Chin. Ann. of Math.
20B: 3(1999),279-296.

EXISTENCE OF MINIMIZING SOLUTIONS AROUND "EXTENDED STATES" FOR A NONLINEARLY ELASTIC CLAMPED PLANE MEMBRANE

D. COUTAND*

Abstract

The formal asymptotic analysis of D. Fox, A. Raoult & J.C. Simo[10] has justified the two-dimensional nonlinear "membrane" equations for a plate made of a Saint Venant-Kirchhoff material.

This model, which retains the material-frame indifference of the original three dimensional problem in the sense that its energy density is invariant under the rotations of $\mathbb{R}^3$, is equivalent to finding the critical points of a functional whose nonlinear part depends on the first fundamental form of the unknown deformed surface.

The author establishes here, by the inverse function theorem, the existence of an injective solution to the clamped membrane problem around particular forces corresponding physically to an "extension" of the membrane. Furthermore, it is proved that the solution found in this fashion is also the unique minimizer to the nonlinear membrane functional, which is not sequentially weakly lower semi-continuous.

Keywords Minimizing solution, Nonlinearly elastic clamped plane membrane, Injective Solution
1991 MR Subject Classification 35J60, 35A05
Chinese Library Classification O175.25

§1. Introduction

A justification of the classical two-dimensional equations of a nonlinearly elastic "membrane" plate and those of a nonlinearly elastic "flexural" plate, as they appear in the mechanical litterature (see [13], for instance), has been given by Fox, Raoult & Simo[10] by means of the method of formal asymptotic expansions applied to the three-dimensional equations of nonlinear elasticity for a Saint Venant-Kirchhoff material.

Those two nonlinear models present two remarkable common features. In both cases, the scalings for the displacements set in the analysis of Fox, Raoult & Simo[10] are of order $O(1)$ with respect to the thickness ε of the plate and their energy density is invariant under the rotations of $\mathbb{R}^3$ as the original three-dimensional energy. For these reasons, they are called "large displacements" and frame-indifferent theories. In consequence, they must be distinguished from the more familiar nonlinear Kirchhoff-Love theory justified, again by a

Manuscript received October 22, 1998. Revised February 9, 1999.
*Université of Pierre et Marie Curie, Laboratoire d'Analyse Numérique, 4, Place Jussieu, 75005 Paris, France. **E-mail:** coutand@ann.jussieu.fr

formal asymptotic analysis, by Ciarlet & Destuynder[4] (see in this respect the extensive presentation given in [3]).

Another approach has been developped by Le Dret & Raoult[14] who have justified another nonlinear "membrane" plate model. By means of Γ-convergence theory their analysis gives a convergence result, as the thickness tends to zero, of quasi-minimizers of the three-dimensional energies towards a minimizer of a two-dimensional "membrane" energy. The existence of a minimizer to this energy is thus de facto established.

The models justified by Fox, Raoult & Simo[10] take the form of critical point problems for the associated energies, which in both cases are expressed in terms of the geometry of the unknown deformed surface.

The stored energy function of a "membrane" plate is a quadratic and positive definite expression (via the two-dimensional elasticity tensor of the plate) in terms of the exact difference between the metric tensor of the unkwown surface and that of the reference configuration.

The stored energy of a "flexural" plate is a quadratic and positive definite expression (again via the two-dimensional elasticity tensor of the plate) in terms of the exact difference between the curvature tensor of the unkwown surface and that of the reference configuration. Another distinctive feature of the "flexural" model is that the critical point problem is formulated over a manifold of admissible deformations which are those that preserve the metric of the undeformed plate and satisfy boundary conditions of clamping or of simple support. The existence of a minimizer to the nonlinear "flexural" plate functional has been established in [5].

Concerning the mathemetical analysis of the nonlinear membrane equations, some results are already known. First, in [7], we have established a local existence result for the clamped membrane plate submitted to "small enough" plane forces and we have shown that the solution found in this fashion is a local minimizer to the associated membrane functional in an "optimal" affine space. Second, when the membrane is submitted to a boundary condition of "tension", introduced by Fox, Raoult & Simo[10], the implicit function theorem provides the existence of an injective solution for "small enough" forces, without restriction on their direction (see [8]). We have also shown that the solution found by this means is also the unique minimizer to the membrane functional over the "whole" affine space of admissible deformations. Furthermore, the behaviour of the membrane as the "tension" goes to infinity has also been investigated, and we have shown in particular that the radius of the ball containing the forces for which we can associate a solution may also go to infinity, in a cubic fashion. Those results hold for any plate with a boundary of class $\mathcal{C}^2$. Third, in the case of a clamped disk, another approach has been developped by Genevey[11]. She has shown that, under some assumptions on the Lamé constants, the inverse function theorem gives the existence of radial solutions in an ad hoc Sobolev weighted space, for radial forces situated around a particular, explicitely given, radial force.

The aim of this paper is to establish other existence results in the case of the boundary condition of clamping for any plate with a boundary of class $\mathcal{C}^2$. Our analysis concerns the properties of the membrane model in a neighborhood of a class of forces identified physically as producing an "extension" at each point of the clamped membrane.

In Section 2, we describe the problem, in terms of a system of partial differential equations or equivalently as a critical point problem. The difficulties inherent to these two formulations are of two kinds : First, the boundary value problem system is quasilinear and not semi-linear as in the case of the nonlinear Kirchhoff-Love theory for instance. Second, as already noted by D. Fox, A. Raoult & J. C. Simo[10], the functional energy associated to the nonlinear membrane model is coercive but not sequentially weakly lower semi-continuous, which forbids to apply the classical theorem of the calculus of variations. For these reasons, the mathematical analysis of these equations is very delicate.

In Section 3, we define the notion of "extended" states, which are the deformations whose metric tensor at each point of the plane membrane is larger, in a certain sense, than the reference configuration one. We show that this class of deformations is not empty for any plate with a boundary of class $\mathcal{C}^2$ and that they can be chosen as "close" to the reference configuration as desired, or as "far" from the undeformed plate as well.

In Section 4, via the inverse function theorem, we establish the existence of an injective solution to the nonlinear clamped membrane problem in a neighborhood of any force corresponding to an"extended state". Those forces, which "extend" the clamped membrane, can be chosen as "small" as desired or as "large" also. Furthermore, we show that the solution found in this fashion possesses the remarkable feature of being the unique minimizer to the associated membrane functional over the "whole" affine space of admissible deformations. Thus, we establish in an indirect way, an existence and uniqueness result for a minimization problem in a case where the standard method of the calculus of variations cannot be applied.

In Section 5, by investigating the behaviour of the clamped plate as some forces corresponding to extended states go to infinity in a certain fashion, we conclude that a "well extended" clamped membrane may undergo large loadings.

§2. The Nonlinear Clamped Plane Membrane Problem

Greek indices and exponents take their values in the set $\{1,2\}$, Latin indices take their values in the set $\{1,2,3\}$, and the summation convention with respect to the repeated indices is used. Vectors of $\mathbb{R}^2$ or $\mathbb{R}^3$ and vector valued functions are written in boldface letters. The Euclidean inner product, the exterior product, and the Euclidean norm of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are denoted by $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \wedge \mathbf{b}$ and $\mid \mathbf{a} \mid$. The standard Euclidean distance between x and y, points of $\mathbb{R}^2$ is denoted by $d(x, y)$.

Let ω be an open, bounded, and connected subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, the set ω being locally on one side of γ.

The usual norm of the Sobolev space $\mathbf{W}^{m,p}(\omega; \mathbb{R}^3)$, $m \in \mathbb{N}$, $p \in]0, +\infty[$, is denoted by $|| \cdot ||_{m,p,\omega}$.

The ball of $\mathbf{W}^{m,p}(\omega; \mathbb{R}^3)$ $(m \in \mathbb{N}, p \in]0, +\infty[)$ centered at $\mathbf{a} \in \mathbf{W}^{m,p}(\omega; \mathbb{R}^3)$ and with radius $R = 840$ is denoted by $B_{m,p}(\mathbf{a}, R)$.

We denote by $\mathbf{id}$ the identity of $\mathbb{R}^2$ and by $\boldsymbol{\iota}$ the mapping defined from $\bar{\omega}$ into $\mathbb{R}^3$ by

$$\boldsymbol{\iota}(x) = (x_1, x_2, 0) \text{ for all } x = (x_1, x_2) \in \bar{\omega}.$$

Let $\boldsymbol{f} \in \mathbf{L}^2(\omega; \mathbb{R}^3)$ be the density of forces acting on the plate. Then the asymptotic analysis of Fox, Raoult & Simo[10] justifies the nonlinear membrane model for the plate clamped over its whole boundary as a critical point problem for the functional

$$I_M(\boldsymbol{f}) = \mathrm{I}_M - L(\boldsymbol{f}),$$

where I_M and $L(\boldsymbol{f})$ are defined on the affine space

$$\big\{\boldsymbol{\iota} + \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)\big\} = \{\boldsymbol{\varphi} \in \mathbf{W}^{1,4}(\omega;\mathbb{R}^3)\ , \boldsymbol{\varphi} = \boldsymbol{\iota} \text{ on } \gamma\},$$

by

$$\mathrm{I}_M(\boldsymbol{\varphi}) = \int_\omega \Big\{ \frac{\lambda\ \mu}{\lambda + 2\mu} \tilde{a}_{\sigma\sigma}(\boldsymbol{\varphi})\tilde{a}_{\tau\tau}(\boldsymbol{\varphi}) + \mu\tilde{a}_{\alpha\beta}(\boldsymbol{\varphi})\tilde{a}_{\alpha\beta}(\boldsymbol{\varphi})\Big\} d\omega,$$

$$L(\boldsymbol{f})(\boldsymbol{\varphi}) = \int_\omega \boldsymbol{f} \cdot \boldsymbol{\varphi} d\omega,$$

where $\tilde{a}_{\alpha\beta}(\boldsymbol{\varphi}) = \partial_\alpha\boldsymbol{\varphi}\cdot\partial_\beta\boldsymbol{\varphi} - \delta_{\alpha\beta}$ are the components of the change of metric tensor between the unkwown deformed surface and the undeformed one.

As already noted by Fox, Raoult & Simo[10], the functional I_M is not sequentially weakly lower semi-continuous over $\mathbf{W}^{1,4}(\omega;\mathbb{R}^3)$, which forbids the use of the classical theorem of the calculus of variations.

Equivalently, this problem can be written under the form of the following boundary value problem: Find $\boldsymbol{\varphi} \in \mathbf{W}^{1,4}(\omega;\mathbb{R}^3)$ such that, in the distributional sense:

$$\begin{cases} -\partial_\alpha\Big\{\Big(\frac{2\lambda\mu}{\lambda+2\mu}\ \delta_{\alpha\beta}\ \tilde{a}_{\sigma\sigma}(\boldsymbol{\varphi}) + 2\mu\ \tilde{a}_{\alpha\beta}(\boldsymbol{\varphi})\Big)\ \partial_\beta\ \boldsymbol{\varphi}\ \Big\} = \boldsymbol{f} \text{ in } \omega, \\ \boldsymbol{\varphi} = \boldsymbol{\iota} \text{ on } \gamma, \end{cases}$$

where $\delta_{\alpha\beta}$ denotes the Kronecker symbol. Note that in this formulation, the unknown $\boldsymbol{\varphi}$ is the deformation of the plate, i.e., the position taken by the plate under the action of the applied forces.

In the next section we define the notion of "extended" states, which are the deformations whose metric tensor is, in a certain sense, larger than the reference configuration one. We then show that the set of extended states is not empty and that they can be chosen as "close" to the reference configuration as desired, or as "far" from the undeformed plate also.

§3. A Class of Deformations Corresponding to an "Extension" of the Membrane

We first introduce the following definition:

Definition 3.1. *We say that* $\bar{\boldsymbol{\varphi}} \in \mathcal{C}^2(\bar{\omega};\mathbb{R}^3)$ *is an extended state if the following conditions hold:*

(i) $\bar{\boldsymbol{\varphi}} = \boldsymbol{\iota}$ *on* γ.

