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ON SYMMETRIC SCALAR CURVATURE ON S^2

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Abstract

Some new results are obtained for the problem of prescribing scalar curvature R on S^2 when R possesses some kinds of symmetries.

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§1. Introduction

Given a continuous function R on the standard sphere S^2, it is an interesting problem whether R can be the scalar curvature of some metric $\tilde{g}$ which is pointwise conformal to the standard metric g_0 on S^2. If we set $\tilde{g} = e^u g_0$, where u is a function on S^2, the problem is equivalent to the solvability of the following PDE:

$$-\triangle_{g_0} u + 2 - R\, e^u = 0, \quad \text{on} \quad S^2. \tag{1.1}$$

Kazdan and Warner[9] pointed out that it may be insolvable. In the last few years, a lot of work has been done to solve problem (1.1), especially when R possesses some kinds of symmetries. After the pioneer work due to Moser[10] for the case of the radial symmetry, Hong[7] considered the case of axisymmetry and Chang Yang[3] considered the case when R is reflection symmetric w.r.t. a plane. For the case of general symmetries, some existence theorems for (1.1) were given by Chen Ding[5].

Let G be a subgroup of the orthogonal transformation group in R^3. Let $B_3 = \{x \in R^3; |x|^2 < 1\}$; $S^2 = \{x \in R^3; |x|^2 = 1\}$ and $f_G := \{x \in S^2; gx = x, \forall g \in G\}$, the set of fixed points on S^2 under the action of G. Throughout this paper, suppose R is G-symmetric, i.e.

$$R(gx) = R(x), \quad \forall x \in S^2, \quad g \in G. \tag{1.2}$$

It is well-known that the solutions of (1.1) can be produced by the critical points of the functional

$$J(u) = \begin{cases} \frac{1}{2}\int_{S^2} |\nabla u|^2 dA - 8\pi \log \int_{S^2} Re^u dA & \text{for } u \in H_*, \\ +\infty & \text{if } u \in H_0 \backslash H_*, \end{cases}$$

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where

$$H_0 = \Big\{ u \in H^1(S^2);\quad \int_{S^2} u dA = 0 \Big\},\quad H_* = \Big\{ u \in H_0;\quad \int_{S^2} Re^u dA > 0 \Big\},$$

with the norm $\|u\| = \big(\int_{S^2} | \bigtriangledown u|^2 dA \big)^{1/2}$. Unfortunately the Palais-Smale condition fails and there does not exist minimum except $R \equiv$ const. for the functional J. The method of all of them is to seek a minimum of J in a smaller space X_* where

$$X = \{ u \in H_0;\ u(gx) = u(x),\ \forall g \in G, x \in S^2 \};\quad X_* = H_* \cap X.$$

Let us briefly recall their idea. Denote $V(u_0) = \{ u \in X_*; P(u) = P(u_0) \}$, where $P : H^1(S^2) \to B_3$ is defined by

$$P(u) = \int_{S^2} x e^{u(x)} dx \Big/ \int_{S^2} e^{u(x)} dx,$$

the mass center of u. It is known that

$$\inf_{X_*} J = \inf_{u_0 \in X_*} \min_{u \in V(u_0)} J(u) \le -8\pi \log 4\pi \big(\max_{f_G} R \big). \tag{1.3}$$

Chen Ding found that if the inequality is strict, then $\inf_{X_*} J$ is attained and (1.1) is solvable thus. This implies their geometric result:

Theorem A. *Assume that $R \in C^2(S^2)$ and $\max_{f_G} R > 0$. If there exists a point $x_0 \in \{ x \in f_G; R(x) = \max_{f_G} R \}$ such that $\triangle R(x_0) > 0$, then (1.1) is solvable.*

In this paper, some new results are given. We find that there seems a kind of duality in this problem. This leads us to some new results. Indeed it is not difficult to prove that

$$\sup_{u_0 \in X_*} \min_{u \in V(u_0)} J(u) \ge -8\pi \log 4\pi \big(\min_{f_G} R \big). \tag{1.4}$$

We find that, if this inequality is strict, then we can establish a minimax principle to solve (1.1) (compare (1.4) with (1.3)), and arrive at

Theorem 1.1. *Suppose $R_0 := \min_{f_G} R > 0$. If there exists $u_0 \in X_*$ such that $\min_{V(u_0)} J > -8\pi \log 4\pi R_0$, then J has a critical point in X_* and (1.1) is solvable.*

Theorem 1.1 enables us to prove the following geometric result.

