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A CRITERION OF ESSENTIALLY COMMUTING TOEPLITZ OPERATORS ON BERGMAN SPACE

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Abstract

On the setting of the unit ball U the author considers Toeplitz operators on Bergman space. The Bergman space $B^p(U)$ $(1 \le p < \infty)$ is the closed subspace of the usual Lebesgue space $L^p(U)$ consisting of holomorphic functions. For a function $\beta \in L^2(U)$, the Toeplitz operator T_β with symbol β is defined by $T_\beta f = \wp(\beta f)$ for function $f \in B^2(U)$. Here $\wp$ is the Bergman projection from $L^2(U)$ onto $B^2(U)$. Two bounded linear operators S, T on the Hilbert H are said to be essentially commuting on H if the commutator $ST - TS$ is compact on H. In this paper, a criterion of essentially Toeplitz operators with the vanishing property is obtained.

Keywords Bergman space, Toeplitz operator, Unit ball
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§1. Introduction

Let m denote the volume measure on the unit ball U normalized to have total mass 1 of the n-dimensional complex space $\mathbf{C}^n$. The Bergman space $B^p(U)$ $(1 \le p < \infty)$ is the closed subspace of the usual Lebesgue space

$$L^p(U) = L^p(U, m)$$

consisting of holomorphic functions. Let $\wp$ be the Hilbert space orthogonal projection. Then

$$\wp : L^2(U) \to B^2(U)$$

is called the Bergman projection. It is well-known that the Bergman projection $\wp$ is given by

$$\wp(\psi)(z) = \int_U \frac{\psi(w)}{(1 - z \cdot \bar{w})^{n+1}} dm(w) \tag{1.1}$$

for $z \in U$ and $\psi \in L^2(U)$. Here the notation

$$z \cdot \bar{w} = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$$

denotes the ordinary Hermitian inner product for points $z, w \in \mathbf{C}^n$.

For a function $\beta \in L^2(U)$, the Toeplitz operator T_β with symbol β is defined by

$$T_\beta f = \wp(\beta f)$$

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for function $f \in B^2(U)$. The operator $T_\beta : B^2(U) \to B^2(U)$ is densely defined and not bounded in general. However, T_β is always bounded for symbols β which we are connerned about in this paper.

We say that two bounded linear operators S, T on a Hilbert space H are essentially commuting on H if the commutator $ST - TS$ is compact on H.

In this paper, we obtain a criterion of essentially commuting Toeplitz operators with the vanishing property. In one dimensional case, K. Stoethoff[3] has obtained a complete description of two harmonic symbols for essentially Toeplitz operators. And Axler and Cucković[1] have obtained a complete description of harmonic symbols of commuting Toeplitz operators. Trying to generalize this characterization to the ball, one may naturally think of pluriharmonic symbols. In this paper, we consider the same characterizing problem on the ball with pluriharmonic symbols.

For $a \in U$, let φ_a denote the canonical automorphism (see Section 2) of U. We let

$$\Phi = \overline{\{\varphi_a : a \in U\}} - \{\varphi_a : a \in U\}.$$

Here the set $\bar{A}$ denotes the closure of the set A. We know[4, Proposition 9] that if a net $\{\varphi_{a_i}\}$ of automorphisms converges to some $\varphi \in \Phi$, then for any bounded pluriharmonic function β, the function $\beta \circ \varphi_{a_i}$ converges to $\beta \circ \varphi$ uniformly on every compact subset of U. Hence $\beta \circ \varphi$ is also a bounded pluriharmonic function on U. For some more information, we refer the reader to [4].

For $f \in C^2(U)$, the invariant Laplacian $\tilde{\Delta} f$ is defined by

$$(\tilde{\Delta} f)(a) = \Delta(f \circ \varphi_a)(0)$$

for $a \in U$. Here

$$\Delta = 4 \sum_{j=1}^{n} \partial^2 / \partial z_j \partial \bar{z}_j$$

is the usual Laplacian. The operator $\tilde{\Delta}$ commutes with automorphisms in the sense that

$$\tilde{\Delta}(f \circ \varphi) = (\tilde{\Delta} f) \circ \varphi$$

for all automorphisms φ of U (see [2, Chapter 4]).