(ii) *There exists a constant* $C_1 > 0$ *such that for all* x *and* y *in* $\bar{\omega}$ *:*

$$|\ \bar{\boldsymbol{\varphi}}(x) - \bar{\boldsymbol{\varphi}}(y)\ | \geq C_1\ d(x,y).$$

(iii) *There exists a constant* $C_2 > 0$ *such that, for all* $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ *and all* $x \in \bar{\omega}$,

$$\frac{2\ \lambda\ \mu}{\lambda + 2\mu}\ \tilde{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})(x)\ \boldsymbol{\xi}_\tau \cdot \boldsymbol{\xi}_\tau + 2\ \mu\ \tilde{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})(x)\ \boldsymbol{\xi}_\alpha \cdot \boldsymbol{\xi}_\beta \geq\ C_2\ (\ |\boldsymbol{\xi}_1|^{\ 2} + |\boldsymbol{\xi}_2|^{\ 2}).$$

We denote by $E(\omega)$ the set of such mappings $\bar{\boldsymbol{\varphi}}$.

Remark. We note that the second condition implies the injectivity of $\bar{\boldsymbol{\varphi}}$. Moreover we infer from (i) that a constant satisfying (ii) necessarily verifies $0 < C_1 \leq 1$.

The adjective extended is justified by condition (iii) which asserts that the metric tensor of an element of $E(\omega)$ at each point of the plate is in a certain sense larger than the reference configuration one. In Section 4 we establish that this condition implies the strong ellipticity of the derivative at any extended state of the nonlinear membrane operator.

Now, for any plate with a boundary of class $\mathcal{C}^2$, we prove the existence of extended states, which in addition can be chosen as close to the identity as desired in the spaces $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)$ $(p>2)$, or of norm arbitrarily large in the same spaces.

We first need two preliminary results.

Lemma 3.1. *Assume that the boundary γ of ω is of class $\mathcal{C}^2$. Then, there exist $\delta(\omega)>0$, $h>0$, and $\psi\in\mathcal{C}^2(\bar\omega)$ such that the following conditions hold*:

(i) $\partial_\sigma\psi\ \partial_\sigma\psi>\delta(\omega)$ *in* $\omega_1=\{x\in\omega;\ d(x,\gamma)<h\}$,

(ii) $\psi=0$ *on* γ, $\psi>0$ *in* ω *and* $\psi>\delta(\omega)$ *in* $\omega-\omega_1$,

(iii) $\partial_\sigma\psi\ \partial_\sigma\psi+\partial_\sigma(\psi\ \boldsymbol{\iota})\cdot\partial_\sigma(\psi\ \boldsymbol{\iota})>\delta(\omega)$ *in* $\bar\omega$.

Proof. The open set ω being locally on one side of its boundary of class $\mathcal{C}^2$, for each $x\in\gamma$ there exists a neighborhood $V(x)$ of x in $\mathbb{R}^2$ such that we can define a $\mathcal{C}^2$-diffeomorphism $\boldsymbol{\theta}$ from $]-1,1[\times]-1,1[$ into $V(x)$ satisfying

$$\begin{cases}\boldsymbol{\theta}(]-1,1[\times]0,1[)=V(x)\cap\omega,\\ \boldsymbol{\theta}(]-1,1[\times\{0\})=V(x)\cap\gamma,\\ \boldsymbol{\theta}(]-1,1[\times]-1,0[)=V(x)\cap\{\mathbb{R}^2-\bar\omega\}.\end{cases}$$

Let $(V_i)_{1\le i\le n}$ be a finite extracted covering of the compact set γ and $(\boldsymbol{\theta}_i)_{1\le i\le n}$ be the corresponding $\mathcal{C}^2$-diffeomorphisms. We denote by $\tilde{\boldsymbol{\theta}}_i$ $(1\le i\le n)$ the inverse mapping of $\boldsymbol{\theta}_i$, which defines a $\mathcal{C}^2$-diffeomorphism from V_i into $]-1,1[\times]-1,1[$.

Since the compact set γ is included in the open set $V=\bigcup\limits_{i=1}^{n}V_i$, let κ be a given real such that $0<\kappa<d(\gamma,\partial V)$, where ∂V denotes the boundary of V.

Let $V_0=\{x\in\omega\ ;\ d(x,\gamma)>\kappa\}$. Then, $\bar\omega\subset\bigcup\limits_{i=0}^{n}V_i$.

Let $(\alpha_i)_{1\le i\le n}$ be a partition of the unity associated to this covering of $\bar\omega$, *i.e.*

$$\begin{cases}\alpha_i\in\mathcal{D}(\mathbb{R}^2),\ S_i=\operatorname{supp}\alpha_i\subset V_i,\ 0\le i\le n,\\ 0\le\alpha_i\le1 \text{ in } \mathbb{R}^2,\ 0\le i\le n,\\ \sum\limits_{i=0}^{n}\alpha_i=1 \text{ in } \bar\omega.\end{cases}$$

For $1\le i\le n$, we denote by ψ_i the mapping defined on $\bar\omega$ by

$$\begin{cases}\psi_i(x)=\alpha_i(x)\ \tilde{\boldsymbol{\theta}}_i(x)\cdot\mathbf{e}_2, \text{ if } x\in V_i\cap\bar\omega,\\ \psi_i(x)=0, \text{ otherwise,}\end{cases}$$

where $\mathbf{e}_2=(0,1)$. Since $S_i\subset V_i$, it follows that $\psi_i\in\mathcal{C}^2(\bar\omega)$. Moreover, from

$$\tilde{\boldsymbol{\theta}}_i(V_i\cap\omega)=]-1,1[\times]0,1[\ ,\ \text{and}\ \tilde{\boldsymbol{\theta}}_i(V_i\cap\gamma)=]-1,1[\times\{0\},\tag{3.1}$$

we infer that $\psi_i\ge0$ in ω and $\psi_i=0$ on γ. We then define the mapping $\psi\in\mathcal{C}^2(\bar\omega)$ by $\psi=\alpha_0+\sum\limits_{i=1}^{n}\psi_i$. Clearly, $\psi\ge0$ in ω and $\psi=0$ on γ.

Now, assume that there exists $x\in\omega$ such that $\psi(x)=0$. Then $\alpha_0(x)=0$ and $\psi_i(x)=0$, $1\le i\le n$. The condition $\alpha_0(x)=0$ implies $\sum\limits_{i=1}^{n}\alpha_i(x)=1$. Let $i\in[1,n]$ be such that

$\alpha_i(x) > 0$, which in turn implies $x \in S_i \cap \omega \subset V_i \cap \omega$. From (3.1) we deduce $\tilde{\boldsymbol{\theta}}_i(x) \cdot \mathbf{e}_2 > 0$ and finally, $\psi_i(x) > 0$, which gives a contradiction. Hence, we infer that

$$\psi > 0 \text{ in } \omega. \tag{3.2}$$

Now let x be any element of γ. Clearly,

$$\partial_\nu \psi(x) = \sum_{i=1}^{n} \partial_\nu \psi_i(x) = \sum_{i \in I(x)} \alpha_i(x)\ \partial_\nu \tilde{\boldsymbol{\theta}}_i(x) \cdot \mathbf{e}_2,$$

where $I(x) = \{i \in \mathbb{N}_n\ ;\ \alpha_i(x) > 0\}$.

Let i be any element in $I(x)$. Since $\mathrm{d}\boldsymbol{\theta}_i(\tilde{\boldsymbol{\theta}}_i(x)) \circ \mathrm{d}\tilde{\boldsymbol{\theta}}_i(x) = \mathbf{id}$, we infer that the rank of the linear operator $\mathrm{d}\tilde{\boldsymbol{\theta}}_i(x)$ is two. Since $\tilde{\boldsymbol{\theta}}_i \cdot \mathbf{e}_2 = 0$ along $V_i \cap \gamma$, we deduce that $\partial_\tau \tilde{\boldsymbol{\theta}}_i(x) \cdot \mathbf{e}_2 = 0$, which gives $\partial_\nu \tilde{\boldsymbol{\theta}}_i(x) \cdot \mathbf{e}_2 \neq 0$. Moreover (3.1) enables us to assert that $\partial_\nu \tilde{\boldsymbol{\theta}}_i(x) \cdot \mathbf{e}_2 < 0$, and finally that

$$\sum_{i \in I(x)} \alpha_i(x)\ \partial_\nu \tilde{\boldsymbol{\theta}}_i(x) \cdot \mathbf{e}_2 < 0,$$

which gives

$$\nabla \psi(x) \neq \mathbf{0} \text{ on } \gamma. \tag{3.3}$$

Since $\psi \in \mathcal{C}^2(\bar{\omega})$, let $\delta_1 > 0$ and $h > 0$ be such that

$$\partial_\sigma\ \psi\ \partial_\sigma \psi > \delta_1 \text{ on } \omega_1 = \{x \in \omega\ ;\ d(x,\gamma) < h\}\ .$$

Since $\psi \in \mathcal{C}^2(\bar{\omega})$ and $\psi > 0$ in ω, let $\delta_2 > 0$ be such that $\psi > \delta_2$ in $\omega - \omega_1$. Now, assume that there exists $x \in \bar{\omega}$ such that

$$\partial_\sigma \psi(x) \partial_\sigma \psi(x) + \partial_\sigma(\psi\ \boldsymbol{\iota})(x) \cdot \partial_\sigma(\psi\ \boldsymbol{\iota})(x) = 0.$$

Since $\partial_\sigma(\psi\ \boldsymbol{\iota})(x) = \partial_\sigma \psi(x)(x_1, x_2, 0) + \psi(x)\ (\delta_{1\sigma}, \delta_{2\sigma}, 0)$, it follows that $\psi(x) = 0$ and then that $x \in \gamma$. But we have seen in (3.3) that on the boundary, $\partial_\sigma \psi(x) \partial_\sigma \psi(x) > 0$, which leads to a contradiction. Hence, since $\psi \in \mathcal{C}^2(\bar{\omega})$, let $\delta_3 > 0$ be such that

$$\partial_\sigma \psi \partial_\sigma \psi + \partial_\sigma(\psi\ \boldsymbol{\iota}) \cdot \partial_\sigma(\psi\ \boldsymbol{\iota}) > \delta_3 \text{ in } \bar{\omega}.$$

Taking $\delta(\omega) = \min(\delta_1, \delta_2, \delta_3) > 0$, we see that ψ satisfies the statement of the lemma.

Lemma 3.2. *Assume that the boundary of ω is of class $\mathcal{C}^2$. Let ψ denote any element satisfying the conditions of Lemma 3.1. There exists $M > 0$ such that for any k_1 and k_2 such that $k_1 > 0$ and $0 < k_2 \leq M\ k_1^2$, the mapping $\bar{\boldsymbol{\varphi}} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ defined by*

$$\bar{\boldsymbol{\varphi}}(x_1, x_2) = \boldsymbol{\iota}(x) + k_1\ (0, 0, \psi(x)) + k_2\ \psi(x)\ (x_1, x_2, 0) \text{ for all } x = (x_1, x_2) \in \bar{\omega},$$

is an element of $E(\omega)$.

Proof. Let $k_1 > 0$ and $k_2 > 0$ be given, and let $\bar{\boldsymbol{\varphi}}$ be defined as in the statement of the lemma.

Step 1. Since $\psi \in \mathcal{C}^2(\bar{\omega})$ and $\psi = 0$ on γ, then $\bar{\boldsymbol{\varphi}} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ and $\bar{\boldsymbol{\varphi}} = \boldsymbol{\iota}$ on γ, which shows that condition (i) of Definition 3.1 is satisfied.