Theorem 1.2. *Assume $R \in C^2(S^2)$ and $\min_{f_G} R > 0$. If there exists a point $x_0 \in \{ x \in f_G; R(x) = \min_{f_G} R \}$ such that $\triangle R(x_0) < 0$, then (1.1) is solvable.*

In comparing it with Chen Ding's Theorem A, a clear duality appears.

Since the subgroup G is allowed to be an arbitrary subgroup, the set f_G may be S^2 itself (if G is the unit group), or an equator, or a pair of poles, or even empty. When G is the unit group, (1.2) is no longer a restriction at all and on R actually there is not any symmetry assumption. When G is the group generated by the reflection w.r.t. XY-plane for example, f_G is the equator $S^2 \cap \{ z = 0 \}$ and R is required to be reflectional symmetric w.r.t. XY-plane. If G is the group consisting of all the rotations around Z-axes, or a discrete subgroup of it, f_G only contains south and north poles in both cases, and R is axisymmetric in the former case but is of course not necessarily axisymmetric in the latter case.

In above theorems, $f_G \neq \emptyset$ is assumed. Actually the case $f_G = \emptyset$ (for example, $G = \{ id., -id. \}$, R is radial symmetric) has been solved (see [5,10]).

In the case that R is axisymmetric, Xu Yang[11] also found the solution of minimax type. But the idea is quite different. Their method is valid only for the case of axisymmetry and the nondegeneracy condition on R is required.

For the general case that R is not necessarily symmetric, there has been some development (see [2,3,4,6,9] and references therein).

§2. Preliminaries

First of all, notice that we only need to find the critical points of the restriction $J|_{X_*}$ since (1.2) implies

$$\langle dJ(u_0),\ v\rangle = 0, \quad \forall u_0 \in X, \quad \forall v \in X^{\perp}, \tag{2.1}$$

where $X^{\perp}$ is the orthogonal complement of X in H_0.

For obtaining our results, we shall use the following facts, quoted from [2] and [6].

Denote

$$Q(u) = P(u)/|P(u)|, \quad d(u) = |Q(u) - P(u)| \quad \text{if} \quad P(u) \neq 0. \tag{2.2}$$

For $\zeta \in S^2, \delta > 0$, denote

$$C_{\zeta,\delta} = \{u \in H^1(S^2); \quad Q(u) = \zeta, d(u) = \delta\}. \tag{2.3}$$

Set

$$I(u) = \frac{1}{2}\int_{S^2} |\bigtriangledown u|^2 dA + 2\int_{S^2} u dA, \quad u \in H^1(S^2). \tag{2.4}$$

Lemma 2.1.[6, Lemma 1.1] *There exists constant C_0 such that for $u \in H^1(S^2)$,*

$$\left|\int_{S^2} \{R(x) - R(Q(u))\} e^{u(x)} dx\right| \le C_0 \sqrt[3]{d(u)} \int_{S^2} e^u dA.$$

