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ON A CONSTRUCTION OF INERTIAL MANIFOLDS FOR A CLASS OF SECOND ORDER IN TIME EVOLUTION EQUATIONS

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Abstract

Inertial manifolds for a class of second order in time dissipative equations are constructed. The author also proves an asymptotic completeness property for the inertial manifolds and characterizes the inertial manifolds as the set of trajectories whose growth is at most of order $O(e^{-\mu t})$ for some $\mu > 0$. As applications, a nonlinear wave equation and a problem of nonlinear oscillations of a shallow shell are considered.

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§0. Introduction

It is well known that inertial manifolds are of prime importance in the study of the long-time behavior of trajectories of nonlinear evolutionary systems. These manifolds, when they exist, are finite dimensional invariant Lipschitz manifolds in the phase space of the systems. They contain the universal attractor and attract exponentially all the trajectories. The dynamics restricted to the inertial manifolds leads to a finite dimensional system of ordinary differential equations which completely describes the long-time behavior of the original systems. In the last two decades, inertial manifolds for parabolic equations have been widely studied (see [1] and the references therein). However, for wave equations we only obtain the existence of attractors[2], exponential attractors[3], invariant manifolds[4] and approximate inertial manifolds[5]. No results on the existence of inertial manifolds have been obtained until now.

The aim of this paper is to construct inertial manifolds for a class of second order evolution equations which includes wave equations. The interesting facts obtained here are the asymptotic completeness property of the inertial manifolds and the characterization of the inertial manifolds as the collection of all complete orbits which do not grow more rapidly than $e^{\mu|t|}$, for some appropriate $\mu > 0$, as $|t| \to \infty$. In Section 1, we set up the notations and present the main result on the existence of inertial manifolds. In Section 2, we prove the main result, and in Section 3, as applications, we consider a nonlinear wave equation and a problem of nonlinear oscillation of a shallow shell.

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§1. Assumptions and Main Result

We consider a dissipative evolution equation of second order in time in a Hilbert space H of the form

$$\delta u_{tt} + u_t + Au = f(u), \quad u(0) = u_0, \quad u_t(0) = u_1, \tag{1.1}$$

where δ is a positive number, A is a positive selfadjoint operator with eigenfunctions $\{w_j\}_{j=1}^{\infty}$ forming a complete basis of H, and eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$,

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \quad j \to +\infty,$$

and where $f(\cdot)$ is a local Lipschitz mapping from $D(A^{\frac{1}{2}})$ into $D(A^{\frac{1}{2}-\beta})$, $0 \le \beta < \frac{1}{2}$.

We also assume that the following properties hold:

(i) Problem (1.1) is well posed in $D(A^{\frac{1}{2}}) \times H$, i.e. there exists the uniquely determined strongly continuous group $\{T(t)\}_{t\in R}$ acting on the space $D(A^{\frac{1}{2}}) \times H$ according to the formula $T(t)(u_0, u_1) = (u(t), u_t(t))$, where $(u(t), u_t(t)) \in C(R, D(A^{\frac{1}{2}}) \times H)$ is the solution of (1.1) with initial condition (u_0, u_1);

(ii) The group $\{T(t)\}_{t\in R}$ is dissipative, i.e. there exists a bounded absorbing set B which lies in a ball of center 0 and of radius R in $D(A^{\frac{1}{2}}) \times H$.

Consider a numerical function $\theta \in C^1([0, +\infty], [0, 1])$ which is equal to 1 in $[0, 1]$ and to 0 in $[2, +\infty]$ and such that $|\theta'(s)| \le 2$ for every $s \ge 0$. Define $F(\cdot)$ by

$$F(u) = \theta\Big(\frac{|u|^2_{D(A^{\frac{1}{2}})}}{R^2}\Big) f(u), \quad \forall u \in D(A^{\frac{1}{2}}).$$

Then, $F(\cdot)$ is Lipschitz and bounded, i.e. there exists $M > 0$, s.t.

$$|F(u) - F(v)|_{D(A^{\frac{1}{2}-\beta})} \le M |u - v|_{D(A^{\frac{1}{2}})}, \quad \forall u, v \in D(A^{\frac{1}{2}}), \tag{1.2}$$

$$|F(u)|_{D(A^{\frac{1}{2}-\beta})} \le M, \quad \forall u \in D(A^{\frac{1}{2}}). \tag{1.3}$$

Problem (1.1) will have the same long-time behavior as the following problem

$$\delta u_{tt} + u_t + Au = F(u), \quad u(0) = u_0, \quad u_t(0) = u_1, \tag{1.4}$$

which is assumed to define a continuous group $\{S(t)\}_{t\in R}$ in $D(A^{\frac{1}{2}}) \times H$.