In Section 2, we collect some results on Toeplitz operators which we need in the proof of Main Theorem. In Section 3, we obtain a basic lemma. In Section 4, we give a characterization of essentially commuting Toeplitz operators (Main theorem).

§2. Basic Facts

For $a \in U,\ a \neq 0$, the explict formula for the canonical automorphism φ_a is given by

$$\varphi_a(b) = \frac{a - |a|^{-2}(b \cdot \bar{a})a - \sqrt{1 - |a|^2}[b - |a|^{-2}(b \cdot \bar{a})a]}{1 - b \cdot \bar{a}}$$

for $b \in U$. Here

$$\varphi_0(b) = -b, \varphi_a(0) = a \ \text{ and } \ \varphi_a(a) = 0.$$

It is well known that $\varphi_a \circ \varphi_a$ is the identity on U. The real Jacobian $J_R \varphi_a$ of φ_a is given by

$$J_R \varphi_a(b) = \left\{ \frac{1 - |a|^2}{|1 - b \cdot \bar{a}|^2} \right\}^{n+1} \tag{2.1}$$

for $b \in U$. In addition, the identity

$$1 - \varphi_a(z) \cdot \overline{\varphi_a(w)} = \frac{(1-|a|^2)(1 - z \cdot \bar{w})}{(1 - z \cdot \bar{a})(1 - a \cdot \bar{w})} \tag{2.2}$$

holds for every $z, w \in U$ (see [2, Chapter 2] for details).

For $a \in U$, we put

$$k_a(z) = \left\{ \frac{\sqrt{1-|a|^2}}{1 - z \cdot \bar{a})} \right\}$$

for $z \in U$. By (2.1) and (2.2), we have a useful change-of-variable formula

$$\int_U h dm = \int_U h \circ \varphi_a |k_a|^2 dm \tag{2.3}$$

for $a \in U$ and for all measurable h on U.

§3. Basic Lemma

In this section, we collect some results which we need in the proof of Main theorem (Section 4). Now we recall the well-known Block space and Hankel operators. The Block space B is the space of all holomorphic functions f on U for which

$$\|f\| = \sup_{w \in U} (1 - |w|^2) |\nabla f(w)| < \infty,$$

where ∇f is the complex gradient of f. Note that $B \subset B^p(U)$ for all $p < \infty$. Moreover, it turns out that, for $1 \leq p < \infty$, there is a positive constant A_p such that

$$A_p^{-1} \|f\| \leq \sup_{a \in U} \|f \circ \varphi_a - f(a)\|_p \leq A_p \|f\| \tag{3.1}$$

for all holomorphic functions f on U (see [3]). Here $\| \, \|_p$ denotes the usual p-norm on $L^p(U)$.

For a function $\beta \in L^2(U)$, the Hankel operator $H_\beta : B^2(U) \to B^2(U)^\perp$ with symbol β is defined by

$$H_\beta f = \beta f - \wp(\beta f)$$

for functions $f \in B^2(U)$. Here $B^2(U)^\perp$ is the orthogonal complement of $B^2(U)$.

There is a connection between Toeplitz operators and Hankel operators. We have the following formula

$$T_\beta T_\delta - T_\delta T_\beta = H_{\bar{\delta}}^* B_\beta - H_{\bar{\beta}}^* H_\delta,$$

which can be verified (see [3, Proposition 6]). Here T_β^* is the adjoint operator of T_β. Note that Hankel operators with holomorphic symbols are the zero operator. Let

$$\beta = f + \bar{g} \quad \text{and} \quad \delta = h + \bar{k}$$

be bounded pluriharmonic symbols for some holomorphic functions f, g, h, k on U. The above formula yields

$$T_\beta T_\delta - T_\delta T_\beta = H_{\bar{h}}^* H_{\bar{g}} - H_{\bar{f}}^* H_{\bar{k}}. \tag{3.2}$$

Recall that if β is a bounded pluriharmonic function and if $\{\varphi_{w_i}\}$ is a net such that $\varphi_{w_i} \to \varphi$ for some $\varphi \in \Phi$, then $\beta \circ \varphi_{w_i} \to \beta \circ \varphi$ uniformly on every compact subset of U and thus $\beta \circ \varphi$ is also a bounded pluriharmonic function.