Lemma 2.2. (See Proposition 2.1 and Proposition 4.3 of [6]) *Assume $\{u_i\} \subset H_*, J(u_i) \le \beta < +\infty, dJ(u_i) \to 0$ as $i \to \infty$. If $|P(u_i)| \le 1 - \gamma < 1$ $(\forall i)$ for a constant γ, then $\{u_i\}$ possesses a strongly convergent subsequence in H_*. If $P(u_i) \to \zeta \in S^2$ $i \to \infty$ with $R(\zeta) > 0$, then there is a subsequence $\{u_{i_k}\}$ such that as $k \to \infty$,*

$$J(u_{i_k}) \to -8\pi \log 4\pi R(\zeta).$$

Lemma 2.3.[2, Corollary 5.1] *Suppose $u \in C_{\zeta,\delta}$ with $I(u) = O(\delta^{\beta})$ for some $\beta > 0$ and δ sufficiently small, $\int_{S^2} e^u dA = 4\pi$. Then for every function $h \in C^2(S^2)$,*

$$\frac{1}{4\pi}\int_{S^2} h e^u dA = h(\zeta) + \frac{1}{2} \triangle h(\zeta)\delta + O(\delta/(-\log \delta)).$$

§3. A Minimax Approach

In this section, we establish a minimax principle to obtain Theorem 1.1.

Proof of Theorem 1.1. First of all, $X_* \subset X$ is open and nonempty since $\max_{S^2} R \ge R_0 > 0$.

Introduce the special functions

$$\varphi_{\lambda,y}(x) = \log \frac{1-\lambda^2}{(1 - \lambda \cos d(x,y))^2}, \quad x \in S^2$$

for $y \in S^2$ and $\lambda \in [0,1)$, where $d(x,y)$ is the distance on (S^2, g_0) between x, y. A straight computation shows (see [8]) that

$$\int_{S^2} e^{\varphi_{\lambda,y}} dA = 4\pi; \quad \frac{1}{2}\|\varphi_{\lambda,y}\|^2 + 2\int_{S^2} \varphi_{\lambda,y} dA = 0, \tag{3.1}$$

$$\int_{S^2} R e^{\varphi_{\lambda,y}} dA \to 4\pi R(y) \quad \text{uniformly in} \quad y \in S^2 \quad \text{as} \quad \lambda \to 1, \tag{3.2}$$

$$P(\varphi_{\lambda,y}) = p(\lambda)y, \quad p(\lambda) = \frac{1}{\lambda} + \frac{1}{2}\left(\frac{1}{\lambda^2} - 1\right)\log\frac{1-\lambda}{1+\lambda} \to 1 \quad \text{as} \quad \lambda \to 1. \tag{3.3}$$

Denote

$$\psi_{\lambda,y} = \varphi_{\lambda,y} - \frac{1}{4\pi}\int_{S^2}\varphi_{\lambda,y}dA. \tag{3.4}$$

We chaim that, for λ close to 1,

$$\psi_{\lambda,y} \in X_* \qquad \forall y \in f_G. \tag{3.5}$$

Indeed, $\forall y \in f_G$, $R(y) \geq R_0 > 0$, then we have by (3.2) that $\int_{S^2} Re^{\varphi_{\lambda,y}}dA > 0$ for λ close to 1, showing $\psi_{\lambda,y} \in H_*$. And for any $g \in G$, $y \in f_G$, since g is an isometric transformation and $gy = y$, we have $d(gx, y) = d(gx, gy) = d(x, y)$, so $\varphi_{\lambda,y}(gx) = \varphi_{\lambda,y}(x)$, $\forall x \in S^2$, showing $\psi_{\lambda,y} \in X$. Thus (3.5) is obtained.

Denote $F_G := \{x \in R^3;\ \ gx = x, \forall g \in G\}$, the set of fixed points in R^3 under the action of G. It is not difficult to see that F_G is a linear subspace of R^3, say, m dimensional. Set $B_m = B_3 \cap F_G$, the unit ball of m dimension on F_G, then $f_G = S^2 \cap F_G = \partial B_m$, which implies by (3.5) that for $y \in \partial B_m, \psi_{\lambda,y} \in X_*$ for λ close to 1. Thus we can define

$$\Sigma_\lambda := \{h \in C^0(\overline{B}_m, X_*);\ \ h(y) = \psi_{\lambda,y},\ \ \forall y \in \partial B_m\}, \quad \mu_\lambda := \inf_{h\in\Sigma_\lambda}\max_{x\in\overline{B}_m} J(h(x))$$

for λ close to 1. We will show that, for λ close to 1,

(1) Σ_λ is nonempty (so $\mu_\lambda < +\infty$);

