Chin. Ann. of Math.
20B: 3(1999),297-304.

DISTORTION THEOREMS FOR BIHOLOMORPHIC CONVEX MAPPINGS ON BOUNDED CONVEX CIRCULAR DOMAINS**

GONG Sheng* LIU Taishun*

Abstract

In terms of Carathéodory metric and Kobayashi metric, distortion theorems for biholomorphic convex mappings on bounded circular convex domains are given.

Keywords Distortion theorem, Convex mappins, Infinitesimal form of Carathéodory metric, Infinitesimal form of Kobayashi-Royden metric, Minkowski functional

1991 MR Subject Classification 32H02

Chinese Library Classification O174.56

§1. Two Different Distortion Theorems

Various distortion theorems for families of univalent functions have been studied since as early as 1907 when Köbe discovered his "Verzerrungssatz", the distortion theorem for the class of univalent functions defined on the unit disk in the complex plane $\mathbf{C}$.

Let $\Omega \subset \mathbf{C}^n$ be a domain and $f(z) = (f_1(z), \cdots, f_n(z))$ be a biholomorphic mapping on Ω which maps Ω to $\mathbf{C}^n$. There are many counter-examples to show that $|\det J_f(z)|$ and $J_f(z)\overline{J_f(z)}'$ have no finite upper bound and no non-zero lower bound, where $J_f(z)$ is the Jacobian of f at point z.

Let $\Omega \subset \mathbf{C}^n$ be a domain and $m \in \Omega, f(z) = (f_1(z), \cdots, f_n(z))$ be a holomorphic mapping on Ω which maps Ω into $\mathbf{C}^n$. We say f is normalized at point m if $f(m) = 0$ and $J_f(m) = I$, where I is the $n \times n$ identity matrix. If $0 \in \Omega$ and $m = 0$, then we say that f is normalized.

Let $\Omega \subset \mathbf{C}^n$ be a bounded homogeneous domain and $0 \in \Omega, \mathrm{Aut}(\Omega)$ be the group of holomorphic automorphism of $\Omega, \mathcal{P}$ be a family of normalized holomorphic mapping on Ω which maps Ω into $\mathbf{C}^n$. For any $f \in \mathcal{P}, \varphi \in \mathrm{Aut}(\Omega)$, we normalize $f(\varphi(z))$ and obtain a normalized holomorphic mapping $F(z)$ on Ω. We say $\mathcal{P}$ is a linear-invariant family if $F(z) \in \mathcal{P}$ for any $f \in \mathcal{P}$ and $\varphi \in \mathrm{Aut}(\Omega)$.

The first affirmative result about the estimations of $|\det J_f(z)|$ for normalized holomorphic mappings was given by Barnard, FitzGerald and Gong[1] in the case $n = 2$ at first, and then Liu Taishun[2] extended it to the general case.

Manuscript received May 25, 1998. Revised December 8, 1998.

*Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.

E-mail: gongs@public.intercom.com.cn

**Project supported by the National Natural Science Foundation of China.

Let

$$f(z)=(f_1(z),\cdots,f_n(z))=z+(aA^1z',\cdots,zA^nz')+\cdots\in\mathbf{S},$$

where $A^i=(a^i_{j,k})_{1\le j,k\le n}, i=1,\cdots,n$ and $\mathbf{S}$ is a linear-invarint family of holomorphic mappings on the unit ball

$$B^n=\Big\{(z_1,\cdots,z_n)\in\mathbf{C}^n:\sum_{i=1}^n|z_i|^2<1\Big\}$$

in $\mathbf{C}^n$. Then

$$\frac{(1-|z|)^{C(S)-\frac{n+1}{2}}}{(1+|z|)^{C(S)+\frac{n+1}{2}}}\le|\det J_f(z)|\le\frac{(1+|z|)^{C(S)-\frac{n+1}{2}}}{(1-|z|)^{C(S)+\frac{n+1}{2}}},$$

where $C(S)=\sup\Big\{\Big|\sum\limits_{i=1}^n a^i_{i1}\Big|:f\in\mathbf{S}\Big\}$. In particular, if $\mathbf{S}$ is the family of normalized biholomorphic convex mappings on $\mathbf{B}^n$, then $\frac{n+1}{2}\le C(S)\le 1+\frac{\sqrt{2}(n-1)}{2}$.