Lemma 3.1. *Let β, δ be two bounded pluriharmonic functions on U and assume*

$$\beta = f + \bar{g}, \quad \delta = h + \bar{k}$$

for some holomorphic functions f, g, h, on U. And let $\{\varphi_{w_i}\}$ be a net such that

$$\varphi_{w_i} \to \varphi \in \Phi.$$

If

$$\beta \circ \varphi = F + \overline{G}, \quad \delta \circ \varphi = V + \overline{W},$$

where F, G, V, W are holomorphic functions on U, then

$$\tilde{\Delta}[f\bar{k}] \circ \varphi_{w_i} \to \tilde{\Delta}[F\overline{W}] \quad and \quad \tilde{\Delta}[h\bar{g}] \circ \varphi_{w_i} \to \tilde{\Delta}[V\overline{G}].$$

Proof. Put $f_i = f \circ \varphi_{w_i} - f(w_i)$ and $k_i = k \circ \varphi_{w_i} - k(w_i)$. Since f_i, k_i are holomorphic, we have

$$\tilde{\Delta}(f_i\bar{k}_i)(z) = 4(1-|z|^2)[\nabla f_i \cdot \overline{\nabla k_i} - Rf_i\overline{Rk_i}] \tag{3.3}$$

for $z \in U$ by [2, Proposition 4.1.3]. Here R denotes the radial differention. Note that

$$\beta \circ \varphi_{w_i} - \beta(w_i) \to \beta \circ \varphi - \beta \circ \varphi(0)$$

uniformly on every compact subset of U. In particular, since β and δ are bounded,

$$\beta \circ \varphi_{w_i} - \beta(w_i) \to \beta \circ \varphi - \beta \circ \varphi(0)$$

in $L^2(U)$. Now, using the $L^2(U)$-boundedness of the Bergman projection $\wp$, we have

$$\wp[\beta \circ \varphi_{w_i} - \beta(w_i)] \to \wp[\beta \circ \varphi - \beta \circ \varphi(0)]$$

in $L^2(U)$. Note $\wp(\bar{\varphi}) = \bar{\varphi}(0)$ for every $\varphi \in B^2(U)$. It follows that

$$\wp[\beta \circ \varphi_{w_i} - \beta(w_i)] = f_i$$

and

$$\wp[\beta \circ \varphi - \beta \circ \varphi(0)] = F - F(0).$$

Hence, $f_i \to F - F(0)$ in $L^2(U)$. Therefore, we have $f_i \to F - F(0)$ uniformly on every compact subset of U and therefore $\nabla f_i \to \nabla F$ and $Rf \to RF$ uniformly on every compact subset of U. Applying the same method to $\bar{\delta}$, we know that the same is true for k_i. Then, by taking the limit in (3.3), we have

$$\tilde{\Delta}(f_i\bar{k}_i) \to \tilde{\Delta}(F\overline{W}). \tag{3.4}$$

On the other hand, since $\tilde{\Delta}$ annihilates (anti)holomorphic functions and commutes with automorphisms, we have

$$\begin{aligned}\tilde{\Delta}(f_i\bar{k}_i) &= \tilde{\Delta}[(f \circ \varphi_{w_i} - f(w_i))(\overline{k \circ \varphi_{w_i} - k(w_i)})]\\ &= \tilde{\Delta}[f \circ \varphi_{w_i}\overline{k \circ \varphi_{w_i}}]\\ &= \tilde{\Delta}[f\bar{k} \circ \varphi_{w_i}].\end{aligned}$$

By (3.4), $\tilde{\Delta}[f\bar{k}] \circ \varphi_{w_i} \to \tilde{\Delta}(F\overline{W})$. Similarly,

$$\tilde{\Delta}[h\bar{g}] \circ \varphi_{w_i} \to \tilde{\Delta}(V\overline{G}).$$

We complete the proof.