Step 2. Let x and y be any points in $\bar{\omega}$. From Schwarz's inequality,

$$\sqrt{3}\ |\ \bar{\boldsymbol{\varphi}}(x) - \bar{\boldsymbol{\varphi}}(y)\ | \geq \sum_\alpha |\ x_\alpha + k_2 \psi(x)\ x_\alpha - y_\alpha - k_2 \psi(y) y_\alpha\ | + k_1\ |\ \psi(x) - \psi(y)\ |\ . \tag{3.4}$$

Writing the right side in a different way, we also obtain

$$\begin{aligned}\sqrt{3} \mid \bar{\boldsymbol{\varphi}}(x) - \bar{\boldsymbol{\varphi}}(y) \mid \geq \sum_\alpha \mid (x_\alpha - y_\alpha)\ (1 + k_2\ \psi(y)) + k_2 x_\alpha\ (\psi(x) - \psi(y)) \mid \\ + k_1 \mid \psi(x) - \psi(y) \mid .\end{aligned} \tag{3.5}$$

Let $R = \max_{x \in \bar{\omega}} d(x, 0) > 0$. If $2k_2 R \mid \psi(x) - \psi(y) \mid \geq d(x, y)$, then from (3.4)

$$\mid \bar{\boldsymbol{\varphi}}(x) - \bar{\boldsymbol{\varphi}}(y) \mid \geq \frac{k_1}{2\sqrt{3}k_2 R} d(x, y). \tag{3.6}$$

If $2k_2 R \mid \psi(x) - \psi(y) \mid \leq d(x, y)$, then from (3.5)

$$\begin{aligned}\sqrt{3} \mid \bar{\boldsymbol{\varphi}}(x) - \bar{\boldsymbol{\varphi}}(y) \mid \geq \sum_\alpha \mid (x_\alpha - y_\alpha)\ (1 + k_2\ \psi(y)) \mid + k_1 \mid \psi(x) - \psi(y) \mid \\ - k_2 \mid \psi(x) - \psi(y) \mid \ \sum_\alpha \mid x_\alpha \mid .\end{aligned}$$

Since $\psi \geq 0$ in ω and $\sum_\alpha \mid x_\alpha \mid \leq \sqrt{2} R$, we obtain

$$\sqrt{3} \mid \bar{\boldsymbol{\varphi}}(x) - \bar{\boldsymbol{\varphi}}(y) \mid \geq \sum_\alpha \mid x_\alpha - y_\alpha \mid - \sqrt{2} k_2 R \mid \psi(x) - \psi(y) \mid .$$

Since $d(x, y) \leq \sum_\alpha \mid x_\alpha - y_\alpha \mid$ and $2\ k_2\ R \mid \psi(x) - \psi(y) \mid \leq d(x, y)$, we infer that

$$\mid \bar{\boldsymbol{\varphi}}(x) - \bar{\boldsymbol{\varphi}}(y) \mid \geq \frac{\sqrt{2} - 1}{\sqrt{6}} d(x, y). \tag{3.7}$$

Let

$$C_1 = \min\Big(\frac{k_1}{2\ \sqrt{3}\ k_2 R}, \frac{\sqrt{2} - 1}{\sqrt{6}}\Big) > 0. \tag{3.8}$$

Then for all x and y in $\bar{\omega}$, we have

$$\mid \bar{\boldsymbol{\varphi}}(x) - \bar{\boldsymbol{\varphi}}(y) \mid \geq C_1\ d(x, y). \tag{3.9}$$

This is precisely condition (ii) of Definition 3.1.

Step 3. Let $\boldsymbol{\eta}$ be defined by $\boldsymbol{\eta} = \bar{\boldsymbol{\varphi}} - \boldsymbol{\iota}$ and let $\boldsymbol{\xi} = (\xi_1, \xi_2)$ be any element of $\mathbb{R}^2$.

Then we have in $\bar{\omega}$ (we no longer mention the dependence on x; we also remind that the summation convention with respect to the repeated indices is used):

$$\begin{aligned}&\frac{2\lambda\ \mu}{\lambda + 2\mu} \bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}}) \boldsymbol{\xi}_\tau \cdot \boldsymbol{\xi}_\tau + 2\ \mu \bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\ \boldsymbol{\xi}_\alpha \cdot \boldsymbol{\xi}_\beta \\ &= \frac{2\ \lambda\ \mu}{\lambda + 2\mu} (k_1^2\ \partial_\sigma \psi \partial_\sigma \psi + k_2^2 \partial_\sigma(\psi\ \boldsymbol{\iota}) \cdot \partial_\sigma(\psi\ \boldsymbol{\iota}) + 2\ k_2 \mathrm{div}(\psi\ \boldsymbol{\iota})) \mid \boldsymbol{\xi} \mid^2 \\ &\quad + 2\mu(\partial_\alpha \boldsymbol{\eta} \cdot \partial_\beta \boldsymbol{\eta} + k_2\ (\partial_\alpha(\psi \boldsymbol{\iota}) \cdot \partial_\beta \boldsymbol{\iota} + \partial_\beta(\psi\ \boldsymbol{\iota}) \cdot \partial_\beta \boldsymbol{\iota}))\xi_\alpha\ \xi_\beta.\end{aligned} \tag{3.10}$$

But $\partial_\alpha \boldsymbol{\eta} \cdot \partial_\beta \boldsymbol{\eta}\ \xi_\alpha\ \xi_\beta = (\xi_\alpha\ \partial_\alpha \boldsymbol{\eta}) \cdot (\xi_\beta\ \partial_\beta \boldsymbol{\eta}) \geq 0$, hence

$$\begin{aligned}&\frac{2\ \lambda\ \mu}{\lambda + 2\mu} \bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}}) \boldsymbol{\xi}_\tau \cdot \boldsymbol{\xi}_\tau + 2\mu \bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}}) \boldsymbol{\xi}_\alpha \cdot \boldsymbol{\xi}_\beta \\ &\geq \frac{2\lambda\mu}{\lambda + 2\mu} (k_1^2 \partial_\sigma \psi \partial_\sigma \psi + k_2^2 \partial_\sigma(\psi\ \boldsymbol{\iota}) \cdot \partial_\sigma(\psi \boldsymbol{\iota}) + 2k_2 \mathrm{div}(\psi \boldsymbol{\iota})) \mid \boldsymbol{\xi} \mid^2 \\ &\quad + 2\mu k_2\ (\partial_\alpha(\psi \boldsymbol{\iota}) \cdot \partial_\beta \boldsymbol{\iota} + \partial_\beta(\psi \boldsymbol{\iota}) \cdot \partial_\alpha \boldsymbol{\iota}) \xi_\alpha\ \xi_\beta.\end{aligned}$$

Equivalently, we have

$$\frac{2\lambda\mu}{\lambda+2\mu}\bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\tau\cdot\boldsymbol{\xi}_\tau+2\mu\bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\alpha\cdot\boldsymbol{\xi}_\beta$$
$$\geq\frac{2\lambda\mu}{\lambda+2\mu}(k_1^2\ \partial_\sigma\psi\partial_\sigma\psi+k_2^2\partial_\sigma(\psi\boldsymbol{\iota})\cdot\partial_\sigma(\psi\boldsymbol{\iota})+4k_2\psi+2k_2\partial_\sigma\psi\ x_\sigma)\mid\boldsymbol{\xi}\mid^2$$
$$+2\mu k_2(2\psi\delta_{\alpha\beta}+\partial_\alpha\psi x_\beta+\partial_\beta\psi x_\alpha)\xi_\alpha\xi_\beta.$$

By Schwarz's inequality,

$$\frac{2\lambda\mu}{\lambda+2\mu}\bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\tau\cdot\boldsymbol{\xi}_\tau+2\mu\bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\alpha\cdot\boldsymbol{\xi}_\beta$$
$$\geq\frac{2\lambda\mu}{\lambda+2\mu}(k_1^2\ \partial_\sigma\psi\partial_\sigma\psi+k_2^2\partial_\sigma(\psi\boldsymbol{\iota})\cdot\partial_\sigma(\psi\boldsymbol{\iota})+4k_2\psi+2k_2\partial_\sigma\psi x_\sigma)\mid\boldsymbol{\xi}\mid^2$$
$$+4\mu k_2\psi\mid\boldsymbol{\xi}\mid^2-4\mu k_2\sum_{\alpha,\beta}\mid\partial_\alpha\psi\ x_\beta\mid\mid\boldsymbol{\xi}\mid^2.$$

Hence

$$\frac{2\lambda\mu}{\lambda+2\mu}\bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\tau\cdot\boldsymbol{\xi}_\tau+2\mu\bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\alpha\cdot\boldsymbol{\xi}_\beta$$
$$\geq\frac{2\lambda\mu}{\lambda+2\mu}\Big(\frac{k_1^2}{2}\partial_\sigma\psi\partial_\sigma\psi+k_2^2\partial_\sigma(\psi\boldsymbol{\iota})\cdot\partial_\sigma(\psi\boldsymbol{\iota})\Big)\mid\boldsymbol{\xi}\mid^2+4\mu k_2\psi\frac{3\ \lambda+2\mu}{\lambda+2\mu}\mid\boldsymbol{\xi}\mid^2$$
$$+\Big[\frac{\lambda\mu}{\lambda+2\mu}k_1^2\partial_\sigma\psi\partial_\sigma\psi+\frac{4\lambda\mu}{\lambda+2\mu}k_2\partial_\sigma\psi x_\sigma-4\mu k_2\sum_{\alpha,\beta}\mid\partial_\alpha\psi x_\beta\mid\Big]\mid\boldsymbol{\xi}\mid^2.$$

Letting $\mid\nabla\psi\mid=\sqrt{\partial_\sigma\psi\partial_\sigma\psi}$ and $m=\min\Big(\frac{k_1^2}{2},k_2^2\Big)$, we have, again by Schwarz's inequality,

$$\frac{2\lambda\mu}{\lambda+2\mu}\bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\tau\cdot\boldsymbol{\xi}_\tau+2\mu\bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\alpha\cdot\boldsymbol{\xi}_\beta$$
$$\geq\frac{2\ \lambda\ \mu}{\lambda+2\mu}m(\partial_\sigma\psi\partial_\sigma\psi+\partial_\sigma(\psi\boldsymbol{\iota})\cdot\partial_\sigma(\psi\ \boldsymbol{\iota}))\mid\boldsymbol{\xi}\mid^2+4\mu k_2\psi\frac{3\lambda+2\ \mu}{\lambda+2\mu}\mid\boldsymbol{\xi}\mid^2$$
$$+\Big[\frac{\lambda\mu}{\lambda+2\mu}k_1^2\mid\nabla\psi\mid^2-4\mu\frac{3\lambda+4\mu}{\lambda+2\mu}k_2R\mid\nabla\psi\mid\Big]\mid\boldsymbol{\xi}\mid^2.$$

From the properties of ψ, we obtain

$$\frac{2\lambda\mu}{\lambda+2\mu}\bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\tau\cdot\boldsymbol{\xi}_\tau+2\mu\bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\alpha\cdot\boldsymbol{\xi}_\beta$$
$$\geq\frac{2\lambda\mu}{\lambda+2\mu}m\delta(\omega)\mid\boldsymbol{\xi}\mid^2+4\mu k_2\psi\frac{3\ \lambda\ +4\ \mu}{\lambda+2\mu}\mid\boldsymbol{\xi}\mid^2$$
$$+\Big[\frac{\lambda\mu}{\lambda+2\mu}k_1^2\mid\nabla\psi\mid^2-4\mu\frac{3\lambda+4\mu}{\lambda+2\mu}k_2R\mid\nabla\psi\mid\Big]\mid\boldsymbol{\xi}\mid^2.\tag{3.11}$$

(i) But, since we are dealing with polynomials of degree two, we have

$$\frac{\lambda\ \mu}{\lambda+2\mu}k_1^2\mid\nabla\psi\mid^2-4\mu\frac{3\lambda+4\mu}{\lambda+2\mu}k_2R\mid\nabla\psi\mid\geq-4R^2\frac{\mu(3\lambda+4\mu)^2}{\lambda(\lambda+2\mu)}\frac{k_2^2}{k_1^2}.\tag{3.12}$$