(2) the maximum $\max\limits_{x\in\overline{B}_m} J(h(x))$ can not be attained on the boundary ∂B_m for any $h \in \Sigma_\lambda$;

(3) there exists constant $\delta > 0$, independent of λ, such that $\mu_\lambda \geq -8\pi\log 4\pi R_0 + \delta$;

(4) the restriction $J|_{X_*}$ satisfies $(\text{PS})_a$ condition for $a \in (-8\pi\log 4\pi R_0, \infty)$.

These four points together yield that μ_λ is a critical value of $J|_{X_*}$ when λ is close to 1, by a generalized mountain pass lemma (see [1]). And μ_λ is also critical for J by (2.1), which implies the conclusion. Thus we only need to verify the points (1)–(4).

For (1), taking a fixed point $y_0 \in \partial B_m$, setting $I(z) = z/|z|, \forall z \in \overline{B}_m\backslash\{0\}$, we construct a continuous map $h_\lambda : \overline{B}_m \to X_*$ as follows: $h_\lambda(0) = \psi_{\lambda,y_0}$ and

$$h_\lambda(z) = \log(|z|e^{\psi_{\lambda,I(z)}} + (1-|z|)e^{\psi_{\lambda,y_0}}) + c_\lambda, \quad \forall z \in \overline{B}_m\backslash\{0\},$$

where c_λ is the constant, so that $\int_{S^2} h_\lambda(z)dA = 0$. The continuity is clear. To show $h_\lambda(z) \in X_*$, we first have $h_\lambda(z) \in X$ by (3.5) and $\partial B_m = f_G$. Secondly (3.1) implies $-\int \varphi_{\lambda,y}dA \geq 0$ and (3.2) implies $\int Re^{\varphi_{\lambda,y}}dA \geq 2\pi R(y) \geq 2\pi R_0 > 0$ for λ close to 1. Then for λ close to 1,

$$\int_{S^2} Re^{\psi_{\lambda,y}}dA \geq 2\pi R_0, \quad \forall y \in \partial B_m,$$

which implies, in turn,

$$e^{-c_\lambda}\int_{S^2} Re^{h_\lambda(z)}dA = |z|\int_{S^2} Re^{\psi_{\lambda,I(z)}}dA + (1-|z|)\int_{S^2} Re^{\psi_{\lambda,y_0}}dA > 0,$$

i.e. $h_\lambda(z) \in H_*$. Thus $h_\lambda(z) \in X_*$. At last for $z \in \partial B_m, |z| = 1, I(z) = z$, we see $h_\lambda(z) = \psi_{\lambda,z}$. Thus $h_\lambda \in \Sigma_\lambda$ for λ close to 1, and (1) is obtained.

Since for any $g \in G$ and $u \in X_*$, g is a linear isometric transformation, and $u(gx) = u(x)$ $(x \in S^2)$, it is easy to verify that $gP(u) = P(u)$, i.e. the mass center $P(u)$ of u is a fixed point under the action of G, so $P(X_*) \subset F_G \cap B_3 = B_m$ and $P \circ h$ is a map from $\overline{B}_m$ into itself $(\forall h \in \Sigma_\lambda)$. For $y \in \partial B_m$ and $h \in \Sigma_\lambda$, from (3.3) it follows that $P(h(y)) = P(\varphi_{\lambda,y}) = p(\lambda)y$, where $p(\lambda) \to 1$ (as $\lambda \to 1$). Now

$$P \circ h : \overline{B}_m \to B_m, \quad P \circ h|_{\partial B_m} = p(\lambda)id., \quad \forall h \in \Sigma_\lambda, \tag{3.6}$$

where $id.$ denotes the identity. Since $p(\lambda) \to 1$ as $\lambda \to 1$, we can take λ sufficiently close to 1 so that $P(u_0) \in p(\lambda)B_m$. By (3.6), for $h \in \Sigma_\lambda$,

$$\deg(P \circ h, B_m, P(u_0)) = \deg(p(\lambda)id., B_m, P(u_0)) = 1,$$

which implies that $(P \circ h)^{-1}(P(u_0))$ is not empty, i.e.