§4. Main Theorem

For $a \in U$, we let K_a be the Bergman kernel given by

$$K_a(z) = (1 - z \cdot \bar{a})^{-(n+1)}$$

for $z \in U$. Then, By (1.1), we have the following reproducing property

$$F(a) = \langle F, K_a \rangle \tag{4.1}$$

for every $F \in B^2(U)$. Here $\langle \ , \ \rangle$ denotes the usual inner product in $L^2(U)$. In this paper, the same letter A will denote various positive constant which may change from one occurrence to the next. Our Main result is

Theorem 4.1. *Let β, δ be two bounded pluriharmonic functions on U and assume*

$$\beta = f + \bar{g}, \quad \delta = h + \bar{k}$$

for some functions f, g, h, k holomorphic on U. Suppose that

$$\lim_{|a|\to 1} \int_U |(f \circ \varphi_a - f(a)(\bar{k} \circ \varphi_a - \bar{k}(a)) - (h \circ \varphi_a - h(a))(\bar{g} \circ \varphi_a - \bar{g}(a))| dm = 0.$$

Then T_β and T_δ are essentially commuting on $B^2(U)$.

Proof. By (3.2), it is sufficient to show the compactness of

$$H^*_{\bar{f}} H_{\bar{k}} - H^*_{\bar{h}} H_{\bar{g}}.$$

We put

$$Q(z, a) = (f(z) - f(a))(\bar{k}(z) - \bar{k}(a)) - (h(z) - h(a))(\overline{g(z)} - \overline{g(a)})$$

for $z, a \in U$. Then, by the Cauchy-Schwarz inequality and (3.1), we can see that for each $1 \le p < \infty$,

$$\sup_{a\in U} \int_U |Q(\varphi_a(z), a)|^p dm \le A(\|f\|^p \|k\|^p + \|h\|^p \|g\|^p) < \infty \tag{4.2}$$

for some constant $A = A(p)$. The last inequality follows from the fact that f, g, h and k are all in B. By assumption, we have

$$\lim_{|a|\to 1} \int_U |Q(\varphi_a(z), a)| dm(z) = 0. \tag{4.3}$$

By the reproducing property (4.1), it is easy to see that

$$\wp(\overline{F} K_a) = \overline{F}(a) K_a$$

for every $F \in B^2(U)$. Let $\psi \in B^2(U)$ and pick a point $a \in U$. It follows from (4.1) that

$$\begin{aligned} H^*_{\bar{f}} H_{\bar{k}} \psi(a) &= \langle H^*_{\bar{f}} H_{\bar{k}} \psi, K_a \rangle = \langle H_{\bar{k}} \psi, H_{\bar{f}} K_a \rangle \\ &= \langle \bar{k}\psi - \wp(\bar{k}\psi, \bar{f} K_a - \wp(\bar{f} K_a) \rangle \\ &= \langle \bar{k}\psi, (\bar{f} - \bar{f}(a)) K_a \rangle. \end{aligned}$$

By (4.1), we obtain

$$\langle \psi, (\bar{f} - \bar{f}(a)) K_a \rangle = \langle \psi(f - f(a)), K_a \rangle = 0.$$

It follows that

$$\begin{aligned} H^*_{\bar{f}} H_{\bar{k}} \psi(a) &= \langle \bar{k} - \bar{k}(a))) \psi, (\bar{f} - \bar{f}(a)) K_a \rangle \\ &= \int_U \frac{(f(z) - f(a))(\bar{k}(z) - \bar{k}(a))}{(1 - a \cdot \bar{z})^{n+1}} \psi(z) dm(z). \end{aligned}$$

Similarly, we have

$$H^*_{\bar{h}} H_{\bar{g}} \psi(a) = \int_U \frac{(h(z) - h(a))(\bar{g}(z) - \bar{g}(a))}{(1 - a \cdot \bar{z})^{n+1}} \psi(z) dm(z).$$