Hence, in $\omega-\omega_1$, since $\psi\geq\delta(\omega)$, we obtain from (3.11) and (3.12)

$$\frac{2\lambda\mu}{\lambda+2\mu}\bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\tau\cdot\boldsymbol{\xi}_\tau+2\mu\bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\alpha\cdot\boldsymbol{\xi}_\beta$$
$$\geq\frac{2\lambda\mu}{\lambda+2\mu}m\delta(\omega)\mid\boldsymbol{\xi}\mid^2+k_2\Big[4\mu\delta(\omega)\frac{3\lambda+2\mu}{\lambda+2\mu}-\frac{4\mu R^2(3\lambda+4\mu)^2}{\lambda(\lambda+2\mu)}\frac{k_2}{k_1^2}\Big]\mid\boldsymbol{\xi}\mid^2.\tag{3.13}$$

Let

$$M_1 = \frac{\lambda\delta(\omega)(3\lambda+2\mu)}{R^2\ (3\lambda+4\mu)^2} > 0.$$

Then, for $0 < k_2 < M_1\ k_1^2$ and for each point of $\omega - \omega 1$ and each $\boldsymbol{\xi} \in \mathbb{R}^2$, the following inequality holds

$$\frac{2\lambda\mu}{\lambda+2\mu}\bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\tau \cdot \boldsymbol{\xi}_\tau + 2\mu\bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\alpha \cdot \boldsymbol{\xi}_\beta \geq \frac{2\ \lambda\ \mu}{\lambda+2\mu} m\delta(\omega) \mid \boldsymbol{\xi} \mid^2 . \tag{3.14}$$

(ii) We know that $\partial_\sigma\psi\partial_\sigma\psi \geq \delta(\omega)$ in $\bar{\omega}_1$. Let

$$M_2 = \frac{\lambda\sqrt{\delta(\omega)}}{4R(3\lambda+4\mu)} > 0.$$

Then, for $0 < k_2 < M_2\ k_1^2$, we have in $\bar{\omega}_1$,

$$\frac{\lambda\ \mu}{\lambda+2\mu}k_1^2 \mid \nabla\psi \mid^2 - 4\mu\frac{3\lambda+4\mu}{\lambda+2\mu}k_2 R \mid \nabla\psi \mid \geq 0.$$

Then, from (3.11) we deduce that for $0 < k_2 \leq M\ k_1^2$, and each point of $\bar{\omega}_1$ and each $\boldsymbol{\xi} \in \mathbb{R}^2$ the following inequality holds:

$$\frac{2\lambda\mu}{\lambda+2\mu}\bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\tau \cdot \boldsymbol{\xi}_\tau + 2\mu\bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\alpha \cdot \boldsymbol{\xi}_\beta \geq \frac{2\lambda\mu}{\lambda+2\mu} m\delta(\omega) \mid \boldsymbol{\xi} \mid^2 . \tag{3.15}$$

(iii) Let $M = \min(M_1, M_2) > 0$. Then, for $k_1 > 0$ and $0 <\mid k_2 \mid \leq M k_1^2$, we have for each point of $\bar{\omega}$ and each $\boldsymbol{\xi} \in \mathbb{R}^2$ the following inequality

$$\frac{2\lambda\mu}{\lambda+2\mu}\bar{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\tau \cdot \boldsymbol{\xi}_\tau + 2\mu\bar{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\boldsymbol{\xi}_\alpha \cdot \boldsymbol{\xi}_\beta \geq \frac{2\lambda\mu}{\lambda+2\mu} m\delta(\omega) \mid \boldsymbol{\xi} \mid^2, \tag{3.16}$$

with

$$m = \min\Big(\frac{k_1^2}{2}, k_2^2\Big) > 0. \tag{3.17}$$

This is precisely condition (iii) of Definition 3.1, with $C_2 = \frac{2\lambda\mu}{\lambda+2\mu} m\delta(\omega)$.

Since we have seen that $\bar{\boldsymbol{\varphi}}$ satisfies all the conditions of Definition 3.1, we conclude that for $k_1 > 0$ and $0 < k_2 \leq M k_1^2$, $\bar{\boldsymbol{\varphi}}$ is an element of $E(\omega)$.

Now, we can state the following theorem:

Theorem 3.1. *Assume that the boundary γ is of class $\mathcal{C}^2$. Then the set $E(\omega)$ is not empty. Moreover, for each $p > 2$ and $\varepsilon > 0$, there exists $\bar{\boldsymbol{\varphi}} \in E(\omega)$ such that*

$$\| \bar{\boldsymbol{\varphi}} - \boldsymbol{\iota} \|_{2,p,\omega} < \varepsilon \quad \text{and} \quad \| \boldsymbol{f}_{\bar{\boldsymbol{\varphi}}} \|_{0,p,\omega} < \varepsilon.$$

Furthermore, for each $A > 0$, there exists $\bar{\boldsymbol{\varphi}} \in E(\omega)$ such that

$$\| \bar{\boldsymbol{\varphi}} - \boldsymbol{\iota} \|_{2,p,\omega} > A \quad \text{and} \quad \| \boldsymbol{f}_{\bar{\boldsymbol{\varphi}}} \|_{0,p,\omega} > A.$$

Proof. The first part of the proposition is a straightforward consequence of Lemma 3.2.

The second part of the proposition follows from the expression of the particular element of $E(\omega)$ given in Lemma 3.2.

Let $\varepsilon > 0$ be given. Then, it is easily seen that there exists $\eta > 0$ such that, if $0 < k_1 < \eta$ and $0 < k_2 < M\ k_1^2$, the corresponding $\bar{\boldsymbol{\varphi}}$ satisfies $\| \bar{\boldsymbol{\varphi}} \|_{2,p,\omega} < \varepsilon$ and $\| \boldsymbol{f}_{\bar{\boldsymbol{\varphi}}} \|_{0,p,\omega} < \varepsilon$.

Let $A > 0$ be given. Then, it is likewise easily seen that there exists $\eta > 0$ such that, if $1 < \eta < k_1$ and $0 < k_2 < M\ k_1 \leq M\ k_1^2$, the corresponding $\bar{\boldsymbol{\varphi}}$ satisfies $\| \bar{\boldsymbol{\varphi}} \|_{2,p,\omega} > A$ and $\|\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}}\|_{0,p,\omega} > A$.

We next show that in a neighborhood of the force corresponding to any extended state, the inverse function theorem provides a solution to the nonlinear clamped membrane problem, which furthermore is the unique minimizer to the nonlinear membrane functional. As a consequence of this result and of the second part of Theorem 3.1 we can assert that the nonlinear clamped membrane problem—whose functional is not sequentially weakly lower semi-continuous—admits a unique minimizer for forces situated in balls of $\mathbf{L}^p(\omega;\mathbb{R}^3)$ arbitrarily close to the origin as desired or as far from $\mathbf{0}$ also.

§4. Existence Results Around the Force Corresponding to any Extended State

Theorem 4.1. *Assume that the boundary γ is of class $\mathcal{C}^2$. Let $\bar{\boldsymbol{\varphi}}$ be any element of $E(\omega)$ and let $C_1>0,\ C_2>0$ be such that Definition 3.1 is satisfied. Let $p>2$ be given and let $\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}}$ be the element of $\mathbf{L}^p(\omega;\mathbb{R}^3)$ defined by*

$$\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}}=-\partial_\alpha\Big\{\Big(\frac{2\lambda\,\mu}{\lambda+2\mu}\,\delta_{\alpha\beta}\,\tilde{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})+2\mu\tilde{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\Big)\partial_\beta\bar{\boldsymbol{\varphi}}\Big\}.$$

Then there exist a neighborhood $\mathbf{F}^p$ of $\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}}$ in $\mathbf{L}^p(\omega;\mathbb{R}^3)$ and a neighborhood $\mathbf{U}^p$ of the origin in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ such that for all $\boldsymbol{f}\in\mathbf{F}^p$, there is a unique $\boldsymbol{u}\in\mathbf{U}^p$ such that $\boldsymbol{\varphi}(\boldsymbol{u})=\bar{\boldsymbol{\varphi}}+\boldsymbol{u}$ satisfies the nonlinear membrane problem. Furthermore, the mapping implicitly defined in this fashion is a $\mathcal{C}^\infty$-diffeomorphism between $\{\bar{\boldsymbol{\varphi}}+\mathbf{U}^p\}$ and $\{\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}}+\mathbf{F}^p\}$.

Proof. For the sake of clarity, the proof is divided into five steps.

Step 1. Consider the nonlinear operator $\mathbf{T}_{\bar{\boldsymbol{\varphi}}}$ from the space $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ into $\mathbf{L}^p(\omega;\mathbb{R}^3)$ defined by

$$\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\boldsymbol{u})=-\partial_\alpha\Big\{\Big(\frac{2\lambda\mu}{\lambda+2\mu}\,\delta_{\alpha\beta}\,\tilde{a}_{\sigma\sigma}(\boldsymbol{\varphi}(\boldsymbol{u}))+2\mu\tilde{a}_{\alpha\beta}(\boldsymbol{\varphi}(\boldsymbol{u}))\Big)\partial_\beta(\boldsymbol{\varphi}(\boldsymbol{u})\Big\},\tag{4.1}$$

where $\boldsymbol{\varphi}(\boldsymbol{u})$ is the element of $\{\boldsymbol{\iota}+\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)\}$ defined by

$$\boldsymbol{\varphi}(\boldsymbol{u})=\bar{\boldsymbol{\varphi}}+\boldsymbol{u},\tag{4.2}$$

for all $\boldsymbol{u}\in\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$.

Let $\boldsymbol{f}\in\mathbf{L}^p(\omega;\mathbb{R}^3)$. By definition, $\boldsymbol{\varphi}\in\{\boldsymbol{\iota}+\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)\}$ is a solution to the nonlinear clamped membrane problem corresponding to $\boldsymbol{f}$ if and only if

$$\mathrm{T}_{\bar{\boldsymbol{\varphi}}}(\boldsymbol{u})=\boldsymbol{f},\tag{4.3}$$

where $\boldsymbol{u}=\bar{\boldsymbol{\varphi}}-\boldsymbol{\varphi}$.

Since $\mathrm{W}^{1,p}(\omega)$ is a Banach algebra ($p>2$), $\mathbf{T}_{\bar{\boldsymbol{\varphi}}}$ is of class $\mathcal{C}^\infty$ between $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ and $\mathbf{L}^p(\omega;\mathbb{R}^3)$ and admits for differential at the origin

$$\begin{aligned}\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})(\boldsymbol{u})=&-\partial_\alpha\Big\{\Big(\frac{2\lambda\mu}{\lambda+2\mu}\delta_{\alpha\beta}\tilde{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})+2\mu\tilde{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\Big)\partial_\beta\boldsymbol{u}\Big\}\\&-\partial_\alpha\Big\{\Big(\frac{4\lambda\mu}{\lambda+2\mu}\delta_{\alpha\beta}(\partial_\sigma\bar{\boldsymbol{\varphi}}\cdot\partial_\sigma\boldsymbol{u})+2\mu(\partial_\alpha\bar{\boldsymbol{\varphi}}\cdot\partial_\beta\boldsymbol{u}+\partial_\beta\bar{\boldsymbol{\varphi}}\cdot\partial_\alpha\boldsymbol{u})\Big)\partial_\beta\bar{\boldsymbol{\varphi}}\Big\}.\end{aligned}\tag{4.4}$$

Step 2. We next show that the linear operator $\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$ associated to boundary conditions of Dirichlet type satisfies the specific assumptions of [1].