We thus consider problem (1.4).

We state our main result as follows:

Theorem. *Under the above assumptions, if for some $n \in N$ the following spectral gap condition is satisfied*

$$\lambda_{n+1} - \lambda_n > 2M(\lambda_{n+1}^{\beta} + \lambda_n^{\beta}), \tag{1.5}$$

then there exists a $\bar{\delta}$ such that, when $\delta \in [0, \bar{\delta}]$, problem (1.4) *possesses an inertial manifold $\mathfrak{M}$, i.e. there exists an n-dimensional Lipschitz manifold $\mathfrak{M} \subset D(A^{\frac{1}{2}}) \times H$, s.t.*

(1) $S(t)\mathfrak{M} = \mathfrak{M}, \quad \forall t \ge 0$;

(2) $$\mathrm{dist}(S(t)U, \mathfrak{M})_{D(A^{\frac{1}{2}})\times H} \le C(U) e^{-\mu t}, \quad \forall t \ge 0, \quad \forall U \in D(A^{\frac{1}{2}}) \times H, \tag{1.6}$$

where $C(U)$ is a positive constant depending on U and μ satisfying

$$\lambda_n + 2M\lambda_n^{\beta} < \mu < \lambda_{n+1} - 2M\lambda_{n+1}^{\beta}.$$

Moreover, $\mathfrak{M}$ has the asymptotic completeness property. More precisely, for every $U \in$

$D(A^{\frac{1}{2}}) \times H$, *there exists* $V \in \mathfrak{M}$ *such that*

$$|S(t)U - S(t)V|_{D(A^{\frac{1}{2}})\times H} \le C(U)e^{-\mu t}, \quad \forall t \ge 0. \tag{1.7}$$

§2. Proof of Theorem

We assume in all this section that all the hypotheses of Section 1 are satisfied. For simplicity of notations, let $P = P_n$ denote the projection from H onto span $\{w_1, w_2, \cdots, w_n\}$, $Q = I - P_n$, $E = D(A^{\frac{1}{2}})$, $F = D(A^{\frac{1}{2}-\beta})$. Here $(\cdot,\cdot)$ and $|\cdot|$ denote the inner product and norm on $H = D(A^0)$.

Given a Banach space X, we introduce the spaces

$$X_\mu = \{u \in C(R, X),\ |u|_{X_\mu} = \sup_{t\in R}\{e^{-\mu|t|}|u(t)|_X\} < +\infty\}, \tag{2.1}$$

and

$$X_\mu^* = \{u \in C(R, X),\ |u|_{X_\mu^*} = \sup_{t\in R}\{e^{\mu t}|u(t)|_X\} < +\infty\}, \tag{2.2}$$

which are Banach spaces when endowed with the norms $|\cdot|_{X_\mu}$ and $|\cdot|_{X_\mu^*}$ respectively.

First, consider the following equation

$$\delta p_{tt} + p_t = f. \tag{2.3}$$

Lemma 2.1. *Let* μ *be a real number,* $0 < \delta < \mu^{-1}$. *Then*

(i) *for every* $f \in PF_\mu$ *and* $x \in PE$, *there exists a unique solution* p *of* (2.3) *with* $p \in PF_\mu$, $p_t \in PH_\mu$, $p(0) = x$, $p = x + R_1 f$, $p_t = R_2 f$, *and*

$$|R_1|_{L(PF_\mu, PE_\mu)} \le \mu^{-1}(1-\delta\mu)^{-1}\cdot\lambda_n^\beta, \tag{2.4}$$

$$|R_2|_{L(PF_\mu, PH_\mu)} \le (1-\delta\mu)^{-1}; \tag{2.5}$$

(ii) *for every* $f \in PF_\mu^*$, *there exists a unique solution* p *of* (2.3) *with* $p \in PF_\mu^*$, $p_t \in PH_\mu^*$, $p = R_1^* f$, $p_t = R_2^* f$, *and*

$$|R_1^*|_{L(PF_\mu^*, PE_\mu^*)} \le \mu^{-1}(1-\delta\mu)^{-1}\cdot\lambda_n^\beta, \tag{2.6}$$

$$|R_2^*|_{L(PF_\mu^*, PH_\mu^*)} \le \mu^{-1}(1-\delta\mu)^{-1}\cdot\lambda_n^\beta. \tag{2.7}$$