Moreover, Barnard, FitzGerald and Gong[1] conjectured that $C(S)=\frac{n+1}{2}$ for the family of normalized biholomorphic convex mappings on $\mathbf{B}^n$. Recently, Pfaltzgraff and Suffridge[3] gave a counter-example to show that the conjecture is not true. To find the exact value of $C(S)$ for the family of normalized biholomorphic convex mappings remains as an open problem. Even we do not know the precise value of $C(S)$ for the family of biholomorphic convex mappings on B^n, but we had the following precise estimations of $J_f(z)\overline{J_f(z)}'$ for the family of normalized biholomorphic convex mappings on B^n which was given by Gong, Wang ang Yu[4] in 1993.

Theorem 1.1. *Let $f:\mathbf{B}^n\to\mathbf{C}^n$ be a normalized biholomorphic convex mapping on $\mathbf{B}^n$ in $\mathbf{C}^n$. Then*

$$\left(\frac{1-|z|}{1+|z|}\right)^2G\le J_f(z)\overline{J_f(z)}\le\left(\frac{1+|z|}{1-|z|}\right)^2G \tag{1.1}$$

holds for every $z=(z_1,\cdots,z_n)\in\mathbf{B}^n$ where

$$G=(g_{i,j})_{1\le i,j\le n}=\left(\frac{(1-|z|^2)\delta_{i,j}+\bar{z}_iz_j}{(1-|z|^2)^2}\right)_{1\le i,j\le n}$$

is the matrix of the Bergman matric of $\mathbf{B}^n$ in $\mathbf{C}^n$. The estimations are precise.

§2. Main Results

In the note, we prove the following main result which extends Theorem 1.1.

Theorem 2.1. *Let $\Omega\subset\mathbf{C}^n$ be a bounded convex circular domain with $0\in\Omega$, and $p(z)(z\in\Omega)$ be its Minkowski functional. Let $f(z):\Omega\to\mathbf{C}^n$ be a normalized biholomorphic convex mapping on Ω. Then for every $z\in\Omega$ and vector $\xi\in\mathbf{C}^n$, the inequalities*

$$\frac{1-p(z)}{1+p(z)}F^\Omega(z,\xi)\le p(J_f(z)\xi)\le\frac{1+p(z)}{1-p(z)}F^\Omega(z,\xi) \tag{2.1}$$

hold, where

$$F^\Omega(z,\xi)=F^\Omega_C(z,\xi)=F^\Omega_K(z,\xi),$$

$F^\Omega_C(z,\xi)$ and $F^\Omega_K(z,\xi)$ are the infinitesimal form of Carathéodory metric and the infinitesimal form of Kobayashi-Royden metric of Ω respectively.

When Ω is the unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$, we have

$$(F^\Omega(z,\xi))^2 = (F_C^\Omega(z,\xi))^2 = (F_K^\Omega(z,\xi))^2 = \frac{|\xi|^2}{1-|z|^2} + \frac{\xi \bar{z}' z \bar{\xi}'}{(1-|z|^2)^2}.$$

(2.1) can be written as (1.1). Theorem 2.1 extends Theorem 1.1.

The original proof of Theorem 1.1 in [4] is based on differential geometry and the detail of the proof is a little bit lengthy, but the proof of Theorem 2.1 is shorter and simple.

Moreover, there are two consequences of the main result.

Theorem 2.2. *Let $\Omega \subset \mathbf{C}^n$ be a bounded convex circular domain with $0 \in \Omega$, and $p(z)(z \in \Omega)$ be its Minkowski functional. Let $f(z) : \Omega \to \mathbf{C}^n$ be a normalized convex biholomorphic mapping on Ω. Then the inequalities*

$$\frac{p(z)}{(1+p(z))^2} \le p(J_f(z)z) \le \frac{p(z)}{(1-p(z))^2} \tag{2.2}$$

hold, where z is a column vector.

