may regard $A \subseteq A\#H^*$. In addition, we write

$$A\#H^{*\mathrm{rat}} = \Big\{ \sum_i a_i\#h_i^* \mid a_i \in A,\, h_i^* \in H^{*\mathrm{rat}} \Big\}.$$

Lemma 2.1. (1) *A is a comodule ideal of $\widetilde{A}$;*

(2) *$A\#H^{*\mathrm{rat}}$ is an ideal of $\widetilde{A}\#H^*$;*

(3) *$A\#H^*$ is an ideal of $\widetilde{A}\#H^*$.*

Proof. Straightforward.

Theorem 2.1. (1) *$H^{*\mathrm{rat}}$ is an essential right (left) ideal of H^*.*

(2) *$\widetilde{A}\#H^{*\mathrm{rat}}$ is an essential right ideal of $\widetilde{A}\#H^*$.*

Proof. (1) Let I be a nonzero right ideal of H^* and $0 \neq f^* \in I$. Then there is an $h \in H$ such that $\langle f^*, h\rangle \neq 0$. Since $(id \otimes \varepsilon)(\triangle\ h - h \otimes 1) = 0$, $\triangle\ h - h\otimes 1 \in H \bigotimes \ker\varepsilon$. Set $\triangle\ h = h\otimes 1 + \sum\limits_{i=1}^{m} x_i \otimes y_i$, $y_i \in \ker\varepsilon$, $1 \leqslant i \leqslant m$, and $y_1, y_2, \cdots, y_m$ are linearly independent. It follows that $1, y_1, y_2, \cdots, y_m$ are linearly independent. Since $H^{*\mathrm{rat}}$ is dense in H^*, there is a $g^* \in H^{*\mathrm{rat}}$ such that $\langle g^*, 1\rangle = 1$, $\langle g^*, y_i\rangle = 0$, $1 \leqslant i \leqslant m$. Hence

$$\langle f^* * g^*, h\rangle = \sum_{i=1}^{m}\langle f^*, x_i\rangle\langle g^*, y_i\rangle + \langle f^*, h\rangle\langle g^*, 1\rangle = \langle f^*, h\rangle\langle g^*, 1\rangle \neq 0.$$

Thus $0 \neq f^* * g^* \in I * H^{*\mathrm{rat}} \subseteq I \bigcap H^{*\mathrm{rat}}$ and $I \bigcap H^{*\mathrm{rat}} \neq 0$. Therefore $H^{*\mathrm{rat}}$ is an essential right ideal of H^*.

Similarly, we can prove that $H^{*\mathrm{rat}}$ is an essential left ideal of H^*.

(2) Let I be a nonzero right ideal of $\widetilde{A}\#H^*$ and $0 \neq \sum\limits_{i=1}^{n} a_i\#h_i^* \in I$, where $h_1^*, h_2^*, \cdots, h_n^*$ are linearly independent elements of H^* and $a_1 \neq 0$. Then there is an $h \in H$, such that $\langle h_1^*, h\rangle = 1$, $\langle h_i^*, h\rangle = 0$, $2 \leqslant i \leqslant n$. Similarly to (1), let $\triangle\ h = h\otimes 1 + \sum\limits_{j=1}^{m} x_j \otimes y_j$ where $1, y_1, y_2, \ldots, y_m$ are linearly independent elements of H. Then there is a $g^* \in H^{*\mathrm{rat}}$ such that $\langle g^*, 1\rangle = 1$ and $\langle g^*, y_j\rangle = 0$, $1 \leqslant j \leqslant m$. Thus $\langle h_1^* * g^*, h\rangle = 1$, $\langle h_i^* * g^*, h\rangle = 0$, $2 \leqslant i \leqslant n$. And it will be seen that the $h_1^* * g^*$ cannot be linearly represented by $h_2^* * g^*, h_3^* * g^*, \ldots, h_n^* * g^*$. Hence $(\sum a_i\#h_i^*)(1\#g^*) = \sum a_i\#(h_i^* * g^*) \neq 0$ and $I\bigcap(\widetilde{A}\#H^{*\mathrm{rat}}) \neq 0$. This completes the proof.

Lemma 2.2. *In $\widetilde{A}\#H^*$,*

(1) *$A\#H^{*\mathrm{rat}} = (1\#H^{*\mathrm{rat}})(A\#\varepsilon)$;* (2) *$A\#H^* = (A\#\varepsilon)(1\#H^*)$.*

Proof. (1) For any $a \in A$, $h^* \in H^*$, we have

$$\begin{aligned} a\#h^* &= \sum a_{(0)}\varepsilon(a_{(1)})\#h^* = \sum a_{(0)}\#[h^* \leftharpoonup \varepsilon(a_{(1)})1_H] \\ &= \sum a_{(0)}\#[(h^* \leftharpoonup (\overline{\mathbf{S}}a_{(2)})) \leftharpoonup a_{(1)}] = \sum [1\#(h^* \leftharpoonup (\overline{\mathbf{S}}a_{(1)}))](a_{(0)}\#\varepsilon). \end{aligned}$$

(2) For any $a \in A$, $h^* \in H^*$, one can easily check that $a\,\#h^* = (a\#\varepsilon)(1\#h^*)$. Thus (2) holds.