Proof. (i) Uniqueness. It suffices to verify that, for the homogeneous problem $\delta p_{tt} + p_t = 0$, $p(0) = 0$, the only solution belonging to PE_μ is $p \equiv 0$; indeed, the general solution of this problem is $p(t) = (1 - e^{-\frac{t}{\delta}})z$, $z \in PE$, but, due to the assumption $\mu < \frac{1}{\delta}$, this function will not belong to PE_μ unless $z = 0$.

Existence. Let

$$p_t(t) = (R_2 f)(t) = \frac{1}{\delta}\int_{-\infty}^t e^{-\frac{1}{\delta}(t-\tau)} f(\tau)d\tau, \quad t \in R,$$

$$(R_1 f)(t) = \int_0^t p_t(\tau)d\tau, \quad t \in R,$$

$$p(t) = x + (R_1 f)(t), \quad t \in R.$$

Then

$$e^{\mu t}|(R_2 f)(t)| \le \frac{1}{\delta}\int_{-\infty}^t e^{(\mu-\frac{1}{\delta})(t-\tau)}\cdot e^{\mu\tau}|f(\tau)|d\tau \le \frac{1}{\delta}\frac{\delta}{(1-\delta\mu)}|f|_{PF_\mu}, \quad \forall t \le 0, \tag{2.8}$$

$$e^{-\mu t}|(R_2f)(t)| \le \frac{1}{\delta}e^{-\mu t}\Big(\int_{-\infty}^{0} e^{-\frac{1}{\delta}(t-\tau)}\cdot e^{-\mu\tau}\cdot e^{\mu\tau}|f(\tau)|d\tau + \int_0^t e^{-\frac{1}{\delta}(t-\tau)}\cdot e^{\mu\tau}\cdot e^{-\mu\tau}|f(\tau)|d\tau\Big) \le (1-\delta\mu)^{-1}\cdot|f|_{PF_\mu}, \quad \forall t\ge 0, \tag{2.9}$$

$$e^{\mu t}|(R_1f)(t)|_{PE} \le \frac{1}{\delta}\int_t^0\int_{-\infty}^{\tau} e^{\mu t}e^{-\frac{1}{\delta}(\tau-s)}\cdot e^{-\mu s}\cdot e^{\mu s}\cdot\lambda_n^{\beta}|f(s)|_{PF}dsd\tau \le \mu^{-1}\cdot(1-\delta\mu)^{-1}\cdot\lambda_n^{\beta}|f|_{PF_\mu}, \quad \forall t\le 0, \tag{2.10}$$

$$e^{-\mu t}|(R_1f)(t)|_{PE} \le \frac{1}{\delta}\int_0^t\Big[\int_{-\infty}^{0} e^{-\mu t}\cdot e^{-\frac{1}{\delta}(\tau-s)}\cdot e^{-\mu s}\cdot e^{\mu s}\cdot|f(s)|_E ds + \int_0^{\tau} e^{-\mu t}\cdot e^{-\frac{1}{\delta}(\tau-s)}\cdot e^{\mu s}\cdot e^{-\mu s}\cdot|f(s)|_E ds\Big]d\tau \le \mu^{-1}\cdot(1-\delta\mu)^{-1}\cdot\lambda_n^{\beta}|f|_{PF_\mu}, \quad \forall t\ge 0. \tag{2.11}$$

Thanks to (2.8)–(2.11), we see that $p(t)$ and $p_t(t)$ defined above make sense. Thus $p(t)$ is a solution of (2.3) and $p(0)=x$ and (2.4)–(2.5) are true.

(ii) Uniqueness. It suffices to prove that the only solution of problem $\delta p_{tt}+p_t=0$ with $p\in PE_\mu^*$, $p_t\in PH_\mu^*$ is $p\equiv 0$.