When Ω is the unit disk Δ in $\mathbf{C}$, (2.2) is the classical growth theorem of normalized starlike function in Δ due to Alexander theorem: $zf'(z)$ is starlike if and only if $f(z)$ is convex. But the Alexander theorem is not true for several complex variables case in general. Liu Taishun and Ren Guangbin[5] already proved the growth theorem for normalized starlike biholomorphic mappings on bounded starlike circular domains as follows.

Let $\Omega \subset \mathbf{C}^n$ be a bounded starlike circular domain with $0 \in \Omega$, and $p(z)(z \in \Omega)$ be its Minkowski functional. Let $g(z) : \Omega \to \mathbf{C}^n$ be a normalized starlike biholomorphic mapping on Ω. Then for any $z \in \Omega$, the inequalities

$$\frac{p(z)}{(1+p(z))^2} \le p(g(z)) \le \frac{p(z)}{(1-p(z))^2} \tag{2.3}$$

hold.

As we already mentioned, the family of normalized starlike biholomorphic mappings and the family of $J_f(z)z$ (z is a column vector) where $f(z)$ is convex are essential different in the case of several complex case. Thus the inequalities (2.2) and (2.3) are two different inequalities.

As another consequence of Theorem 2.1, we have the estimations of the modulus of the eigenvalues of the Jacobian of the convex mapping.

Theorem 2.3. *Let $\Omega \subset \mathbf{C}^n$ be a bounded convex circular domain with $0 \in \Omega$, and $p(z)(z \in \Omega)$ be its Minkowski functional. Let $f(z) : \Omega \to \mathbf{C}^n$ be a normalized convex biholomophic mapping on Ω, and $\lambda_1(z), \cdots, \lambda_n(z)$ be the eigenvalues of $J_f(z)$ with $|\lambda_1(z)| \ge |\lambda_2(z)| \ge \cdots \ge |\lambda_n(z)|$. Then the inequalities*

$$\frac{(1-p(z))}{(1+p(z))^2} \le |\lambda_n(z)| \le \cdots \le |\lambda_2(z)| \le |\lambda_1(z)| \le \frac{1+p(z)}{(1-p(z))^2} \tag{2.4}$$

hold.

§3. Lemmas

We need the following lemmas.

Lemma 3.1. *Let $\Omega \subset \mathbf{C}^n$ be a bounded convex circular domain with $0 \in \Omega$, and $p(z)(z \in$*

Ω) be its Minkowski functional. Then

$$F^{\Omega}(z,\zeta) = F_K^{\Omega}(z,\zeta) = F_C^{\Omega}(z,\zeta) = \frac{1}{1-(p(z))^2} \tag{3.1}$$

holds, where $\zeta = \frac{z}{p(z)} \in \partial\Omega$ is a column vector.

Proof. According to the definition of the infinitesimal form of Kobayashi-Royden metric F_K^{Ω} of Ω,

$$F_K^{\Omega}(z,\xi) = \inf\left\{\frac{|\alpha|}{1-|\lambda|^2} : \exists\varphi \in H(\Delta,\Omega), \exists\lambda \in \Delta,\ \varphi(\lambda) = z, \alpha\varphi'(\lambda) = \xi\right\}, \tag{3.2}$$

where $H(\Delta,\Omega)$ is the family of holomophic mappings on Δ which maps Δ into Ω.