Lemma 2.3. (1) *Let I be a nonzero right ideal of $A\#H^*$ with $I(A\#H^*) \neq 0$. Then*

$$I\bigcap(A\#H^{*\mathrm{rat}}) \neq 0.$$

(2) *If I is a nonzero semiprime ideal of $A\#H^*$, then $I\bigcap(A\#H^{*\mathrm{rat}}) \neq 0$.*

Proof. (1) Since $A\#H^* \lhd \widetilde{A}\#H^*$, $I(A\#H^*)$ is a nonzero right ideal of $\widetilde{A}\#H^*$. By Theroem 2.1(2) we have $[I(A\#H^*)]\bigcap(\widetilde{A}\#H^{*\mathrm{rat}}) \neq 0$. Clearly $I(A\#H^*) \subseteq I$. Thus

$$I\bigcap(A\#H^{*\mathrm{rat}}) = I\bigcap(\widetilde{A}\#H^{*\mathrm{rat}}) \neq 0.$$

(2) Since I is a semiprime ideal of $A\#H^*$ and $A\#H^* \lhd \widetilde{A}\#H^*$, $I \lhd \widetilde{A}\#H^*$. By Theorem 2.1(2) we have

$$I\bigcap(A\#H^{*\mathrm{rat}}) = I\bigcap(\widetilde{A}\#H^{*\mathrm{rat}}) \neq 0.$$

Note that a left (resp. right) R-module M is called unital if $RM = M$ (resp. $MR = M$). An (A, H)-Hopf module M is called unital if it is a unital left A-module.

If M is a left (A, H)-Hopf module, then the right H-comodule structure induces naturally a left $H^{*\mathrm{rat}}$-module structure: $h^* \rightarrow m = \sum\langle h^*, m_{(1)}\rangle m_{(0)}$. Thus it induces naturally a left $A\#H^{*\mathrm{rat}}$-module structure: $(a\#h^*)\cdot m = a\cdot(h^*\rightarrow m)$, $m \in M$, $a \in A$, $h^* \in H^{*\mathrm{rat}}$.

Set $(0 : A\#H^{*\mathrm{rat}})_M = \{\, m \in M \mid A\#H^{*\mathrm{rat}}\cdot m = 0\,\}$.

Theorem 2.2. (1) *If M is a unital left (A, H)-Hopf module, then the left $A\#H^{*\mathrm{rat}}$-module induced above is unital.*

(2) *If M is a unital left $A\#H^{*\mathrm{rat}}$-module with $(0 : A\#H^{*\mathrm{rat}})_M = 0$, then M becomes a unital left (A, H)-Hopf module and the induced left $A\#H^{*\mathrm{rat}}$-module structure coincides with the original one.*

Proof. (1) Since $H^{*\mathrm{rat}}$ is dense in H^*, by [3, Lemma 2.1 (2)], the left $H^{*\mathrm{rat}}$-module M is unital. Thus

$$(A\#H^{*\mathrm{rat}})\cdot M = A\cdot(H^{*\mathrm{rat}} \longrightarrow M) = A\cdot M = M.$$

(2) Since $(A\#H^{*\mathrm{rat}})\cdot M = M$, for any $m \in M$, there are $m_i \in M$, $x_i \in A\#H^{*\mathrm{rat}}$ such that $m = \sum x_i\cdot m_i$. Since $A\#H^{*\mathrm{rat}} \lhd \widetilde{A}\#H^{*\mathrm{rat}}$, for any $y \in \widetilde{A}\#H^{*\mathrm{rat}}$, define $y\cdot m = \sum(yx_i)m_i$. If $\sum x_im_i = 0$, then

$$(A\#H^{*\mathrm{rat}})\Big[\sum(yx_i)m_i\Big] = [(A\#H^{*\mathrm{rat}})y]\Big(\sum x_im_i\Big) = 0,$$

hence $\sum(yx_i)m_i = 0$ and the action $y\cdot m$ is well-defined. Thus the $A\#H^{*\mathrm{rat}}$-module action on M is extended to an $\widetilde{A}\#H^{*\mathrm{rat}}$-module action. By [3, Corollary 3.6 (1)], M becomes a left $(\widetilde{A}, H)$-Hopf module. Hence M is a left (A, H)-Hopf module, too. Furthermore,

$$A\cdot M = A\cdot(H^{*\mathrm{rat}} \longrightarrow M) = (A\#H^{*\mathrm{rat}})\cdot M = M.$$

Lemma 2.4. *If M is a left (A, H)-Hopf module, then*

(1) $(0 : M)_A = \{\, a \in A \mid a\cdot M = 0\,\}$ *is a comodule ideal of A;*

(2) $(0 : A)_M = \{\, m \in M \mid A\cdot m = 0\,\}$ *is an (A, H)-Hopf submodule of M.*

Proof. (1) For any $a \in A$, $m \in M$, $h^* \in H^*$,

$$\begin{aligned} h^* \rightarrow (a\cdot m) &= \sum\langle h^*, a_{(1)}m_{(1)}\rangle a_{(0)}\cdot m_{(0)} \\ &= \sum\langle m_{(1)} \rightharpoonup h^*, a_{(1)}\rangle a_{(0)}\cdot m_{(0)} = \sum[(m_{(1)} \rightharpoonup h^*) \rightarrow a]\cdot m_{(0)}. \end{aligned}$$

Thus for any $a \in (0 : M)_A, m \in M, h^* \in H^*$,

$$\begin{aligned} (h^* \rightarrow a)\cdot m &= \sum\varepsilon(m_{(1)})(h^* \rightarrow a)\cdot m_{(0)} = \sum[(\varepsilon(m_{(1)})1_H \rightharpoonup h^*) \rightarrow a]\cdot m_{(0)} \\ &= \sum[(m_{(1)} \rightharpoonup \mathbf{S}m_{(2)} \rightharpoonup h^*) \rightarrow a]\cdot m_{(0)} = \sum(\mathbf{S}m_{(1)} \rightharpoonup h^*) \rightarrow (a\cdot m_{(0)}) = 0. \end{aligned}$$

This shows that $(0 : M)_A$ is a right H-comodule ideal of A.

