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A C^*-ALGEBRA APPROACH TO THE IRREDUCIBILITY OF COWEN-DOUGLAS OPERATORS**

XUE YIFENG*   WANG ZONGYAO*

Abstract

The authors consider the irreducibility of the Cowen-Douglas operator T. It is proved that T is irreducible iff the unital C^*-algebra generated by some non-zero blocks in the decomposition of T with respect to $\bigoplus\limits_{n=0}^{\infty}(\mathrm{Ker}\,T^{n+1}\ominus\mathrm{Ker}\,T^n)$ is $\mathrm{M}_n(\mathbb{C})$.

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Let H be a separable infinite dimensional Hilbert space over the field $\mathbb{C}$ and let $L(H)$ (resp. $K(H)$) denote the algebra of all bounded linear operators (resp. all compact operators) on H. For $T \in L(H)$, we denote by $\mathrm{R}\,(T)$ (resp. $\mathrm{Ker}\,T$) the range (resp. the null space) of T. According to [2], $B_n(\Omega)$ consists of all Cowen-Douglas operators $T \in L(H)$ satisfying following conditions:

(1) $\Omega \subset \sigma(T)$ is a conected open subset in $\mathbb{C}$;
(2) $\mathrm{R}\,(\omega - T) = H$ and $\dim\mathrm{Ker}\,(\omega - T) = n < \infty,\ \forall\omega \in \Omega$;
(3) $H = \bigvee\limits_{\omega\in\Omega}\mathrm{Ker}\,(\omega - T)$.

We have seen from [5] that the Cowen-Douglas operators play a very important role in studying the strong irreduciblity of operators. But we feel very regretful that the conditons which make $T \in B_n(\Omega)$ strongly irreducible are nuknown till now except $n = 1$. Considering the close relationship between the strong irreducibilty and irreduciblity of operators, that is, $T \in L(H)$ is strongly irreducible iff XTX^{-1} is irreducible for every invertible operator $X \in L(H)$, we will display various equivalent conditions (see Theorem 1) that make a Cowen-Douglas operator irreducible. These will lead us to find some irreducible conditions which are invariant under similar transformation. On the other hand, since the first author has proven that every Cowen-Douglas operator can be decomposed as the direct sum of finite number of irreducible Cowen-Douglas operators (see [7, Theorem 2]), we could only investigate the properties of irreducible Cowen-Douglas operators instead of general Cowen-Douglas operators.

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*Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China.
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After we finished the paper, we recieved Jiang and Li's paper (see [4]). They showed that $T \in B_n(\Omega)$ is irreducible iff the unital C^*-algebra generated by all non-zero blocks in the decomposition of T with respect to $\bigoplus_{n=0}^{\infty}(\operatorname{Ker} T^{n+1} \ominus \operatorname{Ker} T^n)$ is $\mathrm{M}_n(\mathbb{C})$. This result is contained in Theorem 3(4) and, moreover, the methods we used are different.

According to [2, 1.7], we may assume that $0 \in \Omega$ and $\bigvee_{k=1}^{\infty} \operatorname{Ker} T^k = H$ $\forall T \in B_n(\Omega)$. For any $T \in B_n(\Omega)$ let $P_k\colon H \to \operatorname{Ker} T^k$ be the orthogonal projection $k = 1, \cdots$, and $P_0 = 0$. Thus P_k converges to I_H strongly and $P_k \le P_{k+1}$. Consequently, $\sum_{k=0}^{\infty}(P_{k+1} - P_k) = I_H$. We denote by $\mathcal{A}(T,n)$ (resp. $\mathcal{N}(T,n)$) the C^*-algebra (resp. the Von Neumann algebra) generated by $T \in B_n(\Omega)$ and I_H. Note that $\mathrm{R}\,(T^k) = H, \forall k \ge 1$ for $T \in B_n(\Omega)$ indicates that T^kT^{*k} is invertible in $\mathcal{A}(T,n)$. Set $B_k = T^{*k}(T^kT^{*k})^{-1} \in \mathcal{A}(T,n)$. Then $T^kB_k = I_H$ and $B_kT^k = I_H - P_k \in \mathcal{A}(T,n)$, i.e., B_k is the generalized inverse of T^k (see [3]). Now set $\mathcal{J}_n(T) = K(H) \cap \mathcal{A}(T,n)$. Obviously, $\mathcal{J}_n(T)$ is a non-trivial two-side closed ideal of $\mathcal{A}(T,n)$ $(P_1 \in \mathcal{J}_n(T))$.