Fix $z \in \Omega$ and define $\varphi(w) = w\zeta$, where $w \in \Delta$. Then $\varphi \in H(\Delta,\Omega)$ since Ω is a bounded convex circular domain. Let $\lambda = p(z)$, then $\varphi(\lambda) = z$. Let $\alpha = 1$, then $\alpha\varphi'(\lambda) = \zeta$. By the definition (3.2), we have

$$F_K^{\Omega}(z,\zeta) \le \frac{1}{1-(p(z))^2}. \tag{3.3}$$

On the other hand, according to the definition of the infinitesimal form of Carathéodory metric F_C^{Ω} of Ω,

$$F_C^{\Omega}(z,\zeta) = \sup\{|J_{\varphi}(z)\zeta| : \exists\varphi \in H(\Omega,\Delta),\ \varphi(z) = 0\}, \tag{3.4}$$

where $H(\Omega,\Delta)$ is the family of holomorphic mappings in Ω which maps Ω into Δ.

Fix $z \in \Omega$. There exists a continuous linear functional T_z on the Banach space $\mathbf{C}^n$ with the Minkowski functional of Ω as semi-norm, $T_z : \mathbf{C}^n \to \mathbf{C}$, such that $\|T_z\| \le 1$ and $T_z(z) = p(z)$, where $\|\ \|$ is the norm of the dual space of $\mathbf{C}^n$.

Let $\psi_{p(z)}(\lambda)$ be a holomorphic automorphism of Δ which maps $p(z)$ to 0, i.e.,

$$\psi_{p(z)}(\lambda) = \frac{p(z)-\lambda}{1-p(z)\lambda}.$$

Let

$$\varphi(w) = \psi_{p(z)} \circ T_z(w).$$

Then $\varphi \in H(\Omega,\Delta)$ and $\varphi(z) = 0$. Thus

$$J_{\varphi}(z)\zeta = J_{\psi_{p(z)}}(p(z))J_{T_z}(\zeta) = \frac{-1}{1-(p(z))^2}T_z(\zeta) = \frac{-1}{1-(p(z))^2}$$

since $T_z(\zeta) = 1$. By the definition (3.4), we have

$$F_C^{\Omega}(z,\zeta) \ge \frac{1}{1-(p(z))^2}.$$

By the Theorem of Lempert[11], we know that

$$F_C^{\Omega}(z,\xi) = F_K^{\Omega}(z,\xi)$$

when Ω is convex. By (3.3) and (3.5), we obtain (3.1).

From Lemma 3.1 we have

Lemma 3.2. *Let $\Omega \subset \mathbf{C}^n$ be a bounded convex circular domain with $0 \in \Omega$, and $p(z)(z \in \Omega)$ be its Minkowski functional. Then*

$$F^{\Omega}(0,\xi) = F_K^{\Omega}(0,\xi) = F_C^{\Omega}(0,\xi) = p(\xi), \tag{3.6}$$

where $\xi \in \mathbf{C}^n$ is a column vector.

Proof. Let $\zeta = \frac{\xi}{p(\xi)} \in \partial\Omega$ and $0 < t < 1$. Then

$$F^{\Omega}(t\zeta, \zeta) = \frac{1}{1 - (p(t\zeta))^2}$$

by Lemma 3.1. Letting $t \to 0$, we have

$$F^{\Omega}(0, \zeta) = 1$$

by the continuous properties of F^{Ω} and $p(z)$ at point 0.

By the definition of F^{Ω}, we obtain

$$F^{\Omega}(0, \zeta) = F^{\Omega}(0, p(\xi)\zeta) = p(\xi)F^{\Omega}(0, \zeta) = p(\xi).$$

Lemma 3.2 is a known result (see [12]).

Lemma 3.3. *Let $\Omega \subset \mathbf{C}^n$ be a bounded convex circular domain with $0 \in \Omega$, and $p(z)(z \in \Omega)$ be its Minkowski functional. Then for every column vector $z \in \Omega$ and every column vector $\xi \in \partial\Omega$,*

$$\frac{1}{1 + p(z)} \leq F^{\Omega}(z, \xi) \leq \frac{1}{1 - p(z)} \tag{3.7}$$

holds, where $F^{\Omega} = F_K^{\Omega} = F_C^{\Omega}$.

