A C*-ALGEBRA APPROACH TO THE IRREDUCIBILITY OF COWEN-DOUGLAS OPERATORS**

XUE YIFENG* WANG ZONGYAO*

Abstract

The authors consider the irreducibility of the Cowen-Douglas operator $T$. It is proved that $T$ is irreducible if the unital C*-algebra generated by some non-zero blocks in the decomposition of $T$ with respect to $\bigoplus_{n=0}^{\infty} (\text{Ker } T^{n+1} \oplus \text{Ker } T^n)$ is $M_n(\mathbb{C})$.

Keywords C*-algebras, Cowen-Douglas operators, Irreducible operators, Pojections

1991 MR Subject Classification 47B, 46L
Chinese Library Classification O177.1

Let $H$ be a separable infinite dimensional Hilbert space over the field $\mathbb{C}$ and let $L(H)$ (resp. $K(H)$) denote the algebra of all bounded linear operators (resp. all compact operators) on $H$. For $T \in L(H)$, we denote by $R(T)$ (resp. Ker $T$) the range (resp. the null space) of $T$. According to [2], $B_n(\Omega)$ consists of all Cowen-Douglas operators $T \in L(H)$ satisfying the following conditions:

1. $\Omega \subset \sigma(T)$ is a connected open subset in $\mathbb{C}$;
2. $R(\omega - T) = H$ and $\dim \text{Ker } (\omega - T) = n < \infty$, $\forall \omega \in \Omega$;
3. $H = \bigvee_{\omega \in \Omega} \text{Ker } (\omega - T)$.

We have seen from [5] that the Cowen-Douglas operators play a very important role in studying the strong irreducibility of operators. But we feel very regretful that the conditions which make $T \in B_n(\Omega)$ strongly irreducible are unknown till now except $n = 1$. Considering the close relationship between the strong irreducibility and irreducibility of operators, that is, $T \in L(H)$ is strongly irreducible if $XTX^{-1}$ is irreducible for every invertible operator $X \in L(H)$, we will display various equivalent conditions (see Theorem 1) that make a Cowen-Douglas operator irreducible. These will lead us to find some irreducible conditions which are invariant under similar transformation. On the other hand, since the first author has proven that every Cowen-Douglas operator can be decomposed as the direct sum of finite number of irreducible Cowen-Douglas operators (see [7, Theorem 2]), we could only investigate the properties of irreducible Cowen-Douglas operators instead of general Cowen-Douglas operators.


*Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China.

**Project supported by the National Natural Science Foundation of China.
After we finished the paper, we received Jiang and Li's paper (see [4]). They showed that $T \in B_n(\Omega)$ is irreducible iff the unital $C^*$-algebra generated by all non-zero blocks in the decomposition of $T$ with respect to $\bigoplus_{n=0}^{\infty} (\text{Ker} T^{n+1} \ominus \text{Ker} T^n)$ is $M_n(\mathbb{C})$. This result is contained in Theorem 3(4) and, moreover, the methods we used are different.

According to [2, 1.7], we may assume that $0 \in \Omega$ and $\bigvee_{k=1}^{n} \text{Ker} T^{k} = H \forall T \in B_n(\Omega)$. For any $T \in B_n(\Omega)$ let $P_k: H \to \text{Ker} T^k$ be the orthogonal projection $k = 1, \cdots, n$ and $P_0 = 0$. Thus $P_k$ converges to $I_H$ strongly and $P_k \leq P_{k+1}$. Consequently, $\sum_{k=0}^{\infty} (P_{k+1} - P_k) = I_H$.

We denote by $\mathcal{A}(T, n)$ (resp. $\mathcal{N}(T, n)$) the $C^*$-algebra (resp. the Von Neumann algebra) generated by $T \in B_n(\Omega)$ and $I_H$. Note that $R(T^k) = H$, $\forall k \geq 1$ for $T \in B_n(\Omega)$ indicates that $T^kT^k$ is invertible in $\mathcal{A}(T, n)$. Set $B_k = T^kT^k - I_H \in \mathcal{A}(T, n)$. Then $T^kB_k = I_H$ and $B_kT^k = I_H - P_k \in \mathcal{A}(T, n)$, i.e., $B_k$ is the generalized inverse of $T^k$ (see [3]).

Let $\lambda_{P_H}^{\mathcal{N}(T, n)} = \text{spur}\{ Ax | A \in \mathcal{N}(T, n), x \in PH \}$.