Using the same notations as in [1], we have for $x \in \bar{\omega}$, $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $|\,\boldsymbol{\xi}\,|^2 = {\xi_1}^2 + {\xi_2}^2$,

$$l'_{ij}(\boldsymbol{\xi}, x) = \Big\{\frac{2\lambda\mu}{\lambda+2\mu}\tilde{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})(x)\xi_\tau\xi_\tau + 2\mu\tilde{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})(x)\xi_\alpha\ \xi_\beta\Big\}\delta_{ij}$$
$$+ \frac{4\lambda\mu}{\lambda+2\mu}\partial_\sigma\bar{\varphi}_i(x)\partial_\tau\bar{\varphi}_j(x)\xi_\sigma\xi_\tau + 4\mu\partial_\sigma\bar{\varphi}_i(x)\partial_\tau\bar{\varphi}_j(x)\xi_\sigma\xi_\tau, \tag{4.5}$$

where $\{i,j\} \in \{1,2,3\}^2$.

(i) First, we establish the strong ellipticity of the system in the sense that there exists $c > 0$ such that for all $x \in \bar{\omega}$,

$$\forall\boldsymbol{\xi} \in \mathbb{R}^2, \forall\boldsymbol{\eta} \in \mathcal{C}^3, \quad \Re(l'_{ij}(\boldsymbol{\xi}, x)\eta_i\bar{\eta_j}) \geq c|\,\boldsymbol{\xi}\,|^2(|\,\eta_1\,|^2 + |\,\eta_2\,|^2 + |\,\eta_3\,|^2).$$

Since the polynomials $l'_{ij}(\boldsymbol{\xi})$ are real, it suffices to show that there exists $c > 0$ such that for all $x \in \bar{\omega}$,

$$\forall\boldsymbol{\xi} \in \mathbb{R}^2, \forall\boldsymbol{\eta} \in \mathbb{R}^3, \quad l'_{ij}(\boldsymbol{\xi}, x)\eta_i\eta_j \geq c\,|\,\boldsymbol{\xi}\,|^2({\eta_1}^2 + {\eta_2}^2 + {\eta_3}^2).$$

Let $x \in \bar{\omega}$, $\boldsymbol{\xi} \in \mathbb{R}^2$, and $\boldsymbol{\eta} \in \mathbb{R}^3$. By definition, we have

$$l'_{ij}(\boldsymbol{\xi}, x)\eta_i\eta_j = \Big[\frac{2\lambda\mu}{\lambda+2\mu}\tilde{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})(x)\xi_\tau\xi_\tau + 2\mu\tilde{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})(x)\ \xi_\alpha\ \xi_\beta\Big]\eta_i\ \eta_i$$
$$+ \frac{4\lambda\mu}{\lambda+2\mu}(\xi_\sigma\partial_\sigma\bar{\varphi}(x)\cdot\boldsymbol{\eta})(\xi_\tau\ \partial_\tau\bar{\varphi}(x)\cdot\boldsymbol{\eta}) + 4\mu(\xi_\sigma\partial_\sigma\bar{\varphi}(x)\cdot\boldsymbol{\eta})(\xi_\tau\partial_\tau\bar{\varphi}_(x)\cdot\boldsymbol{\eta}).$$

Thus

$$l'_{ij}(\boldsymbol{\xi}, x)\eta_i\eta_j \geq \Big[\frac{2\lambda\mu}{\lambda+2\mu}\tilde{a}_{\sigma\sigma}(\bar{\boldsymbol{\varphi}})(x)\xi_\tau\xi_\tau + 2\mu\tilde{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})(x)\xi_\alpha\ \xi_\beta\Big]|\,\boldsymbol{\eta}\,|^2. \tag{4.6}$$

Since $\bar{\boldsymbol{\varphi}} \in E(\omega)$, condition (iii) of Definition 3.1 and (4.6) imply the strong ellipticity of the system expressed as $l'_{ij}(\boldsymbol{\xi}, x)\eta_i\eta_j \geq C_2|\,\boldsymbol{\xi}\,|^2|\,\boldsymbol{\eta}\,|^2$.

(ii) We next establish that this system is uniformly elliptic in the sense that there exists $c \geq 1$ such that for all $x \in \bar{\omega}$, $\forall\boldsymbol{\xi} \in \mathbb{R}^2$, $c^{-1}|\,\boldsymbol{\xi}\,|^6 \leq L(\boldsymbol{\xi}, x) \leq c|\,\boldsymbol{\xi}\,|^6$, where $L(\boldsymbol{\xi}, x) = \det(l'_{ij}(\boldsymbol{\xi}, x))$.

Let $x \in \bar{\omega}$ and $\boldsymbol{\xi} \in \mathbb{R}^2$. From (i), we deduce that the symmetric matrix $[l'_{ij}(\boldsymbol{\xi}, x)]_{1\leq i,j\leq 3}$ is positive definite, and that its first eigenvalue is $\geq C_2|\,\boldsymbol{\xi}\,|^2$. Hence, its determinant satisfies $L(\boldsymbol{\xi}, x) \geq C_2^3|\,\boldsymbol{\xi}\,|^6$.

On the other hand, each mapping $\boldsymbol{\xi} \in \mathbb{R}^2 \to l'_{ij}(\boldsymbol{\xi}, x) \in \mathbb{R}$ is a quadratic polynomial with coefficients in $\mathcal{C}^0(\bar{\omega})$. This establishes the existence of a constant $C' > 0$ such that, for all $x \in \bar{\omega}$ and all $\boldsymbol{\xi} \in \mathbb{R}^2$, the determinant satisfies $L(\boldsymbol{\xi}, x) \leq C'|\,\boldsymbol{\xi}\,|^6$.

We have therefore established that the system is uniformly elliptic.

(iii) We also have to verify that the system satisfies the supplementary condition, namely that for all $x \in \bar{\omega}$ and for each pair of linearly independent vectors $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ of $\mathbb{R}^2$, the polynomial $\tau \in \mathbb{C} \to L(\boldsymbol{\xi} + \tau\boldsymbol{\xi}', x) \in \mathbb{C}$ has exactly $m = 3$ roots with positive imaginary part, where $2m = 6$ denotes the degree of the polynomial $\xi \in \mathbb{R}^2 \to L(\xi, x) \in \mathbb{R}$. Let $x \in \bar{\omega}$ and let $(\boldsymbol{\xi}, \boldsymbol{\xi}')$ be a given pair of linearly independent vectors of $\mathbb{R}^2$.

Since each application $\tau \in \mathbb{C} \to l'_{ij}(\boldsymbol{\xi} + \tau\boldsymbol{\xi}', x) \in \mathbb{C}$ is a polynomial of degree $\leq$2, the application $\tau \in \mathbb{C} \to L(\boldsymbol{\xi} + \tau\boldsymbol{\xi}', x) \in \mathbb{C}$ is a polynomial of degree $\leq$6. But we have seen in (ii) that $L(\boldsymbol{\xi} + \tau\boldsymbol{\xi}', x) \geq C_2^3|\,\boldsymbol{\xi} + \tau\boldsymbol{\xi}'\,|^6$. Since $\boldsymbol{\xi}'$ is not null, we deduce that the degree of the polynomial $\tau \in \mathbb{C} \to L(\boldsymbol{\xi} + \tau\boldsymbol{\xi}', x) \in \mathbb{C}$ is exactly 6.

Moreover, since $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are linearly independent, the preceding inequality shows that for $\tau \in \mathbb{R}$, $L(\boldsymbol{\xi}+\tau\boldsymbol{\xi}', x) > 0$. Then, the polynomial of degree 6 defined by

$$\tau \in \mathbb{C} \to L(\boldsymbol{\xi}+\tau\boldsymbol{\xi}', x) \in \mathbb{C}$$

has no real root. Since this polynomial has real coefficients, it admits three couples of complex conjugate roots and has exactly three roots with positive imaginary part.

(iv) Finally, the complementary boundary condition also holds, since we know from [1] that this is the case for any system verifying the strong ellipticity property and associated to a boundary condition of Dirichlet type.

Step 3. From the strong ellipticity of the system established in Step 2, from the $\mathbf{H}_0^1(\omega;\mathbb{R}^3)$-ellipticity of the corresponding bilinear form (which is a consequence of the strong ellipticity of the system for the boundary condition considered here), and from the fact that the coefficients $\tilde{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})$ belong to $\mathcal{C}^1(\bar{\omega})$, we deduce from a result of Nečas[15] that $\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$, considered as an operator from $\mathbf{H}^2(\omega;\mathbb{R}^3)\cap\mathbf{H}_0^1(\omega;\mathbb{R}^3)$ into $\mathbf{L}^2(\omega;\mathbb{R}^3)$, defines an isomorphism between those spaces.

Step 4. The results of Step 2 and the fact that the coefficients $\tilde{a}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})$ belong to $\mathcal{C}^1(\bar{\omega})$ furthermore allow us to use a result of Geymonat[12] to deduce that $\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$, seen as an operator from $\mathbf{V}^q(\omega;\mathbb{R}^3)$ into $\mathbf{L}^q(\omega;\mathbb{R}^3)$, where $\mathbf{V}^q(\omega;\mathbb{R}^3) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,q}(\omega;\mathbb{R}^3); \boldsymbol{\psi} = \mathbf{0} \text{ on } \gamma\}$, has an index independent of $q\in]1,+\infty[$.

From Step 3, we then know that this index is equal to 0.

From the imbedding of $\mathbf{W}^{2;p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1;4}(\omega;\mathbb{R}^3)$ into $\mathbf{H}^2(\omega;\mathbb{R}^3)\cap\mathbf{H}_0^1(\omega;\mathbb{R}^3)$ $(p>2)$, and from the injectivity of $\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$ on $\mathbf{H}^2(\omega;\mathbb{R}^3)\cap\mathbf{H}_0^1(\omega;\mathbb{R}^3)$, it follows that $\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$ is injective on the space $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$. The nullity of the index then shows the surjectivity of the linear operator $\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$ from the space $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ into $\mathbf{L}^p(\omega;\mathbb{R}^3)$.

From the open mapping theorem, we then deduce that $\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$ is an isomorphism between $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ and $\mathbf{L}^p(\omega;\mathbb{R}^3)$.

Step 5. From the results of Steps 1 and 4 and the relation $\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(0) = \boldsymbol{f}_{\bar{\boldsymbol{\varphi}}}$, we deduce by the inverse function theorem the existence of a neighborhood $\mathbf{F}^p$ of $\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}}$ in $\mathbf{L}^p(\omega;\mathbb{R}^3)$ and of a neighborhood $\mathbf{U}^p$ of the origin in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ such that $\mathbf{T}_{\bar{\boldsymbol{\varphi}}}$ defines a $\mathcal{C}^\infty$-diffeomorphism between $\mathbf{U}^p$ and $\mathbf{F}^p$.

We can also establish the injectivity of the deformation.

Theorem 4.2. *The same assumptions as in Theorem* 4.1 *concerning* $\bar{\boldsymbol{\varphi}}$, γ *and* p *are made. There exists a neighborhood* $\tilde{\mathbf{F}}^p$ *of* $\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}}$ *in* $\mathbf{L}^p(\omega;\mathbb{R}^3)$ *contained in* $\mathbf{F}^p$, *such that the unique solution in* $\left\{\bar{\boldsymbol{\varphi}}+\mathbf{U}^p\right\}$ *to the nonlinear membrane problem associated by Theorem* 4.1 *to any element of* $\tilde{\mathbf{F}}^p$ *is injective in* $\bar{\omega}$.

Proof. The proof given hereafter starts from some ideas of [2].

Let $d_\omega(x,y)$ denote the geodesic distance between any points x and y in $\bar{\omega}$, i.e. the infimum of the lengths of all continuous arcs contained in $\bar{\omega}$ and joining x and y. Since ω is an open bounded connected set, with a Lipschitzian boundary, we know that d and d_ω are equivalent on $\bar{\omega}$.

Let κ be a real such that $d_\omega(x,y)\le\kappa d(x,y)$ for all $(x,y)\in\bar{\omega}^2$.

Let x and y be given points in $\bar{\omega}$.