Proof. For fixed z and ξ, the mapping

$$\varphi(w) = (\varphi_1(w), \cdots, \varphi_n(w))' = z + w(1 - p(z))\xi$$

is a holomorphic mapping on Δ where $'$ means transpose.

Obviously, $\varphi(0) = z$ and

$$\varphi'(0) = (\varphi_1'(0), \cdots, \varphi_n'(0))' = (1 - p(z))\xi.$$

Moreover,

$$\begin{aligned} p(\varphi(w)) &= p(z + w(1 - p(z))\xi) \leq p(z) + p(w(1 - p(z))\xi) \\ &= p(z) + |w|(1 - p(z))p(\xi) < 1 \end{aligned}$$

since $|w| < 1$ and $p(\xi) = 1$. Thus $\varphi(w) \in \Omega$. This means $\varphi \in H(\Delta, \Omega)$.

By the definition (3.2) of the infinitesimal form of Kobayashi-Royden metric, and letting $\lambda = 0, \alpha = \frac{1}{1-p(z)}$, we have

$$F_K^{\Omega}(z, \xi) \leq \frac{1}{1 - p(z)}.$$

For fixed ξ, there exists a continuous linear functional T_ξ on Banach space $\mathbf{C}^n$ with the Minkowski functional of Ω as semi-norm, $T_\xi : \mathbf{C}^n \to \mathbf{C}$, such that $\|T_\xi\| \leq 1$ and $T_\xi(\xi) = p(\xi) = 1$, where $\| \ \|$ is the norm of the dual space of $\mathbf{C}^n$. Then the function

$$\psi(w) = \frac{T_\xi(w - z)}{1 + p(z)}$$

is a holomorphic function on Ω.

Obviously, $\psi(z) = 0$ since $T_\xi(0) = 0$ and

$$|\psi(w)| = \frac{|T_\xi(w - z)|}{1 + p(z)} \leq \frac{\|T_\xi\| \cdot p(w - z)}{1 + p(z)} \leq \frac{p(w) + p(z)}{1 + p(z)} < 1.$$

Thus $\psi(w) \in \Delta$. This means $\psi \in H(\Omega, \Delta)$. By the definition (3.4) of the infinitesimal form of Carathéodory metric and

$$J_\psi(z)\xi = \frac{T_\xi(\xi)}{1+p(z)} = \frac{1}{1+p(z)},$$

we have

$$F_C^\Omega(z,\xi) \geq \frac{1}{1+p(z)}.$$

By the Theorem of Lempert[11], we know that

$$F_C^\Omega(z,\xi) = F_K^\Omega(z,\xi)$$

when Ω is convex. We obtain (3.7).

§4. Proof of Theorems

Proof of Theorem 2.1. We prove the right side inequality of (2.1) at first.

For any fixed $z \in \Omega$, we consider the line segment which starts from $f(z)$, passes through the origin, and then meets a point P on the boundary of $f(\Omega)$. There exists a point $z^* \in \partial\Omega$, such that $f(z^*) = P$. Since $f(z), 0$ and $f(z^*)$ lie on a line segment, and 0 is a point between $f(z)$ and $f(z^*)$, there exists a real number $\lambda \in (0,1)$, such that

$$\lambda f(z) + (1-\lambda) f(z^*) = 0.$$

From the definition of $p(z)$, we have

$$\lambda p(f(z)) = (1-\lambda) p(f(z^*)).$$

Liu and Ren[6] proved that if $f(z)$ is a normalized biholomorphic convex mapping on a bounded convex circular domain in $\mathbf{C}^n$ with $0 \in \Omega$, then the inequalities of the growth theorem

$$\frac{p(z)}{1+p(z)} \leq p(f(z)) \leq \frac{p(z)}{1-p(z)}$$

hold for $z \in \Omega$. We have

$$\frac{\lambda p(z)}{1-p(z)} \geq \lambda p(f(z)) = (1-\lambda)p(f(z^*)) \geq \frac{1}{2}(1-\lambda).$$

It yields

$$\lambda \geq \frac{1-p(z)}{1+p(z)}. \tag{4.1}$$

Let

$$h(w) = f^{-1}[\lambda f(w) + (1-\lambda) f(z^*).$$

$h(w)$ is well defined when $w \in \Omega$ since f is a convex mapping. Moreover, $h(w)$ is biholomorphic on $\Omega, h(\Omega) \subset \Omega$ and $h(z) = 0$ since f is a normalized biholomorphic mapping.