(2) Let $m \in (0:A)_M$, $a \in A$, $h \in H$,

$$a \cdot (h^* \to m) = \sum \langle h^*, m_{(1)}\rangle a \cdot m_{(0)} = \sum \langle h^*, \varepsilon(a_{(1)}) m_{(1)}\rangle a_{(0)} \cdot m_{(0)}$$
$$= \sum \langle h^* \leftharpoonup \overline{\mathbf{S}} a_{(2)}, a_{(1)} m_{(1)}\rangle a_{(0)} \cdot m_{(0)} = \sum (h^* \leftharpoonup \overline{\mathbf{S}} a_{(1)}) \to (a_{(0)} \cdot m) = 0.$$

Thus $(0:A)_M$ is a subcomodule of M and so it is an (A,H)-Hopf submodule of M.

Theorem 2.3. *M is a left (A,H)-Hopf simple module if and only if left $A\#H^{*\mathrm{rat}}$-module M is simple, i.e. M has no nontrivial $A\#H^{*\mathrm{rat}}$-submodules and $A\#H^{*\mathrm{rat}} \cdot M \neq 0$.*

Proof. Let M be a left (A,H)-Hopf simple module, then M is a left $(\widetilde{A},H)$-Hopf simple module. By [3, Proposition 3.5 (1) and Corollary 3.6 (1)] the left $\widetilde{A}\#H^{*\mathrm{rat}}$-module M is a simple module. By Theorem 2.2(1), $(A\#H^{*\mathrm{rat}}) \cdot M = M$. If N is a proper submodule of the left $A\#H^{*\mathrm{rat}}$-module M, then $(A\#H^{*\mathrm{rat}}) \cdot N$ is a proper submodule of $\widetilde{A}\#H^{*\mathrm{rat}}$-module M. Thus $(A\#H^{*\mathrm{rat}}) \cdot N = 0$. Since $A \cdot N = A \cdot (H^{*\mathrm{rat}} \to N) = (A\#H^{*\mathrm{rat}})N = 0$, $N \subseteq (0:A)_M = 0$. Hence the left $A\#H^{*\mathrm{rat}}$-module M is simple.

Conversely, let M be a left $A\#H^{*\mathrm{rat}}$-simple module. By Theorem 2.2(2), M becomes a left (A,H)-Hopf module and $A \cdot M = M$. It is clear that a left (A,H)-Hopf submodule of M is also a left $A\#H^{*\mathrm{rat}}$-submodule of M. Thus the left (A,H)-Hopf module M is simple.

§3. The Hopf-Jacobson Radical

Definition 3.1. *Let M be a left (A,H)-Hopf simple module. The $(0:M)_A$ is said to be an (A,H)-Hopf primitive ideal of A. A is called an (A,H)-Hopf primitive algebra if there is an (A,H)-Hopf simple module M such that $(0:M)_A = 0$. $\bigcap (0:M)_A$ (M runs over all (A,H)-Hopf simple modules) is called the Hopf-Jacobson radical of A. We write $\mathbf{J}^H(A)$ for it.*

Theorem 3.1. (1) *$\mathbf{J}^H(A)$ is a comodule ideal of A;*

(2) *$\mathbf{J}^H(A)\#H^{*\mathrm{rat}} \subseteq \mathbf{J}(A\#H^{*\mathrm{rat}})$;*

(3) *$\mathbf{J}^H(A) = \{\, a \in A \mid a\#h^* \in \mathbf{J}(A\#H^{*\mathrm{rat}}),$ for all $h^* \in H^{*\mathrm{rat}} \,\}$;*

(4) *If H is a finite dimensional Hopf algebra, then $\mathbf{J}^H(A) = \mathbf{J}(A\#H^*) \bigcap A$.*

Proof. (1) It is easy to check that the intersection of any set of comodule ideals of A is a comodule ideal. By Lemma 2.4, (1) is proved.

(2) Let M be a left $A\#H^{*\mathrm{rat}}$-simple module. Then M is a left (A,H)-Hopf simple module by Theorem 2.3. Therefore for any $a \in \mathbf{J}^H(A)$ and for all $h^* \in H^{*\mathrm{rat}}$,

$$(a\#h^*) \cdot M = a \cdot (h^* \to M) \subseteq a \cdot M = 0$$

and so $a\#h^*$ is contained in $\mathbf{J}(A\#H^{*\mathrm{rat}})$.

(3) Let $a \in A$ and suppose $a\#h^* \in \mathbf{J}(A\#H^{*\mathrm{rat}})$ for all $h^* \in H^{*\mathrm{rat}}$. Let N be a left (A,H)-Hopf simple module. Then N is a left $A\#H^{*\mathrm{rat}}$-simple module and annihilated by $a\#h^*$. Thus

$$a \cdot N = a \cdot (H^{*\mathrm{rat}} \longrightarrow N) = (a\#H^{*\mathrm{rat}}) \cdot N = 0,$$

and so $a \in \mathbf{J}^H(A)$.