Let P be an orthogonal projection in $\mathcal{N}(T,n)$. We write $c(P)$ to denote the orthogonal projection of H onto

$$\chi_{PH}^{\mathcal{N}(T,n)} = \overline{\operatorname{span}}\{\, Ax \mid A \in \mathcal{N}(T,n),\, x \in PH\}.$$

By [6, Lemma 2.6.3], $c(P) = \bigvee_u u^*Pu$, where u is the unitary element in $\mathcal{N}(T,n)$. The following lemma describes explicit relationship among $P_0, P_1, \cdots$.

Lemma 1. *Let $T \in B_n(\Omega)$ and $\{P_k\}_0^{\infty}$ be as above. Then*

(1) *There is a sequence of partial isometries $\{v_k\}_0^{\infty}$ in $\mathcal{A}(T,n)$ such that $v_kv_k^* = P_1, v_k^*v_k = P_{k+1} - P_k, \forall k \ge 0$;*

(2) *$P_k \in \mathcal{J}_n(T)$, $\forall k \ge 0$ and $c(P_1) = I_H$.*

Proof. (1) Set $S_k = T^kP_{k+1}$. Then by the identity $P_kTP_{k+1} = TP_{k+1}$, $\mathrm{R}\,(S_k) \subset P_1H$. Since $\mathrm{R}\,(T^k) = H$, $\forall k \ge 1$, it follows that for each $x \in \operatorname{Ker} T$, there is a $y \in H$ such that $x = T^ky$. Thus $T^{k+1}y = Tx = 0$, i.e., $y \in \operatorname{Ker} T^{k+1}$. Therefore $x \in \mathrm{R}\,(S_k)$ and hence $\mathrm{R}\,(S_k) = P_1H$.

From $\operatorname{Ker} S_k = \{x \in H \mid (P_{k+1} - P_k)x = 0\}$, we have $\mathrm{R}\,(S_k^*) = (P_{k+1} - P_k)H$. Let $S_k = v_k|S_k|$ be the polar decomposition of S_k. Then $v_k \in \mathcal{A}(T,n)$ by [7, Lemma 1] and $v_kv_k^* = P_1$, $v_k^*v_k = P_{k+1} - P_k$ for $k \ge 1$ and $v_0 = P_1$.

(2) By assertion (1), $\dim P_kH = nk$, $k \ge 1$. Thus $P_k \in \mathcal{J}_n(T)$. Noting that $P_1(P_{k+1} - P_k) = 0$, $\forall k \ge 1$, we can define a unitary element u_k in $\mathcal{A}(T,n)$ by $u_k = v_k + v_k^* + I_H - P_1 - (P_{k+1} - P_k), k \ge 1$ and $u_0 = I_H$. Simple computation shows that $P_{k+1} - P_k = u_k^*P_1u_k$, $k \ge 1$ and hence $I_H = \sum_{k=0}^{\infty} u_k^*P_1u_k$. So $c(P_1) = I_H$.