Hence, we can represent

$$H_{\bar f}^* H_{\bar k}\psi(a) - H_{\bar h}^* H_{\bar g}\psi(a)$$

as an integral operator as follows

$$H_{\bar f}^* H_{\bar k}\psi(a) - H_{\bar h}^* H_{\bar g}\psi(a) = \int_U \frac{Q(z,a)}{(1-a\cdot\bar z)^{n+1}}\psi(z)dm(z).$$

For each $r\in(0,1)$, define $E_r : B^2(U)\to L^2(U)$ by

$$E_r\psi(a) = \chi_{rU}(a)\int_U \frac{Q(z,a)}{(1-a\cdot\bar z)^{n+1}}\psi(z)dm(z),$$

where the notation χ_{rU} denotes the usual characteristic function for $rU\subset U$. We show that each E_r is compact. It is sufficient to see that its kernel function is in $L^2(U\times U)$. That is,

$$\int_U\int_U \Big|\frac{\chi_{rU}(a)Q(z,a)}{(1-a\cdot\bar z)^{n+1}}\Big|^2 dm(z)dm(a) < \infty. \tag{4.4}$$

By (2.3), we have

$$\begin{aligned}
&\int_U\int_U \Big|\frac{\chi_{rU}(a)Q(z,a)}{(1-a\cdot\bar z)^{n+1}}\Big|^2 dm(z)dm(a)\\
&=\int_{rU}\int_U \frac{|Q(z,a)|^2|k_a(z)|^2}{(1-|a|^2)^{n+1}}dm(z)dm(a)\\
&=\int_{rU}\int_U \frac{|Q(\varphi_a(z),a)|^2}{(1-|a|^2)^{n+1}}dm(z)dm(a)\\
&\le \frac{1}{(1-r^2)^{n+1}}\int_{rU}\int_U |Q(\varphi_a(z),a)|^2 dm(z)dm(a).
\end{aligned}$$

Then (4.4) follows from (4.2). Hence E_r is compact. Put

$$T_r = H_{\bar f}^* H_{\bar k} - H_{\bar h}^* H_{\bar g} - E_r.$$

Then we have

$$T_r\psi(a) = \chi_r(a)\int_U \frac{Q(z,a)}{(1-a\cdot\bar z)^{n+1}}\psi(z)dm(z),$$

where $\chi_r = \chi_{U-rU}$. By (2.2), (2.3) and simple calculations we obtain

$$\begin{aligned}
&\int_U \frac{|Q(z,a)|^2}{|1-a\cdot\bar z|^{n+1}\sqrt{1-|z|^2}}dm(z)\\
&=\int_U \frac{|Q(\varphi_a(z),a)|^2|k_a(z)|^2}{|1-a\cdot\overline{\varphi_a(z)}|^{n+1}\sqrt{1-|\varphi_a(z)|^2}}dm(z)\\
&=\frac{1}{\sqrt{1-|a|^2}}\int_U \frac{|Q(\varphi_a(z),a)|^2}{|1-a\cdot\bar z|^{n}\sqrt{1-|z|^2}}dm(z)\\
&\le \frac{1}{\sqrt{1-|a|^2}}\Big(\int_U |Q(\varphi_a(z),a)|^{2t}dm(z)\Big)^{1/t}\Big(\int_U \frac{dm(z)}{|1-a\cdot\bar z|^{sn}(1-|z|^2)^{s/2}}\Big)^{1/s},
\end{aligned}$$

where we use Hölder's inequality with

$$s=(4n+3)/(4n+2) \quad\text{and}\quad t=4n+3.$$

By the Cauchy-Schwarz inequality and (4.2)

$$\begin{aligned}&\int_U |Q(\varphi_a(z),a)|^{2t}dm(z)\\ &=\Big(\int_U |Q(\varphi_a(z),a)|dm(z)\Big)^{1/2}\Big(\int_U |Q(\varphi_a(z),a)|^{4t-1}dm(z)\Big)^{1/2}\\ &=A\Big(\int_U |Q(\varphi_a(z),a)|dm(z)\Big)^{1/2}.\end{aligned}$$