Obviously, we have $\frac{d}{dt}(p_te^{\frac{1}{\delta}t})\equiv 0$. Therefore

$$p_t(t)=e^{-\frac{1}{\delta}(t-s)}\cdot p_t(s)=e^{-\frac{1}{\delta}t}\cdot e^{-(\mu-\frac{1}{\delta})s}e^{\mu s}p_t(s), \quad \forall t,s\in R.$$

Since $p_t\in PH_\mu^*$, we have, as $s\to-\infty$,

$$p_t(t)=0, \quad \forall t\in R.$$

So, $p(t)=p(s)=e^{-\mu s}\cdot e^{\mu s}p(s)$, $\forall t,s\in R$. Since $p\in PE_\mu^*$, we have, as $s\to+\infty$,

$$p(t)=0, \quad \forall t\in R.$$

Existence. Let

$$p_t(t)=(R_2^*f)(t)=\frac{1}{\delta}\int_{-\infty}^{t} e^{-\frac{1}{\delta}(t-\tau)}f(\tau)d\tau, \tag{2.12}$$

$$p(t)=(R_1^*f)(t)=-\int_t^{+\infty} p_t(\tau)d\tau. \tag{2.13}$$

In the same way as in (2.8), (2.10), we obtain

$$e^{\mu t}|p_t(t)| \le (1-\delta\mu)^{-1}|f|_{PF_\mu^*}, \quad \forall t\in R,$$
$$e^{\mu t}|p(t)|_E \le \mu^{-1}(1-\delta\mu)^{-1}\cdot\lambda_n^{\beta}|f|_{PF_\mu^*}, \quad \forall t\in R.$$

So, $p(\cdot)$ defined above makes sense and is a solution of (2.3), and (2.6), (2.7) are true.

We consider the following equation

$$\delta q_{tt}+q_t+Aq=g. \tag{2.14}$$

Lemma 2.2. *Let μ be a real number, $0<\mu<\lambda_{n+1}$, $0<\delta<(16\mu)^{-1}$. Then, for every $g\in QF_\mu$, there exists a unique solution q of (2.14), $q\in QE_\mu$, $q_t\in QH_\mu$, and*

$q = S_1 g, q_t = S_2 g$ with

$$|S_1|_{L(QF_\mu, QE_\mu)} \le C(\mu,\delta) \cdot (\lambda_{n+1} - \mu)^{-1} \cdot \lambda_{n+1}^\beta, \tag{2.15}$$

$$|S_2|_{L(QF_\mu, QH_\mu)} \le C(\mu,\delta) \cdot (\lambda_{n+1} - \mu)^{-1} \cdot \lambda_{n+1}^\beta \cdot \delta. \tag{2.16}$$

Moreover, if $g \in QF_\mu^*$, then $q \in QE_\mu^*$, $q_t \in QH_\mu^*$, and

$$|S_1|_{L(QF_\mu^*, QE_\mu^*)} \le C(\mu,\delta) \cdot (\lambda_{n+1} - \mu)^{-1} \cdot \lambda_{n+1}^\beta, \tag{2.17}$$

$$|S_2|_{L(QF_\mu^*, QH_\mu^*)} \le C(\mu,\delta) \cdot (\lambda_{n+1} - \mu)^{-1} \cdot \lambda_{n+1}^\beta \cdot \delta, \tag{2.18}$$

where $C(\mu,\delta)$ is a positive constant depending on μ, δ, and $C(\mu,\delta) \to 1$, as $\delta \to 0$.

Proof. The uniqueness and existence can be proved as in [2, Chapter 4, Proposition 1.3], and the norms (2.15)–(2.18) may be obtained by making a slight modification of corresponding norms in [4] or [7].

From Lemmas 2.1, 2.2, we can define a mapping T :

$$(x, u^0) \mapsto T(x, u^0) = p + q = x + R_1 f + S_1 g \in E_\mu, \quad \forall (x, u^0) \in PE \times E_\mu, \tag{2.19}$$

where p, q are the solutions of the following problems

$$\delta p_{tt} + p_t = -APu^0 + PF(u^0) \triangleq f,$$
$$P(0) = x;$$
$$\delta q_{tt} + q_t + Aq = QF(u^0) \triangleq g.$$