For $T \in B_n(\Omega)$, let $\{P_k\}_0^{\infty}$ be as above. Set $T_{st} = (P_{s+1} - P_s)T(P_{t+1} - P_t)$ and $B_{st} = (P_{s+1} - P_s)B_1(P_{t+1} - P_t)$. Since $P_kTP_k = TP_k$ and $P_{k+1}B_1P_k = B_1P_k$ for $k \ge 0$, it follows that $T_{st} = 0$, $s \ge t$ and

$$T_{k,k+1}B_{k+1,k} = P_{k+1} - P_k,\ B_{k+1,k}T_{k,k+1} = P_{k+2} - P_{k+1}. \qquad (*)$$

Now let $\{W_k\}_0^{\infty}$ be a sequence of partial isometries in $\mathcal{A}(T,n)$ such that $W_k^*W_k = P_{k+1} - P_k$, $W_kW_k^* = P_1$ (W_k is not necessarily equal to v_k in Lemma 1 (1)). Set $\overline{T}_{st} = W_sT_{st}W_t^*$. Then $\overline{T}_{st}$ can be viewed as an operator in $L(P_1H) \cong \mathrm{M}_n(\mathbb{C})$ and, moreover, $\overline{T}_{k,k+1}$ is invertible for each $k \ge 0$ by $(*)$. Thus if we identify H with $\bigoplus_{i=0}^{\infty}(P_{k+1} - P_k)H$, then $W =$

$\bigoplus_{k=0}^{\infty} W_k$ is an isometry of H onto $\bigoplus_{k=0}^{\infty} P_1H$ such that

$$WTW^* = \begin{pmatrix} 0 & \overline{T}_{01} & \overline{T}_{02} & \cdots \\ & 0 & \overline{T}_{12} & \cdots \\ & 0 & 0 & \cdots \\ & & & \ddots \end{pmatrix}. \qquad (**)$$

According to [5, Lemma 2.20], we can choose a sequence $\{W_k\}_0^\infty$ of partial isometries in $\mathcal{A}(T,n)$ such that $\overline{T}_{k,k+1} > 0,\ k \geq 0$.

Recall that an orthogonal projection P in $\mathcal{A}(T,n)$ for $T \in B_n(\Omega)$ is called minimal if for any $a \in \mathcal{A}(T,n)$ there exists a $\lambda \in \mathbb{C}$ such that $PaP = \lambda P$ (see [1, 1.4]). The next lemma comes from [1, Lemma 1.4.1] or [8, Proposition 1.2].

Lemma 2. *Let T, $\mathcal{J}_n(T)$ be as above. Then for each orthogonal projection P in $\mathcal{J}_n(T)$ there are minimal projections $e_1, \cdots, e_r (r < \infty)$ in $\mathcal{J}_n(T)$ such that $e_ie_j = 0$, $i \neq j$ and $p = \sum_{i=1}^{r} e_i$.*

Let $\mathcal{F}$ be a subset of $\{\overline{T}_{st} \,|\, 0 \leq s < t\}$. We denote by $C^*(\mathcal{F})$ the C^*-algebra generated by the operators in $\mathcal{F}$ and P_1 (the unit of $L(P_1H)$).

The following theorem is our main result of the paper.

Theorem 1. *Let $T \in B_n(\Omega)$ $(n \geq 2)$ and $\overline{T}_{st}$ $(0 \leq s < t)$ be as above. Then following statements are equivalent:*

(1) *T is irreducible;*

(2) *For any orthonormal basis* ONB $\{f_i\}_1^n$ *of* $\operatorname{Ker} T$, *there is a sequence $\{A_{ij}\}_1^n$ in $\mathcal{A}(T,n)$ such that $(A_{ij}f_i, f_j) \neq 0$, $i \neq j$;*

(3) *There is a finite subset $\mathcal{F}$ of $\{\overline{T}_{st}|0 \leq s < t\}$ such that $C^*(\mathcal{F}) = L(P_1H)$;*

(4) *$C^*(\{\overline{T}_{st}|0 \leq s < t\}) = L(P_1H)$.*

Proof. (1)⇒(2). Since $\mathcal{A}(T,n)$ is strongly dense in $\mathcal{N}(T,n)$ and since the irreducibility of T is the same as that of $\mathcal{A}(T,n)$, it follows from [6, Theorem 3.14.3] that $f_j \in \chi_{f_i}^{\mathcal{A}(T,n)}$, $i \neq j$. So there is a sequence $\{A_{ij}^{(k)}\}_{k=1}^\infty$ in $\mathcal{A}(T,n)$ such that $f_j = \lim_{k\to\infty} A_{ij}^{(k)} f_i$, $i \neq j$ and $\lim_{k\to\infty} (A_{ij}^{(k)} f_i, f_j) = 1$. Thus we can pick $A_{ij} = A_{ij}^{(k_0)}$ for k_0 large enough such that $(A_{ij}f_i, f_j) \neq 0$, $i \neq j$.