Since $h(z)$ is a biholomorphic mapping on Ω, Carathéodory metric is an invariant metric (see [7, 8, 9, 10]), we have the equality

$$F_C^{h(\Omega)}(0, J_h(z)\xi) = F_C^\Omega(z,\xi).$$

Since $h(\Omega) \subset \Omega$, we have the inequality of Carathéodory metrics of inner mapping,

$$F_C^{h(\Omega)}(0, J_h(z)\xi) \geq F_C^\Omega(0, J_h(z)\xi). \tag{4.2}$$

Hence

$$F_C^\Omega(0, J_h(z)\xi) \leq F_C^\Omega(z, \xi).$$

From (4.2) and according to the definition of h and F_C^Ω, we have $J_h(z) = \lambda J_f(z)$ and

$$\lambda F_C^\Omega(0, J_f(z)\xi) \leq F_C^\Omega(z, \xi). \tag{4.3}$$

Combining the inequalities (4.1) and (4.3), we obtain the right side inequality of (2.1).

Now we prove the left side inequality of (2.1).

For any fixed $z \in \Omega$, we consider a line segment which starts from the origin, passes through $f(z)$, and then meets a point Q in the boundary of $f(\Omega)$. There exists a point $\tilde{z} \in \partial\Omega$, such that $f(\tilde{z}) = Q$. Since $0, f(z)$ and $f(\tilde{z})$ lie on a line segment, and $f(z)$ is a point between 0 and $f(\tilde{z})$, there exists a real number $\mu \in (0, 1)$, such that $f(z) = \mu f(\tilde{z})$. Then

$$z = \tilde{z}(\mu) = f^{-1}(\mu f(\tilde{z})).$$

In [6], Liu and Ren proved that if $z(t) = f^{-1}(tf(z)), 0 \leq t \leq 1$, then

$$\frac{p(z)}{1 + p(z)} \leq \frac{p(z(t))}{t(1 + p(z(t)))}$$

holds. We have

$$\frac{1}{2} = \frac{p(\tilde{z})}{1 + p(\tilde{z})} \leq \frac{p(\tilde{z}(\mu))}{\mu(1 + p(\tilde{z}(\mu)))} = \frac{p(z)}{\mu(1 + p(z))}.$$

It implies

$$\mu \leq \frac{2p(z)}{1 + p(z)}, \quad 1 - \mu \geq \frac{1 - p(z)}{1 + p(z)}. \tag{4.4}$$

Let

$$H(w) = f^{-1}[(1 - \mu)f(w) + \mu f(\tilde{z})].$$

$H(w)$ is well defined when $w \in \Omega$ since f is a convex mapping. Moreover, $H(w)$ is biholomorphic on $\Omega, H(\Omega) \subset \Omega, H(0) = z$ since f is a normalized biholomorphic mapping.

Since $H(z)$ is biholomorphic on Ω, the Carathéodory metric is an invariant metric, we have the equality

$$F_C^{H(\Omega)}(z, J_H(0)\eta) = F_C^\Omega(0, \eta)$$

which holds for any $z \in \Omega, \eta \in \mathbf{C}^n$. Since $H(\Omega) \subset \Omega$, we have the inequality of Carathéodory metrics of the inner mapping,

$$F_C^{H(\Omega)}(z, J_H(0)\eta) \geq F_C^\Omega(z, J_H(0)\eta).$$

Hence

$$F_C^\Omega(z, J_H(0)\eta) \leq F_C^\Omega(0, \eta). \tag{4.5}$$

From (4.5) and according to the definition of H and F_C^Ω, we have $J_H(0) = J_f^{-1}(z)(1 - \mu)$ and

$$(1 - \mu)F_C^\Omega(z, J_f^{-1}(z)\eta) \leq F_C^\Omega(0, \eta). \tag{4.6}$$

Let $\eta = J_f(z)\xi$ in (4.6), it becomes

$$(1 - \mu)F_C^\Omega(z, \xi) \leq F_C^\Omega(0, J(z)\xi). \tag{4.7}$$

Combining (4.4) and (4.7), we obtain the left side inequality of (2.1).