Lemma 2.3 . *The mapping T defined by (2.19) satisfies*

$$\|T(x, u^0) - T(x, v^0)\|_{E_\mu} \le \theta(\delta,\mu) \|u^0 - v^0\|_{E_\mu} \tag{2.20}$$

for every $x \in PE$, $u^0, v^0 \in E_\mu$, where $\|\cdot\|_{E_\mu}$ denotes the norm (equivalent to $|\cdot|_{E_\mu}$) on E_μ defined by

$$\|u\|_{E_\mu} = \max(|Pu|_{E_\mu}, |Qu|_{E_\mu}), \quad \forall u \in E_\mu, \tag{2.21}$$

and

$$\theta(\delta,\mu) = \max((\lambda_n + 2M\lambda_n^\beta) \cdot (1 - \delta\mu)^{-1} \cdot \mu^{-1},\ C(\mu,\delta) \cdot \lambda_{n+1}^\beta \cdot (\lambda_{n+1} - \mu)^{-1}). \tag{2.22}$$

Proof. By definition of the mapping T, taking $u^0, v^0 \in E_\mu$, $x \in PE$, we have

$$\begin{aligned} &\|T(x,u^0) - T(x,v^0)\|_{E_\mu} \\ &\le \max\Big[\frac{1}{1-\delta\mu}\Big(\frac{\lambda_n + M\lambda_n^\beta}{\mu}|Pu^0 - Pv^0|_{E_\mu} + \frac{M\lambda_n^\beta}{\mu}|Qu^0 - Qv^0|_{E_\mu}\Big), \\ &\quad C(\mu,\delta)\Big(\frac{M\lambda_{n+1}^\beta}{\lambda_{n+1} - \mu}|Pu^0 - Pv^0|_{E_\mu} + \frac{M\lambda_{n+1}^\beta}{\lambda_{n+1} - \mu}|Qu^0 - Qv^0|_{E_\mu}\Big)\Big] \\ &\le \max\Big((1-\delta\mu)^{-1}\frac{\lambda_n + 2M\lambda_n^\beta}{\mu}, C(\mu,\delta)\frac{2M\lambda_{n+1}^\beta}{\lambda_{n+1} - \mu}\Big)\|u^0 - v^0\|_{E_\mu}. \end{aligned}$$

Therefore, (2.20) is true.

Now, thanks to the spectral gap condition (1.5), we can choose μ such that

$$\lambda_n + 2M\lambda_n^\beta < \mu < \lambda_{n+1} - 2M\lambda_{n+1}^\beta. \tag{2.23}$$

We then note that for those μ satisfying (2.23) and δ small, we have $\theta(\delta,\mu) < 1$. Hence, Lemma 2.3 says that $T(x,\cdot)$ is a uniform (with respect to $x \in PE$) contraction on E_μ.

Moreover, $T(x,\cdot)$ has a unique fixed point $w(x),\ w(x)=T(x,w(x))$, where $w(x)$ is a solution of (1.4) with $Pw(x)(0)=x, w(x)(t)\in E_\mu,\ w(x)_t(t)\in H_\mu$.

It is easy to see that $w(x), w(x)_t$ is Lipschitz in x in E_μ and H_μ respectively.

Now, we define the manifold $\mathfrak{M}\subset E\times H$ as follows:

$$\mathfrak{M}=\{(w(x)(0), w(x)_t(0)),\quad x\in PE\}. \tag{2.24}$$

Lemma 2.4. *Consider the manifold* $\mathfrak{M}$ *as in* (2.24). *Then,* $\mathfrak{M}$ *can be characterized as*

$$\begin{aligned}\mathfrak{M}=\{&(u_0,v_0)\in E\times H,\ (u_0,v_0) \text{ belongs to a complete orbit } (u(t),u_t(t))_{t\in R}\\ &\text{solution of } (1.4) \text{ with } |(u(t),u_t(t))|_{E\times H}=O(e^{\mu|t|}) \text{ as } t\to\infty\},\end{aligned} \tag{2.25}$$

where μ *is an arbitrary number satisfying* (2.23).

Proof. To see that $\mathfrak{M}$ has the characterization (2.25), we fix μ satisfying (2.23) and denote by $\mathfrak{M}_\mu$ the right hand side of (2.25). Obviously, $\mathfrak{M}\subseteq\mathfrak{M}_\mu$. Next, we prove that $\mathfrak{M}_\mu\subseteq\mathfrak{M}$.