Let $(x_i)_{1\le i\le p}$ be points in $\bar{\omega}$ such that

$$\sum_{1\le i\le p-1} d(x_i, x_{i+1}) \le 2 d_\omega(x,y),$$
$$x_1 = x, \qquad [x_i, x_{i+1}] \subset \bar{\omega} \quad (1\le i\le p-1), \qquad x_p = y.$$

Then, denoting by $\boldsymbol{w}$ any element of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)$ and using Taylor's formula on each segment $[x_i, x_{i+1}]$, we obtain the inequality

$$| \boldsymbol{w}(x) - \boldsymbol{w}(y) | \le 2\ d_\omega(x,y) \ || \nabla \boldsymbol{w} ||_{0,\infty,\omega} .$$

Then, by Sobolev's inequalities and the definition of κ, we infer the existence of a constant $C > 0$ such that

$$| \boldsymbol{w}(x) - \boldsymbol{w}(y) | \le 2C\ \kappa d(x,y) \ || \boldsymbol{w} ||_{2,p,\omega} . \tag{4.7}$$

By Theorem 4.1, we infer the existence of a neighborhood $\tilde{\mathbf{F}}^p$ of $\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}}$ in $\mathbf{L}^p(\omega;\mathbb{R}^3)$ contained in $\mathbf{F}^p$ such that the solution associated to an element of $\tilde{\mathbf{F}}^p$ satisfies $|| \boldsymbol{u} ||_{2,p,\omega} < (2C\kappa)^{-1} C_1$. We deduce that for all $x \ne y$ in $\bar{\omega}$, the solution $\boldsymbol{\varphi}(\boldsymbol{u}) = \bar{\boldsymbol{\varphi}} + \boldsymbol{u}$ satisfies

$$| \boldsymbol{\varphi}(\boldsymbol{u})(x) - \boldsymbol{\varphi}(\boldsymbol{u})(y) - (\bar{\boldsymbol{\varphi}}(x) - \bar{\boldsymbol{\varphi}}(y)) | < C_1\ d(x,y).$$

The conclusion follows from this inequality, since from the second condition of Definition 3.1 we have $| \bar{\boldsymbol{\varphi}}(x) - \bar{\boldsymbol{\varphi}}(y) | \ge C_1 d(x,y)$.

Next, we establish that the solution given by Theorem 4.1 is also the unique minimizer of the functional associated to the nonlinear membrane problem.

Theorem 4.3. *With the same assumptions as in Theorem* 4.1*, there exists a neighborhood* $\check{\mathbf{F}}^p$ *of* $\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}}$ *in* $\mathbf{L}^p(\omega;\mathbb{R}^3)$ *contained in* $\mathbf{F}^p$ *such that the unique solution in* $\left\{\bar{\boldsymbol{\varphi}} + \mathbf{U}^p\right\}$ *to the nonlinear clamped membrane problem associated by Theorem* 4.1 *to any element* $\boldsymbol{f}$ *of* $\check{\mathbf{F}}^p$ *is the unique minimizer of the functional* $I_M(\boldsymbol{f})$ *over the affine space* $\left\{\boldsymbol{\iota} + \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)\right\}$.

Proof. Let $\boldsymbol{f}$ be an element of $\mathbf{F}^p$ and denote by $\boldsymbol{\varphi}_{\boldsymbol{f}}$ the element of $\left\{\bar{\boldsymbol{\varphi}} + \mathbf{U}^p\right\}$ satisfying the nonlinear membrane problem (Theorem 4.1).

Let $\boldsymbol{\varphi}$ be any element in the affine space $\left\{\boldsymbol{\iota} + \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)\right\}$ and let $\boldsymbol{v}$ be the element of the space $\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ defined by $\boldsymbol{v} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_{\boldsymbol{f}}$. Then a computation shows that

$$\begin{aligned} I_M(\boldsymbol{f})(\boldsymbol{\varphi}) = {} & I_M(\boldsymbol{f})(\boldsymbol{\varphi}_{\boldsymbol{f}}) + \mathrm{d}I_M(\boldsymbol{f})(\boldsymbol{\varphi}_{\boldsymbol{f}})(\boldsymbol{v}) \\ & + \frac{\lambda\ \mu}{\lambda + 2\mu} \int_\omega (\partial_\sigma \boldsymbol{v}\cdot\partial_\sigma \boldsymbol{v} + 2\partial_\sigma \boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\sigma \boldsymbol{v})(\partial_\tau \boldsymbol{v}\cdot\partial_\tau \boldsymbol{v} + 2\partial_\tau \boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\tau \boldsymbol{v}) d\omega \\ & + \mu \int_\omega (\partial_\alpha \boldsymbol{v}\cdot\partial_\beta \boldsymbol{v} + 2\partial_\alpha \boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\beta \boldsymbol{v})(\partial_\alpha \boldsymbol{v}\cdot\partial_\beta \boldsymbol{v} + 2\partial_\alpha \boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\beta \boldsymbol{v}) d\omega \\ & + \frac{2\lambda\mu}{\lambda + 2\mu} \int_\omega (\partial_\sigma \boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\sigma \boldsymbol{\varphi}_{\boldsymbol{f}} - \delta_{\sigma\sigma})\partial_\tau \boldsymbol{v}\cdot\partial_\tau \boldsymbol{v}\ d\omega \\ & + 2\mu \int_\omega (\partial_\alpha \boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\beta \boldsymbol{\varphi}_{\boldsymbol{f}} - \delta_{\alpha\beta})\partial_\alpha \boldsymbol{v}\cdot\partial_\beta \boldsymbol{v}\ d\omega. \end{aligned} \tag{4.8}$$

Since $\boldsymbol{\varphi}_{\boldsymbol{f}}$ is a critical point of the functional $I_M(\boldsymbol{f})$, it follows that

$$\begin{aligned} I_M(\boldsymbol{f})(\boldsymbol{\varphi}) \ge {} & I_M(\boldsymbol{f})(\boldsymbol{\varphi}_{\boldsymbol{f}}) + \frac{2\lambda\mu}{\lambda + 2\mu} \int_\omega (\partial_\sigma \boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\sigma \boldsymbol{\varphi}_{\boldsymbol{f}} - \delta_{\sigma\sigma})\partial_\tau \boldsymbol{v}\cdot\partial_\tau \boldsymbol{v} d\omega \\ & + 2\mu \int_\omega (\partial_\alpha \boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\beta \boldsymbol{\varphi}_{\boldsymbol{f}} - \delta_{\alpha\beta})\partial_\alpha \boldsymbol{v}\cdot\partial_\beta \boldsymbol{v}\ d\omega. \end{aligned} \tag{4.9}$$

From Theorem 4.1 and the continuity of the imbedding of $\mathrm{W}^{2,p}(\omega)$ in $\mathcal{C}^1(\bar{\omega})$, we deduce the existence of a neighborhood $\check{\mathbf{F}}^p$ contained in $\mathbf{F}^p$ such that the solution $\boldsymbol{\varphi}_{\boldsymbol{f}} \in \{\bar{\boldsymbol{\varphi}} + \mathbf{U}^p\}$ associated to $\boldsymbol{f} \in \check{\mathbf{F}}^p$ satisfies the following inequality for all $\boldsymbol{w} \in \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$:

$$C'||\boldsymbol{w}||^2_{1,2,\omega} \leq \frac{2\lambda\mu}{\lambda+2\mu}\int_\omega (\partial_\sigma\boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\sigma\boldsymbol{\varphi}_{\boldsymbol{f}} - \delta_{\sigma\sigma})\partial_\tau\boldsymbol{w}\cdot\partial_\tau\boldsymbol{w} d\omega + 2\mu\int_\omega (\partial_\alpha\boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\beta\boldsymbol{\varphi}_{\boldsymbol{f}} - \delta_{\alpha\beta})\partial_\alpha\boldsymbol{w}\cdot\partial_\beta\boldsymbol{w}\, d\omega, \tag{4.10}$$

where the constant C' satisfies $0 < C' \leq C_2$. Hence, it follows from (4.9) and (4.10) that

$$I_M(\boldsymbol{f})(\boldsymbol{\varphi}) \geq I_M(\boldsymbol{f})(\boldsymbol{\varphi}_{\boldsymbol{f}}) + C'||\boldsymbol{v}||^2_{1,2,\omega}.$$

This last inequality gives the announced result.

Remark. The neighborhood $\check{\mathbf{F}}^p$ is in fact defined in such a way that the associated deformation satisfies the first and third conditions of Definition 3.1.

In the next section we prove that a "well extended " clamped membrane plate can undergo large loadings by giving an asymptotic estimate for the neighborhood $\mathbf{F}^p$ of Theorem 4.1 as some extended states go to infinity in a certain fashion.

§5. Behaviour of the Membrane as some Extended States go to Infinity

In this section ψ denotes a given mapping satisfying the conditions of Lemma 3.1.

From Lemma 3.2, we have the existence of a constant $\bar{k}_0 > 0$ such that $\bar{\boldsymbol{\varphi}}_k \in \mathcal{C}^2(\bar{\omega};\mathbb{R}^3)$ defined by

$$\bar{\boldsymbol{\varphi}}_k(x_1,x_2) = \boldsymbol{\iota}(x) + k\psi(x)\ (x_1,x_2,1) \text{ for all } x = (x_1,x_2)\in\bar{\omega}, \tag{5.1}$$

is an element of $E(\omega)$ for any $k \geq \bar{k}_0$.

In the following, we give an asymptotic estimate for the neighborhood $\mathbf{F}^p$ associated to $\bar{\boldsymbol{\varphi}}_k \in E(\omega)$ by Theorem 4.1 as the parameter k tends to infinity.

We first need two preliminary results. In the following, we denote by $0 < \kappa < 1$ a given real and we denote by $\boldsymbol{\eta}^0$ the element of $\mathcal{C}^2(\bar{\omega};\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ defined by

$$\boldsymbol{\eta}^0(x) = \psi(x)(x_1,x_2,1) \text{ for all } x = (x_1,x_2)\in\bar{\omega}. \tag{5.2}$$

We then have $\bar{\boldsymbol{\varphi}}_k = \boldsymbol{\iota} + k\boldsymbol{\eta}^0$ for $k =\geq \bar{k}_0$.

Lemma 5.1. *Assume that the boundary of* ω *is of class* $\mathcal{C}^2$ *and that* $p > 2$ *is given. There exist* $\delta_0 > 0$ *and* $k_0 > 0$ *such that, if* $0 < \delta < \delta_0$ *and* $k > k_0$*, then for any* $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ *satisfying*

$$||\,\boldsymbol{u}\,||_{2,p,\omega} < \delta k, \quad ||\,\boldsymbol{v}\,||_{2,p,\omega} < \delta\ k\ ,$$

$$\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\mathbf{0})(\boldsymbol{w}-\boldsymbol{v}) = \mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\mathbf{0})(\boldsymbol{v}-\boldsymbol{u}) - (\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\boldsymbol{v}) - \mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\boldsymbol{u})),$$

we have the following inequality: $||\,\boldsymbol{w}-\boldsymbol{v}\,||_{2,p,\omega} \leq \kappa\,||\,\boldsymbol{u}-\boldsymbol{v}\,||_{2,p,\omega}$.