By [6, Lemma 2.6.3], $c(P) = \bigvee u^* Pa$, where $u$ is the unitary element in $\mathcal{N}(T, n)$. The following lemma describes explicit relationship among $P_0, P_1, \cdots$.

**Lemma 1.** Let $T \in B_n(\Omega)$ and $\{P_k\}^\infty_{k=0}$ be as above. Then

1. There is a sequence of partial isometries $\{v_k\}^\infty_{k=0}$ in $\mathcal{A}(T, n)$ such that $v_kv_k^* = P_1, v_kv_k = P_{k+1} - P_k, \forall k \geq 0$;
2. $P_k \in \mathcal{J}_n(T), \forall k \geq 0$ and $c(P_1) = I_H$.

**Proof.** (1) Set $S_k = T^kP_{k+1}$. Then by the identity $P_kTP_{k+1} = TP_{k+1}, R(S_k) \subset P_kH$. Since $R(T^k) = H, \forall k \geq 1$, it follows that for each $x \in \text{Ker} T$, there is a $y \in H$ such that $x = T^ky$. Thus $T^{k+1}y = Tx = 0$, i.e., $y \in \text{Ker} T^{k+1}$. Therefore $x \in R(S_k)$ and hence $R(S_k) = P_1H$.

From $\text{Ker} S_k = \{ x \in H | (P_{k+1} - P_k)x = 0 \}$, we have $R(S_k^*) = (P_{k+1} - P_k)H$. Let $S_k = v_k|S_k|$ be the polar decomposition of $S_k$. Then $v_k \in \mathcal{A}(T, n)$ by [7, Lemma 1] and $v_kv_k^* = P_1, v_k^*v_k = P_{k+1} - P_k$ for $k \geq 1$ and $v_0 = P_1$.

(2) By assertion (1), $\dim P_kH = nk, k \geq 1$. Thus $P_k \in \mathcal{J}_n(T)$. Noting that $P_1(P_{k+1} - P_k) = 0, \forall k \geq 1$, we can define a unitary element $u_k$ in $\mathcal{A}(T, n)$ by $u_k = v_k + v_k^* + I_H - P_1 - (P_{k+1} - P_k), k \geq 1$ and $u_0 = I_H$. Simple computation shows that $P_{k+1} - P_k = u_k^*P_ku_k, k \geq 1$ and hence $I_H = \sum_{k=0}^{\infty} u_k^*P_ku_k$. So $c(P_1) = I_H$.

For $T \in B_n(\Omega)$, let $\{P_k\}^\infty_{k=0}$ be as above. Set $T_{st} = (P_{s+1} - P_s)T(P_{t+1} - P_t)$ and $B_{st} = (P_{s+1} - P_s)B_1(P_{t+1} - P_t)$. Since $P_kTP_k = TTP_k$ and $P_{k+1}B_1P_k = B_1P_k$ for $k \geq 0$, it follows that $T_{st} = 0, s \geq t$ and

$$T_{k,k+1}B_{k+1,k} = P_{k+1} - P_k, B_{k+1,k}T_{k,k+1} = P_{k+2} - P_{k+1}.$$  \hspace{1cm} (1)

Now let $\{W_k\}^\infty_{k=0}$ be a sequence of partial isometries in $\mathcal{A}(T, n)$ such that $W_k^*W_k = P_{k+1} - P_k, W_kW_k^* = P_1$ ($W_k$ is not necessarily equal to $v_k$ in Lemma 1 (1)). Set $T_{st} = W_sT_{st}W_t$. Then $T_{st}$ can be viewed as an operator in $L(P_kH) \cong M_n(\mathbb{C})$ and, moreover, $T_{k,k+1}$ is invertible for each $k \geq 0$ by (1). Thus if we identify $H$ with $\bigoplus_{i=0}^{\infty} (P_{k+1} - P_k)H$, then $W = \cdots$
\[ W_k \] is an isometry of \( H \) onto \( \bigoplus_{k=0}^{\infty} P_k H \) such that

\[
WTW^* = \begin{pmatrix}
0 & T_{01} & T_{02} & \cdots \\
0 & 0 & T_{12} & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

(\(*\))

According to [5, Lemma 2.20], we can choose a sequence \( \{W_k\}_0^\infty \) of partial isometries in \( \mathcal{A}(T,n) \) such that \( T_{k,k+1} > 0, \; k \geq 0 \).