(4) When H is a finite dimensional Hopf algebra, $H^* = H^{*\mathrm{rat}}$. The result follows immediately from (3).

Theorem 3.2. *$\mathbf{J}^H(A)$ is the largest right H-comodule ideal I of A with $I\#H^{*\mathrm{rat}} \subseteq \mathbf{J}(A\#H^{*\mathrm{rat}})$.*

Proof. It follows from Theorem 3.1(3).

Lemma 3.1. (1) *If $a\#\varepsilon$ is invertible in $\widetilde{A}\#H^*$, then a is also invertible in $\widetilde{A}$;*

(2) *If $a \in \widetilde{A}^{\mathrm{co}H}$ is invertible in $\widetilde{A}$, then a is also invertible in $\widetilde{A}^{\mathrm{co}H}$;*

(3) $\mathbf{J}(A\#H^*)\bigcap A \subseteq \mathbf{J}(A)$; (4) $\mathbf{J}(A)\bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(A^{\mathrm{co}H})$.

Proof. (1) Let $a\#\varepsilon$ have an inverse $\sum b_i\#h_i^*$, where h_i^*'s are linearly independent and $h_1^* = \varepsilon$. Then $(a\#\varepsilon)(\sum b_i\#h_i^*) = \sum ab_i\#h_i^* = 1\#\varepsilon$. Thus $ab_1 = 1$ and $(a\#\varepsilon)(b_1\#\varepsilon) = 1\#\varepsilon$. This shows that $\sum b_i\#h_i^* = b_1\ \#\varepsilon$ and b_1 is the inverse of a.

(2) Let $b = a^{-1}$ with $a \in \widetilde{A}^{\mathrm{co}H}$ in $\widetilde{A}$. Then $b \otimes 1_H = (b\otimes 1)\rho_{\widetilde{A}}(ab) = (b\otimes 1)\rho_{\widetilde{A}}(a)\rho_{\widetilde{A}}(b) = (ba\otimes 1)\rho_{\widetilde{A}}(b) = \rho_{\widetilde{A}}(b)$, and hence $b \in \widetilde{A}^{\mathrm{co}H}$.

(3) Let $a \in \mathbf{J}(\widetilde{A}\#H^*)\bigcap\widetilde{A}$. Then $1+a$ is invertible in $\widetilde{A}\#H^*$. This implies that $1+a$ is invertible in $\widetilde{A}$. Thus $\mathbf{J}(\widetilde{A}\#H^*)\bigcap\widetilde{A}$ is a quasi-regular ideal of $\widetilde{A}$, and we have $\mathbf{J}(\widetilde{A}\#H^*)\bigcap\widetilde{A} \subseteq \mathbf{J}(\widetilde{A})$. Thus

$$\mathbf{J}(A\#H^*)\bigcap A \subseteq \mathbf{J}(\widetilde{A}\#H^*)\bigcap \widetilde{A}\bigcap A \subseteq \mathbf{J}(\widetilde{A})\bigcap A = \mathbf{J}(A).$$

(4) By (2), $\mathbf{J}(\widetilde{A})\bigcap\widetilde{A}^{\mathrm{co}H} \subseteq \mathbf{J}(\widetilde{A}^{\mathrm{co}H})$. Thus

$$\mathbf{J}(A)\bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(\widetilde{A})\bigcap \widetilde{A}^{\mathrm{co}H}\bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(\widetilde{A}^{\mathrm{co}H})\bigcap A^{\mathrm{co}H} = \mathbf{J}(A^{\mathrm{co}H}).$$

Proposition 3.1. *If I is a comodule ideal and $I \subseteq \mathbf{J}(A)$, then $I \subseteq \mathbf{J}^H(A)$.*

Proof. Let M be a left (A,H)-Hopf simple module. Suppose N is a subcomodule of M generated by one element. Then N is finite dimensional. Clearly, $A\cdot N$ is an (A,H)-Hopf submodule of M. Since M is simple, $(0:A)_M = 0$ and $A\cdot N = M$. Thus M is a finitely generated left A-module. Let V be a maximal A-submodule of M, then M/V is a left A-simple module. If I is a right H-comodule ideal of A and $I \subseteq \mathbf{J}(A)$, then $I\cdot M \subseteq V \neq M$ and $I\cdot M$ is an (A,H)-Hopf submodule of M. By the simplicity of M, we have $I\cdot M = 0$. Thus $I \subseteq \mathbf{J}^H(A)$.

Theorem 3.3. (1) *If $\mathbf{J}(A)$ is a comodule ideal of A, then $\mathbf{J}(A) \subseteq \mathbf{J}^H(A)$;*

(2) $\mathbf{J}(A\#H^{*\mathrm{rat}})\bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(A^{\mathrm{co}H})$;

If H is a finite dimensional Hopf algebra, then

(3) $\mathbf{J}^H(A) \subseteq \mathbf{J}(A)$;

(4) $\mathbf{J}^H(A)$ *is the largest one of comodule ideals which are contained in* $\mathbf{J}(A)$;

(5) $\mathbf{J}^H(A)\bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(A^{\mathrm{co}H})$.

Proof. (1) Follows immediately from Proposition 3.1.