Proof. (i) Let $k > \bar{k}_0$, $\delta > 0$ be given. Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be given elements of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ satisfying

$$||\,\boldsymbol{u}\,||_{2,p,\omega} < \delta k, \quad ||\,\boldsymbol{v}\,||_{2,p,\omega} < \delta k, \tag{5.3}$$

$$\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\mathbf{0})(\boldsymbol{w}-\boldsymbol{v}) = \mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\mathbf{0})(\boldsymbol{v}-\boldsymbol{u}) - (\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\boldsymbol{v}) - \mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\boldsymbol{u})). \tag{5.4}$$

We remark that (5.4) also reads

$$k^2\mathbf{L}_2(\boldsymbol{w}-\boldsymbol{v})+k\mathbf{L}_1(\boldsymbol{w}-\boldsymbol{v})+\mathbf{L}_0(\boldsymbol{w}-\boldsymbol{v})=k(\mathbf{A}(\boldsymbol{v})-\mathbf{A}(\boldsymbol{u}))+(\mathbf{B}(\boldsymbol{v})-\mathbf{B}(\boldsymbol{u})), \tag{5.5}$$

where $\mathbf{L}_0$, $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{A}$ and $\mathbf{B}$ are the operators defined from the space $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ into $\mathbf{L}^p(\omega;\mathbb{R}^3)$ by

$$\mathbf{L}_0(\boldsymbol{\eta})=-\partial_\alpha\Big\{\Big[\frac{4\lambda\mu}{\lambda+2\mu}\delta_{\alpha\beta}\partial_\sigma\eta_\sigma+2\mu(\partial_\alpha\eta_\beta+\partial_\beta\eta_\alpha)\Big]\partial_\beta\,\boldsymbol{\iota}\Big\}, \tag{5.6}$$

$$\begin{aligned}\mathbf{L}_1(\boldsymbol{\eta})=&-\partial_\alpha\Big\{\Big[\frac{4\lambda\ \mu}{\lambda+2\mu}\delta_{\alpha\beta}\ \partial_\sigma\eta^0_\sigma+2\mu(\partial_\alpha\eta^0_\beta+\partial_\beta\eta^0_\alpha)\Big]\partial_\beta\boldsymbol{\eta}\Big\}\\&-\partial_\alpha\Big\{\Big[\frac{4\lambda\ \mu}{\lambda+2\mu}\delta_{\alpha\beta}\partial_\sigma\boldsymbol{\eta}^0\cdot\partial_\sigma\boldsymbol{\eta}+2\mu(\partial_\alpha\boldsymbol{\eta}^0\cdot\partial_\beta\eta+\partial_\beta\boldsymbol{\eta}^0\cdot\partial_\alpha\eta)\Big]\partial_\beta\boldsymbol{\iota}\Big\}\\&-\partial_\alpha\Big\{\Big[\frac{4\lambda\ \mu}{\lambda+2\mu}\delta_{\alpha\beta}\partial_\sigma\eta_\sigma+2\mu(\partial_\alpha\eta_\beta+\partial_\beta\eta_\alpha)\Big]\partial_\beta\boldsymbol{\eta}^0\},\end{aligned} \tag{5.7}$$

$$\begin{aligned}\mathbf{L}_2(\boldsymbol{\eta})=&-\partial_\alpha\Big\{\Big[\frac{2\lambda\ \mu}{\lambda+2\mu}\delta_{\alpha\beta}\ \partial_\sigma\boldsymbol{\eta}^0\cdot\partial_\sigma\boldsymbol{\eta}^0+2\mu\partial_\alpha\boldsymbol{\eta}^0\cdot\partial_\beta\boldsymbol{\eta}^0\Big]\partial_\beta\boldsymbol{\eta}\Big\}\\&-\partial_\alpha\Big\{\Big[\frac{4\lambda\ \mu}{\lambda+2\mu}\delta_{\alpha\beta}\ \partial_\sigma\boldsymbol{\eta}^0\cdot\partial_\sigma\boldsymbol{\eta}+2\mu(\partial_\alpha\boldsymbol{\eta}^0\cdot\partial_\beta\eta+\partial_\beta\boldsymbol{\eta}^0\cdot\partial_\alpha\eta)\Big]\partial_\beta\boldsymbol{\eta}^0\Big\},\end{aligned} \tag{5.8}$$

$$\begin{aligned}\mathbf{A}(\boldsymbol{\eta})=&-\partial_\alpha\Big\{\Big[\frac{4\lambda\ \mu}{\lambda+2\mu}\delta_{\alpha\beta}\ \partial_\sigma\boldsymbol{\eta}\cdot\partial_\sigma\boldsymbol{\eta}+2\mu\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\eta}\Big]\partial_\beta\boldsymbol{\eta}^0\Big\}\\&-\partial_\alpha\Big\{\Big[\frac{4\lambda\ \mu}{\lambda+2\mu}\delta_{\alpha\beta}\partial_\sigma\boldsymbol{\eta}^0\cdot\partial_\sigma\boldsymbol{\eta}+2\mu(\partial_\alpha\boldsymbol{\eta}^0\cdot\partial_\beta\boldsymbol{\eta}+\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\eta}^0)\Big]\partial_\beta\boldsymbol{\eta}\Big\},\end{aligned} \tag{5.9}$$

$$\begin{aligned}\mathbf{B}(\boldsymbol{\eta})=&-\partial_\alpha\Big\{\Big[\frac{2\lambda\ \mu}{\lambda+2\mu}\delta_{\alpha\beta}\partial_\sigma\boldsymbol{\eta}\cdot\partial_\sigma\boldsymbol{\eta}+2\mu\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\eta}\Big]\partial_\beta\boldsymbol{\eta}\\&-\partial_\alpha\Big\{\Big[\frac{4\lambda\ \mu}{\lambda+2\mu}\delta_{\alpha\beta}\ \partial_\sigma\eta_\sigma+2\mu(\partial_\alpha\eta_\beta+\partial_\beta\eta_\alpha)\Big]\partial_\beta\boldsymbol{\eta}\Big\},\end{aligned} \tag{5.10}$$

for all $\eta\in\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$.

(ii) Since $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)$ is a Banach algebra ($p>2$), we infer that the operators $\mathbf{A}$ and $\mathbf{B}$ are of class $\mathcal{C}^\infty$.

It can be seen that there exists a constant $C_1>0$ such that for each $R>0$ and any $\boldsymbol{\eta}\in B_{2,p}(\mathbf{0},R)$ the following inequalities hold:

$$\begin{aligned}&\|\ \mathrm{d}\mathbf{A}(\boldsymbol{\eta})(\boldsymbol{w})\ \|_{0,p,\omega}\le C_1\ R\ \ \|\ \boldsymbol{w}\ \|_{2,p,\omega},\\&\|\ \mathrm{d}\mathbf{B}(\boldsymbol{\eta})(\boldsymbol{w})\ \|_{0,p,\omega}\le C_1\ (R+R^=2)\ \|\ \boldsymbol{w}\ \|_{2,p,\omega}\end{aligned}$$

for all $\boldsymbol{w}\in\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$.

Since $\mathbf{L}^p(\omega;\mathbb{R}^3)$ is a reflexive Banach space ($1<2<p<+\infty$), it has the Radon-Nikodým property with respect to the Bochner integral (see [9] for instance). Let $R>0$ be given and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be any elements of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ situated in $B_{2,p}(\mathbf{0},R)$. From the Radon-Nikodým property we have

$$\mathbf{A}(\boldsymbol{v})-\mathbf{A}(\boldsymbol{u})=\int_0^1\mathrm{d}\mathbf{A}(\boldsymbol{u}+t(\boldsymbol{v}-\boldsymbol{u}))(\boldsymbol{v}-\boldsymbol{u})\ dt.$$

From the usual properties of the Bochner integral, we deduce

$$\|\ \mathbf{A}(\boldsymbol{v})-\mathbf{A}(\boldsymbol{u})\ \|_{0,p,\omega}\le\int_0^1\|\ \mathrm{d}\mathbf{A}(\boldsymbol{u}+t(\boldsymbol{v}-\boldsymbol{u}))(\boldsymbol{v}-\boldsymbol{u})\ \|_{0,p,\omega}dt,$$

and then that

$$|| \mathbf{A}(\boldsymbol{v}) - \mathbf{A}(\boldsymbol{u}) ||_{0,p,\omega} \leq C_1 R \, || \boldsymbol{v} - \boldsymbol{u} ||_{2,p,\omega} .$$

In the same way, we have

$$|| \mathbf{B}(\boldsymbol{v}) - \mathbf{B}(\boldsymbol{u}) ||_{0,p,\omega} \leq C_1 (R + R^2) \, || \boldsymbol{v} - \boldsymbol{u} ||_{2,p,\omega} .$$

It can be proved that like $\mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$ in Theorem 4.1, the linear operator $\mathbf{L}_2$ defines an isomorphism between $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ and $\mathbf{L}^p(\omega;\mathbb{R}^3)$. For the sake of conciseness, we omit the proof; we only mention that the third condition on ψ in Lemma 3.1 is used in a fundamental way to establish the strong ellipticity of the system. Hence, let $C > 0$ be such that

$$|| \mathbf{L}_2^{-1}(\boldsymbol{f}) ||_{2,p,\omega} \leq C \, || \boldsymbol{f} ||_{0,p,\omega} \quad \text{for all } \boldsymbol{f} \in \mathbf{L}^p(\omega;\mathbb{R}^3),$$

and let $c_1 > 0$ be such that, for any $\boldsymbol{\eta} \in \mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$,

$$|| \mathbf{L}_0(\boldsymbol{\eta}) ||_{0,p,\omega} \leq c_1 \, || \boldsymbol{\eta} ||_{2,p,\omega} \quad \text{and} \quad || \mathbf{L}_1(\boldsymbol{\eta}) ||_{0,p,\omega} \leq c_1 \, || \boldsymbol{\eta} ||_{2,p,\omega} .$$

(iii) Since (5.5) can be also written as

$$k^2(\boldsymbol{w} - \boldsymbol{v}) = \mathbf{L}_2^{-1}\Big((k \, \mathbf{L}_1 + \mathbf{L}_0)(\boldsymbol{v} - \boldsymbol{w}) + k \, (\mathbf{A}(\boldsymbol{v}) - \mathbf{A}(\boldsymbol{u})) \; + \mathbf{B}(\boldsymbol{v}) - \mathbf{B}(\boldsymbol{u}) \Big),$$

we infer that

$$(k^2 - (k+1)Cc_1) \, || \boldsymbol{w} - \boldsymbol{v} ||_{2,p,\omega} \leq C \, \big(k \, || \mathbf{A}(\boldsymbol{v}) - \mathbf{A}(\boldsymbol{u}) ||_{2,p,\omega} + || \mathbf{B}(\boldsymbol{v}) - \mathbf{B}(\boldsymbol{u}) ||_{2,p,\omega} \big).$$

Hence

$$(k^2 - (k+1)Cc_1) \, || \boldsymbol{w} - \boldsymbol{v} ||_{2,p,\omega} \leq CC_1(\delta(k + k^2) + \delta^2 k^2) \, || \boldsymbol{v} - \boldsymbol{u} ||_{2,p,\omega} . \tag{5.11}$$

Let $\delta_0 > 0$ be given such that $\kappa - CC_1(\delta_0^2 + 2\delta_0) > 0$, and let

$$k_0 = \max\Big(\frac{2\kappa C c_1}{\kappa - CC_1(\delta_0^2 + 2\delta_0)}, 1 + 2Cc_1, k_0 \Big) > 0.$$

Then, for $0 < \delta < \delta_0$ and $k > k_0$, we have $\frac{C \; C_1(\delta(k+k^2)+\delta^2 k^2)}{k^2-(k+1)C \; c_1} < \kappa$.