Recall that an orthogonal projection \( P \) in \( \mathcal{A}(T,n) \) for \( T \in B_n(\Omega) \) is called minimal if for any \( a \in \mathcal{A}(T,n) \) there exists a \( \lambda \in \mathbb{C} \) such that \( PA^*P = \lambda P \) (see [1, 1.4.1]). The next lemma comes from [1, Lemma 1.4.1] or [8, Proposition 1.2].

**Lemma 2.** Let \( T, \mathcal{J}_n(T) \) be as above. Then for each orthogonal projection \( P \) in \( \mathcal{J}_n(T) \) there are minimal projections \( e_1, \ldots, e_r (r < \infty) \) in \( \mathcal{J}_n(T) \) such that \( e_i e_j = 0, \; i \neq j \) and

\[
p = \sum_{i=1}^{r} e_i.
\]

Let \( F \) be a subset of \( \{T_{st} | 0 \leq s < t\} \). We denote by \( C^*(F) \) the \( C^* \)-algebra generated by the operators in \( F \) and \( P_1 \) (the unit of \( L(P_1 H) \)).

The following theorem is our main result of the paper.

**Theorem 1.** Let \( T \in B_n(\Omega) \) \( (n \geq 2) \) and \( T_{st} \) \( (0 \leq s < t) \) be as above. Then following statements are equivalent:

1. \( T \) is irreducible;
2. For any orthonormal basis ONB \( \{f_i\}_1^n \) of \( \text{Ker} T \), there is a sequence \( \{A_{ij}\}_1^n \) in \( \mathcal{A}(T,n) \) such that \( A_{ij} f_i, f_j \neq 0, \; i \neq j \);
3. There is a finite subset \( F \) of \( \{T_{st} | 0 \leq s < t\} \) such that \( C^*(F) = L(P_1 H) \);
4. \( C^*(\{T_{st} | 0 \leq s < t\}) = L(P_1 H) \).

**Proof.** (1)\(\Rightarrow\)(2). Since \( \mathcal{A}(T,n) \) is strongly dense in \( \mathcal{N}(T,n) \) and since the irreducibility of \( T \) is the same as that of \( \mathcal{A}(T,n) \), it follows from [6, Theorem 3.14.3] that \( f_j \in \chi_{A_{ij}}^{T,\mathcal{A}(T,n)}, \; i \neq j \). So there is a sequence \( \{A_{ij}\}_1^{\infty} \in \mathcal{A}(T,n) \) such that \( f_j = \lim_{k \to \infty} A_{ij}^{(k)} f_i, \; i \neq j \) and \( \lim_{k \to \infty} (A_{ij}^{(k)} f_i, f_j) = 1 \). Thus we can pick \( A_{ij} = A_{ij}^{(k_0)} \) for \( k_0 \) large enough such that \( (A_{ij} f_i, f_j) \neq 0, \; i \neq j \).

(2)\(\Rightarrow\)(1). Let \( \{f_i\}_1^n \) be an ONB of \( \text{Ker} T \). Let \( \{f_{n_i}\}_1^n \) be a subset of \( \{f_i\}_1^n \), \( 2 \leq k \leq n \) and set \( Q_k = \sum_{i=1}^{k} f_{n_i} \otimes f_{n_i}, \; \text{where} \; f \otimes g \in K(H) \) is defined by \( (f \otimes g)x = (x, f)g, \; f, g, x \in H \).

Then \( Q_k \) is an orthogonal projection majoring by \( P_1 \).

Since there is an \( A_{n_i,n_j} \in \mathcal{A}(T,n) \) such that \( (A_{n_i,n_j} f_{n_i}, f_{n_j}) \neq 0 (i \neq j, \; 1 \leq i, j \leq k) \), we get that \( Q_k A_{n_i,n_j} W Q_k \neq 0 \) for each \( \lambda \in \mathbb{C} \). Consequently, by Lemma 2, there exist minimal projections \( e_1, \ldots, e_r \) in \( \mathcal{J}_n(T)(r < \infty) \) such that \( e_i e_j = 0, \; i \neq 0 \) and \( P_1 = \sum_{i=1}^{r} e_i \).

Using the same argument as above to \( e_1, \ldots, e_r \), we conclude that \( e_i = h_i \otimes h_i \) and \( r = n \).

Otherwise, \( e_i \) will not be minimal. So \( P_1 = \sum_{i=1}^{n} h_i \otimes h_i \).