(2) By Lemma 3.1(3), (4), we have

$$\mathbf{J}(A\#H^{*\mathrm{rat}})\bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(A\#H^*)\bigcap A\bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(A)\bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(A^{\mathrm{co}H}).$$

(3) By Theorem 3.1(4) and Lemma 3.1(3), we have

$$\mathbf{J}^H(A) = \mathbf{J}(A\#H^*)\bigcap A \subseteq J(A).$$

(4) By Theorem 3.1(1), $\mathbf{J}^H(A)$ is a comodule ideal of A. The result follows immediately from (3) and Proposition 3.1.

(5) It follows immediately from (3) and Lemma 3.1(4).

Theorem 3.4. *If $A^{\mathrm{co}H}$ is a direct summand of the right $A^{\mathrm{co}H}$-module A as a submodule, then $\mathbf{J}^H(A)\bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(A^{\mathrm{co}H})$.*

Proof. First consider the special case of a right H-comodule algebra A with identity 1. Let N be a left $A^{\mathrm{co}H}$-simple module. Since the right $A^{\mathrm{co}H}$-module exact sequence:

$0 \longrightarrow A^{\mathrm{co}H} \longrightarrow A$ is split, the sequence: $0 \longrightarrow A^{\mathrm{co}H} \bigotimes_{A^{\mathrm{co}H}} N \longrightarrow A \bigotimes_{A^{\mathrm{co}H}} N$ is exact. Write $M = A \bigotimes_{A^{\mathrm{co}H}} N$, then $N \cong A^{\mathrm{co}H} \bigotimes_{A^{\mathrm{co}H}} N$ is a submodule of the left $A^{\mathrm{co}H}$-module M and $M = A \cdot N$. $M = A \bigotimes_{A^{\mathrm{co}H}} N$ becomes naturally a left (A, H)-Hopf module with the comodule structure map given by $a \otimes n \longmapsto \sum (a_{(0)} \otimes n) \otimes a_{(1)}$, where $\rho_A(a) = \sum a_{(0)} \otimes a_{(1)}$.

Let V be a Hopf submodule of M maximal with respect to $N \bigcap V = 0$. We claim that V is a maximal (A, H)-Hopf submodule of M. If not, there exists a left (A, H)-Hopf submodule W of M with $V \subsetneqq W \subsetneqq M$. Then $W \bigcap N \neq 0$, hence there is an $x \in W \bigcap N$ such that $A^{\mathrm{co}H} \cdot x = N$. Thus $A \cdot x = M$ and $W = M$. That is, V is maximal. Thus M/V is an (A, H)-Hopf simple module, and $\mathbf{J}^H(A) \cdot (M/V) = 0$. We obtain $\mathbf{J}^H(A) \cdot M \subseteq V$ and hence $[\mathbf{J}^H(A) \bigcap A^{\mathrm{co}H}] \cdot N \subseteq N \bigcap V = 0$. Thus $\mathbf{J}^H(A) \bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(A^{\mathrm{co}H})$.

If A doesn't contain an identity, then the right $\widetilde{A}^{\mathrm{co}H}$-submodule $\widetilde{A}^{\mathrm{co}H}$ is also a direct summand of $\widetilde{A}$. By Theorem 3.1(3), $\mathbf{J}^H(A) \subseteq \mathbf{J}^H(\widetilde{A})$. Hence

$$\mathbf{J}^H(A) \bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}^H(\widetilde{A}) \bigcap \widetilde{A}^{\mathrm{co}H} \bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(\widetilde{A}^{\mathrm{co}H}) \bigcap A^{\mathrm{co}H} = \mathbf{J}(A^{\mathrm{co}H}).$$

Corollary 3.1. *If H is cosemisimple, then $\mathbf{J}^H(A) \bigcap A^{\mathrm{co}H} \subseteq \mathbf{J}(A^{\mathrm{co}H})$.*

Proof. By [4, Lemma 1.3 (1)], $A^{\mathrm{co}H} = \varepsilon_e \rightarrow A$, where ε_e is an idempotent in H^*. It is easy to check that $A = (\varepsilon_e \rightarrow A) \bigoplus [(\varepsilon - \varepsilon_e) \rightarrow A]$. This shows that $A^{\mathrm{co}H}$ is a direct summand of the right $A^{\mathrm{co}H}$-module A. The corollary follows by Theorem 3.4.

Theorem 3.5. *If α is a hereditary radical, then*

(1) *If A is a right H-comodule algebra, then $\widetilde{A} \# H^{*\mathrm{rat}}$ is α-semisimple if and only if $\widetilde{A} \# H^*$ is α-semisimple;*

(2) *If α is a supernilpotent radical, then $A \# H^{*\mathrm{rat}}$ is α-semisimple if and only if $A \# H^*$ is α-semisimple.*

Proof. (1) Since α is hereditary and $\widetilde{A} \# H^{*\mathrm{rat}} \lhd \widetilde{A} \# H^*$,

$$\alpha(\widetilde{A} \# H^{*\mathrm{rat}}) = \alpha(\widetilde{A} \# H^*) \bigcap (\widetilde{A} \# H^{*\mathrm{rat}}).$$

By Theorem 2.1, $\alpha(\widetilde{A} \# H^{*\mathrm{rat}}) = 0$ if and only if $\alpha(\widetilde{A} \# H^*) = 0$. This completes the proof.

(2) If α is a supernilpotent radical, then $\alpha(A \# H^*)$ is a semiprime ideal of $A \# H^*$. By Lemma 2.3(2), $\alpha(A \# H^{*\mathrm{rat}}) = \alpha(A \# H^*) \bigcap (A \# H^{*\mathrm{rat}}) = 0$ if and only if $\alpha(A \# H^*) = 0$.