Hence (5.11) shows that for $0 < \delta < \delta_0$ and $k > k_0$,

$$|| \boldsymbol{w} - \boldsymbol{v} ||_{2,p,\omega} \leq \kappa \, || \boldsymbol{u} - \boldsymbol{v} ||_{2,p,\omega} .$$

Thanks to Lemma 5.1, we can establish as in [8, Lemma 4.2] the following result:

Lemma 5.2. *With the same assumptions on γ and p as in Lemma* 5.1*, let C, c_1, k_0 and δ_0 be as in Lemma* 5.1.. *Let*

$$M = \frac{1-\kappa}{C}\Big(1 - \frac{C \; c_1 \; (k_0 + 1)}{k_0^2} \Big) > 0.$$

Then, for any $k \geq k_0$, if $|| \boldsymbol{f} ||_{0,p,\omega} < M\delta k^3$, the sequence of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ recursively defined by

$$\begin{cases} \boldsymbol{u}_0 = \mathbf{0}, \\ \mathrm{d}\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\mathbf{0})(\boldsymbol{u}_{n+1} - \boldsymbol{u}_n) = f - \mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\boldsymbol{u}_n), \; n \geq 0, \end{cases}$$

satisfies

$$\begin{cases} || \boldsymbol{u}_n ||_{2,p,\omega} \leq \delta k, \\ || \boldsymbol{u}_{n+2} - \boldsymbol{u}_{n+1} ||_{2,p,\omega} \leq \kappa \, || \boldsymbol{u}_{n+1} - \boldsymbol{u}_n ||_{2,p,\omega}, \; n \geq 0. \end{cases}$$

Now, concluding as in the proof of the inverse function theorem based on the Banach contraction principle, we infer from Lemmas 5.1 and 5.2 that $\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}$ defines a $\mathcal{C}^\infty$-diffeomorphism between a neighborhood of $\mathbf{0}$ in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ contained in the ball $B_{2,p}(\mathbf{0},\delta k)$ and the ball $B_{0,p}(\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}_k}, M\delta k^3)$ of $\mathbf{L}^p(\omega;\mathbb{R}^3)$. This enables us to state the following result:

Theorem 5.1. *The assumptions concerning γ and p are the same as in Theorem 4.1. Then there exist $\delta_0>0$ and $k_0=\bar{k}_0$ such that if $0<\delta<\delta_0$ and $k>k_0$ the neighborhood $\mathbf{F}^p$ of Theorem 4.1 contains a ball centered at $\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}_k}$ in $\mathbf{L}^p(\omega;\mathbb{R}^3)$ of radius $M\delta k^3$, where $M>0$ is independent of δ. Furthermore, for any $\boldsymbol{f}$ in this ball, the element $\boldsymbol{u}$ satisfying $\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}(\boldsymbol{u})=\boldsymbol{f}$ belongs to the ball with radius δk centered at $\mathbf{0}$ in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$.*

Now we establish that the solution given by Theorem 5.1 is also the unique minimizer of the associated functional over the whole affine space $\{\boldsymbol{\iota}+\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)\}$.

Theorem 5.2. *With the same notations and hypotheses as in the preceding theorem, there exists $0<\delta_1\le\delta_0$ such that if $0<\delta<\delta_1$, the unique solution to the nonlinear clamped membrane problem in $\{\bar{\boldsymbol{\varphi}}_k+B_{2,p}(\mathbf{0},\delta\, k)\}$ associated to any element $\boldsymbol{f}$ in $B_{0,p}(\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}_k}, M\,\delta\,k^3)$, $k>k_0$, is the unique minimizer of the functional $I_M(\boldsymbol{f})$ over the affine space $\{\boldsymbol{\iota}+\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)\}$.*

Proof. Let $k>k_0$ and $0<\delta<\delta_0$ be given.

Consider any element $\boldsymbol{f}$ of $B_{0,p}(\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}_k}, M\delta k^3)$, and denote by $\boldsymbol{\varphi}_{\boldsymbol{f}}$ the associated solution to the nonlinear clamped membrane problem in $\{\bar{\boldsymbol{\varphi}}_k+B_{2,p}(\mathbf{0},\delta\,k)\}$.

Let $\boldsymbol{\varphi}$ be any element in the affine space $\{\boldsymbol{\iota}+\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)\}$ and let $\boldsymbol{v}$ be the element of $\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ defined by $\boldsymbol{v}=\boldsymbol{\varphi}-\boldsymbol{\varphi}_{\boldsymbol{f}}$. Then from (4.9), we get

$$I_M(\boldsymbol{f})(\boldsymbol{\varphi})\ge I_M(\boldsymbol{f})(\boldsymbol{\varphi}_{\boldsymbol{f}})+\frac{2\lambda\mu}{\lambda+2\mu}\int_\omega(\partial_\sigma\boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\sigma\boldsymbol{\varphi}_{\boldsymbol{f}}-\delta_{\sigma\sigma})\partial_\tau\boldsymbol{v}\cdot\partial_\tau\boldsymbol{v}\,d\omega$$
$$+2\,\mu\int_\omega(\partial_\alpha\boldsymbol{\varphi}_{\boldsymbol{f}}\cdot\partial_\beta\boldsymbol{\varphi}_{\boldsymbol{f}}-\delta_{\alpha\beta})\partial_\alpha\boldsymbol{v}\cdot\partial_\beta\boldsymbol{v}d\omega. \tag{5.12}$$

But from (3.16) and (3.17) for any $\boldsymbol{w}\in\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ the following inequality holds:

$$C'k^2\,\|\,\boldsymbol{w}\,\|_{1,2,\omega}^2\le\frac{2\lambda\mu}{\lambda+2\mu}\int_\omega(\partial_\sigma\bar{\boldsymbol{\varphi}}_k\cdot\partial_\sigma\bar{\boldsymbol{\varphi}}_k-\delta_{\sigma\sigma})\partial_\tau\boldsymbol{w}\cdot\partial_\tau\boldsymbol{w}\,d\omega$$
$$+2\mu\int_\omega(\partial_\alpha\bar{\boldsymbol{\varphi}}_k\cdot\partial_\beta\bar{\boldsymbol{\varphi}}_k-\delta_{\alpha\beta})\partial_\alpha\boldsymbol{w}\cdot\partial_\beta\boldsymbol{w}d\omega, \tag{5.13}$$

where $C'>0$ is independent of $k>k_0$.

Writing $\boldsymbol{\varphi}_{\boldsymbol{f}}=\ \bar{\boldsymbol{\varphi}}_k+\delta\,k\,\boldsymbol{u}$ with $\boldsymbol{u}$ in the unit ball of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$, it follows from (5.12) and (5.13) that there exists $0<\delta_1\le\delta_0$ such that, if $0<\delta<\delta_1$, then

$$I_M(\boldsymbol{f})(\boldsymbol{\varphi})\ge I_M(\boldsymbol{f})(\boldsymbol{\varphi}_{\boldsymbol{f}})+C''k^2||\boldsymbol{v}||_{1,2,\omega}^2,$$

where $0<C''<C'$.

This last inequality gives the announced result.

About the injectivity, we also have the following result.

Theorem 5.3. *With the same notations and hypotheses as in Theorem 5.1, there exists $0<\delta_2\le\delta_0$ such that, if $0<\delta<\delta_2$, the unique solution to the nonlinear clamped membrane problem associated in $\{\bar{\boldsymbol{\varphi}}_k+B_{2,p}(\mathbf{0},\delta\,k)\}$ to any element $\boldsymbol{f}$ of $B_{0,p}(\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}_k}, M\delta\,k^2)$, $k>k_0$, is injective in $\bar{\omega}$.*

Proof. Let $k>k_0$ and $\bar{\delta}<\min(\delta_0\,k_0,\delta_0)$ be given. Then set

$$\delta=\frac{\bar{\delta}}{k}. \tag{5.14}$$

Since $0 < \bar{\delta} < \delta_0\ k_0$, we have

$$0 < \delta < \delta_0. \tag{5.15}$$

Then, from Theorem 5.1 we infer that $\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}$ defines a $\mathcal{C}^\infty$-diffeomorphism between a neighborhood of $\mathbf{0}$ in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ contained in the ball $B_{2,p}(\mathbf{0},\delta\ k)$ and the ball $B_{0,p}(\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}_k}, M\delta k^3)$ in $\mathbf{L}^p(\omega;\mathbb{R}^3)$. This establishes that $\mathbf{T}_{\bar{\boldsymbol{\varphi}}_k}$ defines a $\mathcal{C}^\infty$-diffeomorphism between a neighborhood of $\mathbf{0}$ in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ contained in the ball $B_{2,p}(\mathbf{0},\bar{\delta})$ and the ball $B_{0,p}(\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}_k}, M\bar{\delta}k^2)$ in $\mathbf{L}^p(\omega;\mathbb{R}^3)$.

Now, let $k > k_0$ and $0 < \bar{\delta} < \min(\delta_0 k_0, \delta_0)$ be given. Then, the solution associated by Theorem 5.1 to any element of $B_{0,p}(\boldsymbol{f}_{\bar{\boldsymbol{\varphi}}_k}, M\bar{\delta}\ k^2)$ of $\mathbf{L}^p(\omega;\mathbb{R}^3)$ can be written as $\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}}_k + \bar{\delta}\ \boldsymbol{u}$, with $\|\ \boldsymbol{u}\ \|_{2,p,\omega} < 1$. From (4.7), we know that there exists $C(\omega) > 0$ such that for any $\boldsymbol{w} \in \mathbf{W}^{2,p}(\omega;\mathbb{R}^3)$, we have

$$|\ \boldsymbol{w}(x) - \boldsymbol{w}(y)\ | \le C(\omega)\ \|\ \boldsymbol{w}\ \|_{2,p,\omega}\ \ d(x,y) \text{ for all } (x,y) \in \bar{\omega}^2.$$

From (3.8) and (3.9), we infer the existence of $C_0 > 0$ independent of k such that

$$|\ \bar{\boldsymbol{\varphi}}_k(x) - \bar{\boldsymbol{\varphi}}_k(y)\ | \ge C_0\ d(x,y) \text{ for all } (x,y) \in \bar{\omega}^2. \tag{5.16}$$

Let $\delta_2 = \min\Big(\delta_0, \delta_0\ k_0, \frac{C_0}{C(\omega)}\Big) > 0$. Then for $0 < \bar{\delta} < \delta_2$, we deduce that for all $x \neq y$ in $\bar{\omega}$, the solution $\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}}_k + \bar{\delta}\boldsymbol{u}$ satisfies

$$|\ \boldsymbol{\varphi}(x) - \boldsymbol{\varphi}(y) - (\bar{\boldsymbol{\varphi}}_k(x) - \bar{\boldsymbol{\varphi}}_k(y))\ | < C_0\ d(x,y). \tag{5.17}$$

The conclusion follows from (5.16) and (5.17).

Ackowledgement. This work is part of the Human Capital and Mobility Program "Shells: Mathemetical Modeling and Analysis, Scientific Computing" of the Commission of the European Communities (Contract No. ERBCHRXCT 940536).

References

[1] Agmon, S., Douglis, A. & Nirenberg, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II, *Comm. Pure Appl. Math.,* **17**(1964), 35–92.
[2] Ciarlet, P. G., Mathematics elasticity, I, Three-dimensional elasticity, North Holland, 1988.
[3] Ciarlet, P. G., Mathematics elasticity, II, Theory of plates, North Holland, 1997.
[4] Ciarlet, P. G. & Destuynder, P., A justification of a nonlinear model in plate theory, *Comp. Meth. Appl. Mech. Engrg.,* **17/18**(1979), 227–258.
[5] Coutand, D., Existence d'un minimiseur pour le modéle "proprement invariant" de plaque 'en flexion" non linéairement élastique, *C. R. Acad. Sci. Paris, seris I,* **324**(1997), 245–248.
[6] Coutand, D., Théorèmes d'existence pourun modéle "proprement invariant" de plaque membraaire non linéairement élastique, *C. R. Acad. Sci. Paris, seris I,* **324**(1997), 1181–1184.
[7] Coutand, D., Existence of a solution for a nonlinear elastic plance menbrance subject to plance forces, to appear.
[8] Coutand, D., Existence of a solution for a nonlinear elastic plance menbrane "under tension", to appear.
[9] Diestel, J. & Uhl, J. J. Jr., Vector measures, *Math. Surveys,* **15**(1977).
[10] Fox, D., Raoult, A. & Simo, J. C., A justification of nonlinear properly invariant plate theories, *Arch Rational Mech. Anal.,* **124**(1993), 157–199.
[11] Genevey, K., Existence d'une solution radials pour une membrance non linéaire, to appear.
[12] Geymonat, G., Sui problemi ai limiti per i sistemi lineari ellitici, *Ann. math. Pura. Appl.,* **69**(1965), 207–284.
[13] Green, A. E. & Zerna, W., Theoretical elasticity, Oxford University Press, 1968.
[14] Le Dret, H. & Raoult, A., The nonlinear membrance model as variational limit of nonlinear three dimensional elasticity, *J. Math. Pures. Appl.,* **74**(1995), 549–578.
[15] Nečas, J., Les Méthodes directes en Théorie des equation elliptiques, masson, Paris, 1967.