(2)⇒(1). Let $\{f_i\}_1^n$ be an ONB of $\operatorname{Ker} T$. Let $\{f_{n_i}\}_1^k$ be a subset of $\{f_i\}_1^n$, $2 \leq k \leq n$ and set $Q_k = \sum_{i=1}^{k} f_{n_i} \otimes f_{n_i}$, where $f \otimes g \in K(H)$ is defined by $(f \otimes g)x = (x, f)g$, $f, g, x \in H$. Then Q_k is an orthogonal projection majored by P_1.

Since there is an A_{n_i,n_j} in $\mathcal{A}(T,n)$ such that $(A_{n_i,n_j} f_{n_i}, f_{n_j}) \neq 0\,(i \neq j,\ 1 \leq i,j \leq k)$, we get that $Q_kA_{n_1,n_2}WQ_k \neq \lambda Q_k$ for each $\lambda \in \mathbb{C}$. Consequently, by Lemma 2, there exist minimal projections $e_1, \cdots, e_r$ in $\mathcal{J}_n(T)\,(r < \infty)$ such that $e_ie_j = 0$, $i \neq 0$ and $P_1 = \sum_{i=1}^{r} e_i$. Using the same argument as above to $e_1, \cdots, e_r$, we conclude that $e_i = h_i \otimes h_i$ and $r = n$. Otherwise, e_i will not be minimal. So $P_1 = \sum_{i=1}^{n} h_i \otimes h_i$.

For above ONB $\{h_i\}$ of $\operatorname{Ker} T$, there is a sequence $\{A_{j-1}\}_2^n$ in $\mathcal{A}(T,n)$ such that $(A_{j-1}h_1, h_j) \neq 0$, $j = 2, \cdots, n$. Put

$$C_j = \frac{1}{(A_{j-1}h_1, h_j)}(h_j \otimes h_j)A_{j-1}(h_1 \otimes h_1) \in \mathcal{J}_n(T).$$

Then $C_j h_1 = h_j$, $j = 2, \cdots, n$. Thus by Lemma 1(2),

$$H = \chi_{P_1 H}^{\mathcal{N}(T,n)} = \chi_{P_1 H}^{\mathcal{A}(T,n)} \subset \chi_{h_1}^{\mathcal{A}(T,n)} \subset H.$$

Therefore for every two non-zero elements $f, g \in H$, there are two sequences $\{B_m\}$, $\{C_m\}$ in $\mathcal{A}(T, m)$ such that

$$f = \lim_{m\to\infty} B_m h_1,\ g = \lim_{m\to\infty} C_m h_1 \ \text{ and } f \otimes g = \lim_{m\to\infty} B_m (h_1 \otimes h_1) C_m^* \in \mathcal{J}_n(T)$$

with operator norm. This indicates that $K(H) = \mathcal{J}_n(T)$ and hance T is irreducible.

(1)$\Rightarrow$(3). Suppose that for each finite subset $\mathcal{F}$ of $\{\overline{T}_{st}\,|0 \le s < t\}$, $C^*(\mathcal{F}) \ne L(P_1 H)$. Since $\dim L(P_1 H) = n^2$, we can choose a subset $\mathcal{F}_k$ of $\{\overline{T}_{st}\,|0 \le s < t\}$ such that every element in $\{\overline{T}_{st}\,|0 \le s < t\}$ is a linear combination of the elements in $\mathcal{F}_k$ and the elements in $\mathcal{F}_k$ are linear independent. Since $C^*(\mathcal{F}_k) \ne L(P_1 H)$ by assumption, it follows that the commutator of $C^*(\mathcal{F}_k)$ is not equal to $\{\lambda P_1 \,|\, \lambda \in \mathbb{C}\}$. Thus there is an orthogonal projection Q in $L(P_1 H)$ such that $Q < P_1$ and Q commutes with every element in $C^*(\mathcal{F}_k)$. Consequently, $Q\overline{T}_{st} = \overline{T}_{st} Q$, $0 \le s < t$. Put $\overline{Q} = \bigoplus_{k=0}^{\infty} Q$. Then $\overline{Q}$ commutes with WTW^* by $(**)$ which is contrary to the hypothesis that T is irreducible.