Lemma 3.2. (1) *If M is an $A \# H^{*\mathrm{rat}}$-simple module, then M is an $A \# H^*$-simple module.*

(2) *If M is a faithful left $A \# H^{*\mathrm{rat}}$-simple module, then M is a faithful left $A \# H^*$-module.*

Proof. (1) Since M is an $A \# H^{*\mathrm{rat}}$-simple module, $(A \# H^{*\mathrm{rat}})M = M$. Thus M is a left $A \# H^*$-module. Since every $A \# H^*$-submodule of M is an $A \# H^{*\mathrm{rat}}$-submodule, M is an $A \# H^*$-simple module.

(2) Let $I = (0 : M)_{A \# H^*}$, then I is a prime ideal of $A \# H^*$. If $I \neq 0$, then by Lemma 2.3(2), $(0 : M)_{A \# H^{*\mathrm{rat}}} = I \bigcap (A \# H^{*\mathrm{rat}}) \neq 0$. But this is contrary to the $A \# H^{*\mathrm{rat}}$-faithfulness of M.

Theorem 3.6. *If $A \# H^{*\mathrm{rat}}$ is a primitive algebra, then $A \# H^*$ is primitive.*

Proof. It follows immediately from Lemma 3.2.

§4. The Density Theorem of Comodule Algebras

Let M, N be left (A, H)-Hopf modules, φ be a k linear map from M to N, ρ_M and ρ_N be the comodule structure maps of M, N respectively. If φ is a left A-module map and a right H-comodule map, then φ is called an (A,H)-Hopf map. Clearly, if φ is an (A,H)-Hopf map, then $\mathrm{Im}(\varphi)$ is an (A,H)-Hopf submodule of N.

Define $\mathrm{Hom}_A^H(M,N)$ to be the set of all (A,H)-Hopf maps. If $M=N$, then write $\mathrm{End}_A^H(M)$ for $\mathrm{Hom}_A^H(M,M)$.

For any $\varphi\in\mathrm{Hom}_A(M,N)$, $m\in M$, the action φ on m is written as $(m)\varphi$, $(\rho_N\circ\varphi)(m)=\rho_N[(m)\varphi]$ and $[(\varphi\otimes\mathrm{id})\circ\rho_M](m)=\sum(m_{(0)})\varphi\otimes m_{(1)}$.

Lemma 4.1. *Let M, N be left (A,H)-Hopf modules, then $\mathrm{Hom}_A(M,N)$ is a left H^*-module via:*

$$(m)(h^*\cdot\varphi)=\sum[(m_{(0)})\varphi]_{(0)}\langle h^*,(\overline{\mathbf{S}}m_{(1)})[(m_{(0)})\varphi]_{(1)}\rangle,$$

where $h^*\in H^*$, $m\in M$, $\varphi\in\mathrm{Hom}_A(M,N)$.

Proof. First, for any $h^*\in H^*$, $a\in A$, $m\in M$, $\varphi\in\mathrm{Hom}_A(M,N)$,

$$\begin{aligned}(am)(h^*\cdot\varphi)&=\sum\{[(am)_{(0)}]\varphi\}_{(0)}\langle h^*,\overline{\mathbf{S}}(am)_{(1)}\{[(am)_{(0)}]\varphi\}_{(1)}\rangle\\&=\sum[(a_{(0)}m_{(0)})\varphi]_{(0)}\langle h^*,\overline{\mathbf{S}}(a_{(1)}m_{(1)})[(a_{(0)}m_{(0)})\varphi]_{(1)}\rangle\\&=\sum a_{(0)}[(m_{(0)})\varphi]_{(0)}\langle h^*,(\overline{\mathbf{S}}m_{(1)})(\overline{\mathbf{S}}a_{(2)})a_{(1)}[(m_{(0)})\varphi]_{(1)}\rangle\\&=\sum a[(m_{(0)})\varphi]_{(0)}\langle h^*,(\overline{\mathbf{S}}m_{(1)})[(m_{(0)})\varphi]_{(1)}\rangle\\&=a[(m)(h^*\cdot\varphi)].\end{aligned}$$

Hence $h^*\cdot\varphi\in\mathrm{Hom}_A(M,N)$. It is clear that the action $h^*\cdot\varphi$ is linear with respect to h^* and φ.

Next, for any $h^*,g^*\in H^*$, $m\in M$,

$$\begin{aligned}(m)[g^*\cdot(h^*\cdot\varphi)]&=\sum[(m_{(0)})(h^*\cdot\varphi)]_{(0)}\langle g^*,(\overline{\mathbf{S}}m_{(1)})[(m_{(0)})(h^*\cdot\varphi)]_{(1)}\rangle\\&=\sum[(m_{(0)})\varphi]_{(0)}\langle h^*,(\overline{\mathbf{S}}m_{(1)})[(m_{(0)})\varphi]_{(2)}\rangle\langle g^*,(\overline{\mathbf{S}}m_{(2)})[(m_{(0)})\varphi]_{(1)}\rangle\\&=\sum[(m_{(0)})\varphi]\langle g^* * h^*,(\overline{\mathbf{S}}m_{(1)})[(m_{(0)})\varphi]_{(1)}\rangle\\&=(m)[(g^* * h^*)\cdot\varphi].\end{aligned}$$

This shows that the action of H^* on $\mathrm{Hom}_A(M,N)$ is associative. Thus $\mathrm{Hom}_A(M,N)$ is a left H^*-module.