The Theorem of Lempet[11] tells us that

$$F_C^{\Omega}(z,\xi) = F_K^{\Omega}(z,\xi)$$

when Ω is convex.

We have proved the inequality (2.1) by Lemma 3.2.

By Theorem 2.1 and Lemma 3.1, we have Theorem 2.2.

Proof of Theorem 2.3. By Theorem 2.1 and Lemma 3.3, we have

$$\frac{1-p(z)}{(1+p(z))^2} \le p(J_f(z)\xi) \le \frac{1+p(z)}{(1-p(z))^2}$$

when $\xi \in \partial\Omega$ is a column vector. If $\lambda(z)$ is the eigenvalue of $J_f(z)$, then

$$J_f(z)\eta = \lambda(z)\eta,$$

where η is the eigenvector of $\lambda(z)$. Let $\xi = \frac{\eta}{p(\eta)}$. Then $\xi \in \partial\Omega$ and

$$p(J_f(z)\xi) = p(\lambda(z)\xi) = |\lambda(z)|p(\xi) = |\lambda(z)|.$$

Hence

$$\frac{1-p(z)}{(1+p(z))^2} \le |\lambda(z)| \le \frac{1+p(z)}{(1-p(z))^2}$$

hold for all eigenvalues of $J_f(z)$. Thus we have proved (2.4).

References

[1] Barnard, R. M., FitzGerald, C. H. & Gong Sheng, Distortion theorem for biholomorphic mappings in $\mathbf{C}^n$, *Tran. Amer. Math. Soc.,* **344**(1994), 907–924.

[2] Liu Taishun, The distortion theorems, growth theorems and covering theorems for biholomorphic mappings on classical domains, University of Science and Technology of Science of China, Doctor Thesis, 1989.

[3] Pfaltzgraff, J. A. & Suffridge, T. S., Linear invariant, order and convex mappings in $\mathbf{C}^n$, Research report, 96–6, 1996, Univ. of Kentucky.

[4] Gong Sheng, Wang Shikun & Yu Qihuang, Biholomorphic convex mappings of ball in $\mathbf{C}^n$, *Pacific Jour. of Math.,* **161**(1993), 287–306.

[5] Liu Taishun & Ren Guangbin, The growth and 1/4-covering theorems for the normalized starlike mappings on bounded starlike circular domains, *Chin. Ann. of Math.,* **19B:**4(1998), 401–408.

[6] Liu Taishun & Ren Guangbin, The growth theorem of convex mappings on bounded convex circular domains, *Science in China* (*series A*)*,* **41**(1998).

[7] Yau ShingTung & Schoen, Richard, Differential geometry, Science Press, 1991.

[8] Krantz, S. G., Function theory of several complex variables, 2nd edition, Wadsworth & Brooks/col, 1992.

[9] Poletskii, E. A. & Shabat, B V., Invariant metrices, Several complex variables III, G. M. Khenkin (Ed.) Springer-Verlag, 1989, pp.63–11.

[10] Kobayashi, S., Hyperbolic manifolds and holomorphic mappings, Dekker, New York, 1970.

[11] Lempert, L., La metrique Kobayashi et las representation des domains sur la boule, *Bull. Soc. Math. France,* **109**(1981), 427–474.

[12] Jarnicki, M. & Pflug, P., Invariant distances and metrics in complex analysis, de Gruyter, Berlin-New York, 1993.