By Proposition 1.4.10 of [2],

$$\int_U \frac{dm(z)}{|1-a\cdot\bar z|^{sn}(1-|z|^2)^{s/2}} \le A$$

for some constant A independent of $a\in U$. Then we have

$$\int_U \frac{|Q(z,a)|^2}{|1-a\cdot\bar z|^{n+1}\sqrt{1-|z|^2}}dm(z) \le \frac{A}{\sqrt{1-|a|^2}}\Big(\int_U |Q(\varphi_a(z),a)|dm(z)\Big)^{1/2t}$$

for some constant A independent of $a\in U$. The Cauchy-Schwarz inequality yields

$$\begin{aligned}&|T_r\psi(a)|^2\\ &\le \Big(\chi_r(a)\int_U \frac{|Q(z,a)\psi(z)|}{(1-a\cdot\bar z)^{n+1}}dm(z)\Big)^2\\ &\le \Big(\int_U \frac{\chi_r(a)|Q(z,a)|^2}{(1-a\cdot|\bar z|^{n+1}\sqrt{1-|z|^2}}dm(z)\Big)\Big(\int_U \frac{\sqrt{1-|z|^2}}{|1-a\cdot\bar z|^{n+1}}|\psi(z)|^2dm(z)\Big)\\ &\le A\frac{\chi_r(a)}{\sqrt{1-|a|^2}}\Big(\int_U |Q(\varphi_a(z),a)|dm(z)\Big)^{1/2t}\Big(\int_U \frac{\sqrt{1-|z|^2}}{|1-a\cdot\bar z|^{n+1}}|\psi(z)|^2dm(z)\Big)\end{aligned}$$

for some constant A independent of $a\in U$. By Fubini's theorem, we have

$$\begin{aligned}\int_U |T_r\psi|^2 \le A\sup_{a\in U-rU}\Big(\int_U |Q(\varphi_a(z),a)|dm(z)\Big)^{1/2t}\\ \times\int_U \sqrt{1-|z|^2}\psi(z)|^2\int_U \frac{dm(z)}{|1-a\cdot\bar z|^{n+1}\sqrt{1-|a|^2}}dm(z)\end{aligned}$$

for some constant A independent of r. By Proposition 1.4.10 of [2], we have

$$\int_U \frac{dm(z)}{|1-a\cdot\bar z|^{n+1}\sqrt{1-|a|^2}} \le \frac{A}{\sqrt{1-|z|^2}}$$

for $z\in U$ and for some constant A independent of $z\in U$. Therefore, we have

$$\int_U |T_r\psi|^2dm \le A\sup_{a\in U-rU}\Big(\int_U |Q(\varphi_a(z),a)|dm(z)\Big)^{1/2t}\int_U |\psi|^2dm$$

for some constand A independent of r. On the other hand we have

$$\|T_r\|^2 \le A\sup_{a\in U-rU}\Big(\int_U |Q(\varphi_a(z),a)|dm(z)\Big)^{1/(8n+6)}$$

for some constant A independent of r. Letting $r\to 1$, we have $T_r\to 0$ in the operator norm by (4.3). Hence,

$$H^*_{\bar f}H_{\bar k} - H^*_{\bar h}H_{\bar g}$$

can be approximated by compact operators. Therefore it is compact. This is what we want.

Remark. The proof in [3] shows that the converse of the similar result of Theorem 4.1 is also true in one dimensional case. Unfortunately, we were not able to prove or disprove the converse of Theorem 4.1 on the ball in general. However, Theorem 4.1 is enought to product a simple characterization.

We say that a bounded linear operator L on a Hilbert space is essentially normal if L and its adjoint operator L^* are essentially commuting. We conclude this section with a simple application on essentially normal Toeplitz operators.

Corollary 4.1. *Let β be a bounded pluriharmonic symbol on U and assume*

$$\beta = f + \bar{g}$$

for some function f, g holomorphic on U. Suppose that

$$\lim_{|a|\to 1} \int_U (|f \circ \varphi_a - f(a)|^2 - |g \circ \varphi_a - g(a)|^2) dm = 0.$$

Then T_β is essentially normal on $B^2(U)$.

Proof. The result is a consequence of Theorem 4.1.

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