The left H^*-module $\mathrm{Hom}_A(M,N)$ has a unique maximal rational submodule

$$\mathrm{Hom}_A(M,N)^{\mathrm{rat}}=\theta^{-1}(\mathrm{Im}\,\mu),$$

where $\theta:\mathrm{Hom}_A(M,N)\longrightarrow\mathrm{Hom}(H^*,\mathrm{Hom}_A(M,N))$ by $\theta(\varphi)(h^*)=h^*\cdot\varphi$,

$$\mu:\mathrm{Hom}_A(M,N)\bigotimes H\longrightarrow\mathrm{Hom}(H^*,\mathrm{Hom}_A(M,N))$$

by $\mu(\varphi\otimes h)(h^*)=\langle h^*,h\rangle\varphi$. We write $[\mathrm{Hom}_A(M,N)]^{\mathrm{rat}}=\mathrm{HOM}_A(M,N)$.

Lemma 4.2. *Let $\varphi\in\mathrm{Hom}_A(M,N)$. Then $\varphi\in\mathrm{HOM}_A(M,N)$ if and only if there is a $\sum\varphi_{(0)}\otimes\varphi_{(1)}\in\mathrm{Hom}_A(M,N)\bigotimes H$ such that*

$$\sum[(m_{(0)})\varphi]_{(0)}\otimes(\overline{\mathbf{S}}m_{(1)})[(m_{(0)})\varphi]_{(1)}=\sum(m)\varphi_{(0)}\otimes\varphi_{(1)},\quad\text{for any } m\in M.$$

In that case, $\theta(\varphi)=\sum\varphi_{(0)}\otimes\varphi_{(1)}$ and

$$\sum(m_{(0)})\varphi_{(0)}\otimes m_{(1)}\varphi_{(1)}=\sum[(m)\varphi]_{(0)}\otimes[(m)\varphi]_{(1)},$$

where we regard μ as an embedding.

Proof. $\varphi \in \mathrm{HOM}_A(M,N) \Longleftrightarrow$ there is a $\sum \varphi_{(0)} \otimes \varphi_{(1)} \in \mathrm{Hom}_A(M,N) \bigotimes H$ such that $\theta(\varphi) = \sum \varphi_{(0)} \otimes \varphi_{(1)} \Longleftrightarrow h^* \cdot \varphi = \sum \varphi_{(0)} \langle h^*, \varphi_{(1)} \rangle$ for all $h^* \in H^*$; i.e.

$$\sum [(m_{(0)})\varphi]_{(0)} \langle h^*, (\overline{\mathbf{S}}m_{(1)})[(m_{(0)})\varphi]_{(1)} \rangle = \sum (m)\varphi_{(0)} \langle h^*, \varphi_{(1)} \rangle$$

for all $m \in M$, $h^* \in H^*$; this is equivalent to that

$$\sum [(m_{(0)})\varphi]_{(0)} \otimes (\overline{\mathbf{S}}m_{(1)})[(m_{(0)})\varphi]_{(1)} = \sum (m)\varphi_{(0)} \otimes \varphi_{(1)}$$

holds for all $m \in M$. Since

$$\sum [(m_{(0)})\varphi]_{(0)} \otimes (\overline{\mathbf{S}}m_{(1)})[(m_{(0)})\varphi]_{(1)} = \sum (m)\varphi_{(0)} \otimes \varphi_{(1)},$$

$$\begin{aligned}\sum (m_{(0)})\varphi_{(0)} \otimes m_{(1)}\varphi_{(1)} &= \sum [(m_{(0)})\varphi]_{(0)} \otimes m_{(2)}(\overline{\mathbf{S}}m_{(1)})[(m_{(0)})\varphi]_{(1)} \\ &= \sum [(m)\varphi]_{(0)} \otimes [(m)\varphi]_{(1)}.\end{aligned}$$

This completes the proof.

Remark 4.1. If M, N are right (A,H)-Hopf modules, we can similarly show that $\mathrm{Hom}_{\mathrm{A}}(M,N)$ is a left H^*-module. And when A has an identity 1, $[\mathrm{Hom}_A(M,N)]^{\mathrm{rat}}$ is just $\mathrm{HOM}_A(M,N)$ in [11].

Proposition 4.1. *Let M be a left (A,H)-Hopf module, then*

(1) $\mathrm{END}_A(M)$ *is a right H-comodule algebra;*

(2) *M is a right* $(\mathrm{END}_A(M),H)$*-Hopf module;*

(3) $[\mathrm{END}_A(M)]^{\mathrm{co}H} = \mathrm{End}_A^H(M)$*;*

(4) *When left A-module M is unital, we have* $[\mathrm{END}_A(M)]^{\mathrm{co}H} = \mathrm{End}_{A\#H^{*\mathrm{rat}}}(M)$*.*