By definiton, $(u_0,v_0)\in\mathfrak{M}_\mu$ if and only if there exists a solution u of (1.4) with $u(0)=u_0,\ u_t(0)=v_0,\ u\in E_\mu, u_t\in H_\mu$. Since the mapping $T(Pu(0),\cdot)$ has a unique fixed point $w(Pu(0)), w(Pu(0))$ is a solution of (1.4), $w(Pu(0))\in E_\mu, w(Pu(0))_t\in H_\mu$, and since u is the fixed point of $T(Pu(0),\cdot)$, it follows that

$$w(Pu(0))=T(Pu(0), w(Pu(0)))=u.$$

By definition of $\mathfrak{M}$, we have

$$(u_0,v_0)=(u(0),u_t(0))=(w(Pu(0))(0), w(Pu(0))_t(0))\in\mathfrak{M}.$$

Hence, $\mathfrak{M}_\mu\subseteq\mathfrak{M}$, and thus $\mathfrak{M}=\mathfrak{M}_\mu$.

It follows from (2.25) that $\mathfrak{M}$ is invariant.

Lemma 2.5. *The manifold* $\mathfrak{M}$ *defined as in* (2.24) *has the asymptotic completeness property, i.e. for every* $U\in E\times H$, *there exists a* $V\in\mathfrak{M}$ *such that*

$$|S(t)U-S(t)V|_{E\times H}\le C(U)e^{-\mu t},\quad \forall t\ge 0,$$

where μ *is any number satisfying* (2.23) *and* $C(U)$ *is a positive constant depending on* U.

Proof. Let $u(t)$ be a solution of (1.4) with $(u(0),u_t(0))=U,\ u(t)$ defined on R. Let $\bar u(t)=\Phi(t)u(t)$, where $\Phi(t)$ is a smooth function, $|\Phi_{tt}|\le 2$ and

$$\Phi(t)=\begin{cases}1, & t\ge 1,\\ \in[0,1], & t\in[0,1],\\ 0, & t\le 0.\end{cases}$$

We know that problem (1.4) is dissipative, i.e. there exist a $t^*>0$ and a bounded set $B\subset E\times H$ such that $(u(t),u_t(t))\subset B,\ \forall t\ge t_0^*$. Thus, $\bar u\in E_\mu$, and $\bar u_t\in H_\mu$. $\bar u$ satisfies

$$\delta\bar u_{tt}+\bar u_t+A\bar u=F(\bar u)+r(t), \tag{2.26}$$

where

$$r(t)=\Phi F(u)-F(\Phi u)+(\delta\Phi_{tt}+\Phi_t)u+2\Phi_t u_t,\ r(t)=0,\ \text{ as } t\le 0 \text{ or } t>1.$$

Define the mapping T^* as

$$u^0\to T^*(u^0)=p+q=R_1^*f+S_1g\in E_\mu^*,\ \ \forall u^0\in E_\mu^*,$$

where $p \in PE^*_\mu$, $q \in QH^*_\mu$ are respectively the solutions of the following equations

$$p_{tt} + p_t = -APu^0 + PF(u^0 + \bar{u}) - PF(\bar{u}) - Pr(t) \triangleq f,$$
$$q_{tt} + q_t + Aq = QF(u^0 + \bar{u}) - QF(\bar{u}) - Qr(t) \triangleq g.$$

Now, consider the following problem in E^*_μ:

$$\delta\tilde{u}_{tt} + \tilde{u}_t + A\tilde{u} = F(\tilde{u} + \bar{u}) - F(\bar{u}) - r(t). \tag{2.27}$$

This problem is equivalent to

$$\tilde{u} = T^*\tilde{u} = R^*_1(-AP\tilde{u}+PF(\tilde{u}+\bar{u})-PF(\bar{u})-Pr(t))+S_1(QF(\tilde{u}+\bar{u})-QF(\bar{u})-Qr(t)). \tag{2.28}$$