Since the remaining implications (3)$\Rightarrow$(2), (4)$\Rightarrow$(2) and (3)$\Rightarrow$(4) are obivous, we have finished the proof.

Corollary 1. *Let* $T \in B_n(\Omega)\,(n \ge 2)$ *and* $\{P_k\}_0^\infty$, $\{T_{st}\,|\,0 \le s < t\}$ *be as in Theorem* 1. *If* T *satisfies one of the following conditions, then* T *is irreducible.*

(1) *There are* $0 \le s_0 < t_0$ *such that* $T_{s_0 t_0}$ *is irreducible;*

(2) *There is an* $m \ge 2$ *such that* $P_1 T P_{m+1} T^{*m}|_{P_1 H}$ *is irreducible.*

Notice that Corollary 1 generalizes Lemma 2.21 of [5].

Corollary 2. *Let* $T \in B_2(\Omega)$. *Then* T *is irreducible iff for each* ONB $\{f_1, f_2\}$ *of* $\operatorname{Ker} T$, *there are* $k_1, k_2 \ge 0$ *such that* $(B_1^{k_1} f_1, B_1^{k_2} f_2) \ne 0$.

Proof. The "if" part comes directly from Theorem 1(2). We now prove the "only if" part.

Suppose that there is an ONB $\{f_1, f_2\}$ of $\operatorname{Ker} T$ such that $(B_1^{k_1} f_1, B_1^{k_2} f_2) = 0$ for all $k_1, k_2 \ge 0$. Then two closed subspaces $Z_1 = \bigvee_{k=0}^{\infty} \{B_1^k f_1\}$ and $Z_2 = \bigvee_{k=0}^{\infty} \{B_1^k f_2\}$ are mutually orthogonal. Since $\{f_1, f_2, B_1 f_1, B_1 f_2, \cdots, B_1^{k-1} f_1, B_1^{k-1} f_2\}$ is a basis of $\operatorname{Ker} T^k$ and $\bigvee_{k=1}^{\infty} \operatorname{Ker} T^k = H$, we have $H = Z_1 \oplus Z_2$. Therefore $TZ_i \subset Z_i$, $i = 1, 2$ implies that T is reducible, which contradicts the assumptiom that T is irreducible.

References

[1] Arveson, W., An invitation to C^*-algebra (1978), Springer-Verlag.

[2] Cowen, M. J. & Douglas, R. G., Complex geometry and operator theory, *Acta Math.,* **141** (1978), 187–261.

[3] Chen, G., Wei, M. & Xue, Y., Perturbation analysis of the least solution in Hilbert spaces, *Linear Algebra Appl.,* **244**(1996), 69–80.

[4] Jiang, C. & Li, J., The reducible decompositions of Cowen-Douglas operators and wieghted shift operators, preprint.

[5] Jiang, C. & Wang, Z., Strongly irreducible operators on Hiblert spaces, π Pitman Research Notes in Mathematics Series 389, Addison-Wesley-Longman Company, 1998.

[6] Pederson, G. K., C^*-algebras and their automorphism groups, Academic Press, 1979.

[7] Xue, Y., Some peoperties of the C^*-algebras generated by $B_n(\Omega)$ class operators, *Chin. Math. of Ann.,* **10A:**4(1989).

[8] Xue, Y., On AK_0-C^*-algebras (I), *J. East China Normal Univ.,* **1**(1990), 7–12.