For above ONB \( \{h_i\}_1^n \) of \( \text{Ker} T \), there is a sequence \( \{A_{j-1}\}_1^n \in \mathcal{A}(T,n) \) such that \( (A_{j-1} h_1, h_j) \neq 0, \; j = 2, \ldots, n \). Put

\[
C_j = \frac{1}{(A_{j-1} h_1, h_j)} (h_j \otimes h_j) A_{j-1} (h_1 \otimes h_1) \in \mathcal{J}_n(T).
\]
Then $C_j h_1 = h_j$, $j = 2, \cdots, n$. Thus by Lemma 1(2),

$$H = \chi_{F/H}^T \subset A(T, n) \subset H.$$  

Therefore for every two non-zero elements $f, g \in H$, there are two sequences $\{B_m\}, \{C_m\}$ in $A(T, m)$ such that

$$f = \lim_{m \to \infty} B_m h_1, \quad g = \lim_{m \to \infty} C_m h_1 \quad \text{and} \quad f \otimes g = \lim_{m \to \infty} B_m (h_1 \otimes h_1) C_m \in J_n(T)$$

with operator norm. This indicates that $K(H) = J_n(T)$ and hence $T$ is irreducible.

(1)$\Rightarrow$(3). Suppose that for each finite subset $F$ of $\{T_{st} \mid 0 \leq s < t\}$, $C^*(F) \neq L(P_1 H)$. Since $\dim L(P_1 H) = n^2$, we can choose a subset $F_k$ of $\{T_{st} \mid 0 \leq s < t\}$ such that every element in $\{T_{st} \mid 0 \leq s < t\}$ is a linear combination of the elements in $F_k$ and the elements in $F_k$ are linear independent. Since $C^*(F_k) \neq L(P_1 H)$ by assumption, it follows that the commutator of $C^*(F_k)$ is not equal to $\{\lambda P_1 \mid \lambda \in \mathbb{C}\}$. Thus there is an orthogonal projection $Q$ in $L(P_1 H)$ such that $Q < P_1$ and $Q$ commutes with every element in $C^*(F_k)$.

Consequently, $Q T_{st} = T_{st} Q$, $0 \leq s < t$. Put $\overline{Q} = \bigoplus_{k=0}^{\infty} Q$. Then $\overline{Q}$ commutes with $WTW^*$ by (**) which is contrary to the hypothesis that $T$ is irreducible.

Since the remaining implications (3)$\Rightarrow$(2), (4)$\Rightarrow$(2) and (3)$\Rightarrow$(4) are obvious, we have finished the proof.

**Corollary 1.** Let $T \in B_0(\Omega)$ $(n \geq 2)$ and $\{P_k\}_{\infty}^{\infty}$, $\{T_{st} \mid 0 \leq s < t\}$ be as in Theorem 1. If $T$ satisfies one of the following conditions, then $T$ is irreducible.

1. There are $0 \leq s_0 < t_0$ such that $T_{st_0}$ is irreducible;
2. There is an $m \geq 2$ such that $P_1 T P_{m+1} T^{*m} | P_1 H$ is irreducible.

Notice that Corollary 1 generalizes Lemma 2.21 of [5].

**Corollary 2.** Let $T \in B_2(\Omega)$. Then $T$ is irreducible iff for each ONB $\{f_1, f_2\}$ of Ker $T$, there are $k_1, k_2 \geq 0$ such that $(B_k^{k_1} f_1, B_k^{k_2} f_2) \neq 0$.

**Proof.** The “if” part comes directly from Theorem 1(2). We now prove the “only if” part.

Suppose that there is an ONB $\{f_1, f_2\}$ of Ker $T$ such that $(B_k^{k_1} f_1, B_k^{k_2} f_2) = 0$ for all $k_1, k_2 \geq 0$. Then two closed subspaces $Z_1 = \bigvee_{k=0}^{\infty} (B_k^{k_1} f_1)$ and $Z_2 = \bigvee_{k=0}^{\infty} (B_k^{k_2} f_2)$ are mutually orthogonal. Since $\{f_1, f_2, B_1 f_1, B_1 f_2, \cdots, B_k^{k_1-1} f_1, B_k^{k_2-1} f_2\}$ is a basis of Ker $T_k$ and $\bigvee_{k=1}^{\infty} \text{Ker} T_k = H$, we have $H = Z_1 \oplus Z_2$. Therefore $TZ_i \subset Z_i$, $i = 1, 2$ implies that $T$ is reducible, which contradicts the assumption that $T$ is irreducible.

**References**