Proof. (1) By definition, $\mathrm{END}_A(M)$ is a right H-comodule. Next, let $\varphi, \psi \in \mathrm{END}_A(M)$. Then $\theta(\varphi) = \sum \varphi_{(0)} \otimes \varphi_{(1)}$, $\theta(\psi) = \sum \psi_{(0)} \otimes \psi_{(1)}$ and

$$\begin{aligned}&\sum [(m_{(0)})(\varphi\psi)]_{(0)} \otimes \overline{\mathbf{S}}m_{(1)}[(m_{(0)})(\varphi\psi)]_{(1)} \\ =& \sum [(m_{(0)})\varphi]_{(0)}\psi_{(0)} \otimes \overline{\mathbf{S}}m_{(1)}[(m_{(0)})\varphi]_{(1)}\psi_{(1)} \\ =& \sum [(m_{(0)})\varphi_{(0)}]\psi_{(0)} \otimes \overline{\mathbf{S}}m_{(2)}(m_{(1)}\varphi_{(1)})\psi_{(1)} \\ =& \sum (m)(\varphi_{(0)}\psi_{(0)}) \otimes \varphi_{(1)}\psi_{(1)}.\end{aligned}$$

Hence $\varphi, \psi \in \mathrm{END}_A(M)$ and $\theta(\varphi\psi) = \sum \varphi_{(0)}\psi_{(0)} \otimes \varphi_{(1)}\psi_{(1)}$ by Lemma 4.2.

(2) By Lemma 4.2, we have

$$\sum [(m)\varphi]_{(0)} \otimes [(m)\varphi]_{(1)} = \sum (m_{(0)})\varphi_{(0)} \otimes m_{(1)}\varphi_{(1)}.$$

Thus (2) holds.

(3) $\varphi \in [\mathrm{END}_A(M)]^{\mathrm{co}H} \Longleftrightarrow \varphi \otimes 1 = \sum \varphi_{(0)} \otimes \varphi_{(1)} \in \mathrm{End}_A(M) \bigotimes H$

$\Longleftrightarrow \sum (m_{(0)})\varphi_{(0)} \otimes m_{(1)}\varphi_{(1)} = \sum (m_{(0)})\varphi \otimes m_{(1)}$ for all $m \in M$

$\Longleftrightarrow \sum [(m)\varphi]_{(0)} \otimes [(m)\varphi]_{(1)} = \sum (m_{(0)})\varphi \otimes m_{(1)}$ for all $m \in M$ by Lemma 4.2.

Thus $\varphi \in [\mathrm{END}_A(M)]^{\mathrm{co}H}$ if and only if $\varphi \in \mathrm{End}_A^H(M)$.

(4) If φ is an (A,H)-Hopf module map of M, φ is a left A-module map and left H^*-module map as well. Thus φ is an $A\#H^{*\mathrm{rat}}$-module map.

Now, let φ be a left $A\#H^{*\mathrm{rat}}$-module map. For any $m \in M$, there is an $h^* \in H^{*\mathrm{rat}}$ such that $h^* \rightarrow m = m$ and $h \rightarrow [(m)\varphi] = (m)\varphi$. Since

$$\begin{aligned}(a \cdot m)\varphi &= [a \cdot (h^* \rightarrow m)]\varphi = [(a\,\#h^*) \cdot m]\varphi \\ &= (a\,\#h^*) \cdot [(m)\varphi] = a \cdot \{h^* \rightarrow [(m)\varphi]\} = a \cdot [(m)\varphi],\end{aligned}$$

φ is a left A-module map. Since M is unital as a left $A\#H^{*\mathrm{rat}}$-module, there are $x_i \in A\#H^{*\mathrm{rat}}$, $m_i \in M$ such that $m = \sum_i x_i \cdot m_i$. Thus for any $g^* \in H^*$,

$$\begin{aligned}(g^* \to m)\varphi &= \sum_i \{[(1\ \#g^*)x_i] \cdot m_i\}\varphi = \sum_i [(1\ \#g^*)x_i] \cdot [(m_i)\varphi] \\ &= g^* \to [(\sum_i x_i \cdot m_i)\varphi] = g^* \to [(m)\varphi].\end{aligned}$$

Hence φ is also a left H^*-module map, and so it is a right (A, H)-Hopf module map. This, together with (3), completes the proof.

Theorem 4.1. (*A version of the density theorem for comodule algebra*) *Let M be a left (A, H)-Hopf simple module and $D = \mathrm{End}_A^H(M)$. Then*

(1) *D is a division algebra over k;*

(2) *$M^{\mathrm{co}H} = \{ m \in M \mid \rho_M(m) = m \otimes 1 \}$ is a vector space over D;*

(3) *Suppose $x_1, x_2, \cdots, x_n$ are D-linearly indepentent in $M^{\mathrm{co}H}$, then for any $y_1, y_2, \cdots, y_n \in M$, there exists an $a \in A$ such that $a \cdot x_i = y_i$, $1 \leqslant i \leqslant n$.*

Proof. The proofs of (1) and (2) are direct.

(3) By Proposition 4.1, $D \cong \mathrm{End}_{A\#H^{*\mathrm{rat}}}(M)$ is a k-division algebra. By [6, Theorem 19.22], there exists a $\sum_j a_j\ \#h_j^* \in A\#H^{*\mathrm{rat}}$ such that $\sum_j (a_j\ \#h_j^*) \cdot x_i = y_i$, $1 \leqslant i \leqslant n$. Since $x_i \in M^{\mathrm{co}H}$, we have

$$\sum_j (a_j\ \#h_j^*)x_i = \sum_j a_j \cdot (h_j^* \to x_i) = \sum_j a_j \langle h_j^*, 1\rangle x_i, \quad 1 \leqslant i \leqslant n.$$

Let $a = \sum_j \langle h_j^*, 1\rangle a_j$, then $a \cdot x_i = y_i$, $1 \leqslant i \leqslant n$.

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