Since $F(\cdot)$ is Lipschitz, it follows from Lemmas 2.1 and 2.2 that T^* maps E^*_μ into E^*_μ, and as in Lemma 2.3, we obtain that T^* is a contraction on E^*_μ under the assumption (2.23) and when δ is small. Therefore (2.28) and also (2.27) is solvable in E^*_μ, i.e. there exists a unique solution $\tilde{u}$ of (2.27) with $\tilde{u} \in E^*_\mu$, $\tilde{u}_t \in H^*_\mu$. Let $u^* = \tilde{u} + \bar{u}$ Then $u^* \in E_\mu$, $u^*_t \in H_\mu$ and u^* satisfies

$$\delta u^*_{tt} + u^*_t + Au^* = F(u^*).$$

From (2.25), it follows that $(u^*(t), u^*_t(t)) \in \mathfrak{M}$, $t \geq 0$. Let $V = (u^*(0), u^*_t(0))$. Then, $V \in \mathfrak{M}$, and

$$\begin{aligned}|S(t)U - S(t)V|_{E\times H} &\leq |u(t), u_t(t)) - (u^*(t), u^*_t(t))|_{E\times H}\\ &= |\tilde{u}(t), \tilde{u}_t(t))|_{E\times H} \leq \max\{|\tilde{u}|_{E^*_\mu}, |\tilde{u}_t|_{H^*_\mu}\}e^{-\mu t}, \quad t \geq 1.\end{aligned} \tag{2.29}$$

Since $[0,1]$ is a finite interval, we can find a constant $C_1(U)$ such that

$$|S(t)U - S(t)V|_{E\times H} \leq C_1(U)e^{-\mu t}, \quad t \in [0,1]. \tag{2.30}$$

From (2.29) and (2.30), we have

$$|S(t)U - S(t)V|_{E\times H} \leq C(U)e^{-\mu t}, \quad \forall t \geq 0. \tag{2.31}$$

Lemma 2.5 is completed.

From Lemmas 2.4 and 2.5 we complete the proof of Theorem.

§3. Application

We consider a nonlinear wave equation

$$\begin{aligned}&\delta u_{tt} + u_t - u_{xx} + f(u) = 0, \quad x \in (0,\pi), \quad t > 0,\\ &u|_{x=0} = u|_{x=\pi} = 0, \quad u(0) = u_0(x), \quad u_t(0) = u_1(x).\end{aligned} \tag{3.1}$$

We assume that

$$\liminf_{|s|\to\infty} s^{-2}\int_0^s f(\sigma)d\sigma \geq 0;$$

and that there exist $c_1, c_2 > 0$ such that

$$\liminf_{|s|\to\infty} s^{-2}(sf(s) - c_1\int_0^s f(\sigma)d\sigma \geq 0;$$
$$|f'(s)| \leq c_2(1 + |s|^r), \quad 0 \leq r < \infty.$$

Let $Au = -u_{xx}$. Then

$$D(A^{\frac{1}{2}}) = H^1_0(0,\pi), \quad H = L^2(0,\pi).$$

Equation (3.1) is dissipative in $D(A^{\frac{1}{2}})\times H$ (see [2]), and $f: D(A^{\frac{1}{2}})\mapsto D(A^{\frac{1}{2}-\beta})$ for $\beta=0$. The eigenvalues of A satisfy $\lambda_n\sim\lambda_1 n^2$; the spectral gap condition is thus obviously satisfied. So, all the assumptions of Theorem are satisfied. When δ is small, we furthermore have that (3.1) possesses a finite dimensional inertial manifold.

The next example is the following system:

$$\delta u_{tt}+u_t+u_{xxxx}-\Big(\Gamma-\int_\Omega |u_x(x,t)|^2dx\Big)u_{xx}=g(x),$$
$$u|_{\partial\Omega}=u_{xx}|_{\partial\Omega}=0,\quad u(0)=u_0,\quad u_t(0)=u_1. \tag{3.2}$$

Here δ and Γ are positive constants and $\Omega=(0,\pi)$. This system appeared in [8] for the study of nonlinear oscillations of an elastic shell.

Let $H=L^2(\Omega)$, $Au=u_{xxxx}$. Then

$$D(A^{\frac{1}{2}})=H^2(\Omega)\cap H_0^1(\Omega),$$

(3.2) is dissipative in $D(A^{\frac{1}{2}})\times H$ (see [8]), and

$$f(u)=\Big(\Gamma-\int_\Omega |u_x(x,t)|^2dx\Big)u_{xx}+g(x).$$

The function f is Lipschitz from $D(A^{\frac{1}{2}})$ into $D(A^{\frac{1}{2}-\beta})$ for $\beta=\frac{1}{2}$. The eigenvalues of A satisfy $\lambda_n\sim\lambda_1 n^4$ and the spectral gap condition is satisfied. From Theorem, (3.2) has a finite dimensional inertial manifold when δ is small.

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