$$P \circ h(B_m) \cap \{P(u_0)\} \neq \phi,$$

in other words,

$$h(B_m) \cap V(u_0) \neq \phi.$$

Thus

$$\max_{\overline{B}_m} J \circ h \geq \min_{V(u_0)} J, \qquad \forall h \in \Sigma_\lambda. \tag{3.7}$$

However, for $h \in \Sigma_\lambda$ and $y \in \partial B_m$,

$$J \circ h(y) = J(\psi_{\lambda,y}) = -8\pi \log \int_{S^2} Re^{\varphi_{\lambda,y}} dA \to -8\pi \log 4\pi R(y)$$

uniformly in y as $\lambda \to 1$ (by (3.1) (3.2)), so

$$\max_{\partial B_m} J \circ h \leq -8\pi \log 4\pi R_0 + \varepsilon_\lambda \quad \text{with} \quad \varepsilon_\lambda \to 0 \quad \text{as} \quad \lambda \to 1. \tag{3.8}$$

Combining (3.7), (3.8) and the condition that $\min_{V(u_0)} J > -8\pi \log 4\pi R_0$, we obtain (2) and (3) for λ close to 1.

For verifying (4), let $\{u_i\} \subset X_*$ satisfy $J(u_i) \to a$ and $dJ|_{X_*}(u_i) \to 0$ as $i \to \infty$ where $a \in (-8\pi \log 4\pi R_0, +\infty)$. We have $dJ(u_i) \to 0$ by (2.1). On account of Lemma 2.2, it suffices to show that $|P(u_i)| \leq 1 - \gamma$ for a constant $\gamma > 0$. Suppose by contradiction that, for a subsequence, still denoted by $\{u_i\}, P(u_i)$ tends to some $\zeta \in S^2$ as $i \to \infty$. Since $P(X_*) \subset B_m$, we have $\zeta \in \partial B_m$, implying from $f_G = \partial B_m$ that $\zeta \in f_G$, so $R(\zeta) \geq R_0 > 0$. By Lemma 2.2, there is a subsequence $\{u_{i_k}\}$ such that

$$\lim_{i_k \to \infty} J(u_{i_k}) = -8\pi \log 4\pi R(\zeta) \leq -8\pi \log 4\pi R_0,$$

which contradicts $\lim_{i\to\infty} J(u_i) = a > -8\pi \log 4\pi R_0$. Now (4) is obtained.

The proof is finished.

§4. An Application

In this section we apply Theorem 1.1 to obtain Theorem 1.2.

Proof of Theorem 1.2. By assumptions, let $x_0 \in f_G$ such that $R(x_0) = \min\limits_{f_G} R > 0$ and $\triangle R(x_0) < 0$. According to Theorem 1.1, it sufficies to verify that there exists a $u_0 \in X_*$ such that $\min_{V(u_0)} J > -8\pi \log 4\pi R(x_0)$, where $V(u_0) = \{u \in X_*; P(u) = P(u_0)\}$. In fact, we are going to show that $u_0 = \psi_{\lambda,x_0}$ (see (3.4), (3.5) for definition) has such property, where λ is close to 1 and to be determined, by using Lemma 2.1 and Lemma 2.3.

Let $u_\lambda \in V(\psi_{\lambda,x_0})$ attain the minimum of J in $V(\psi_{\lambda,x_0})$. It suffices to prove

$$J(u_\lambda) > -8\pi \log 4\pi R(x_0) \tag{4.1}$$

for some λ being close to 1. Obviously $J(u_\lambda) \leq J(\psi_{\lambda,x_0}), P(u_\lambda) = P(\psi_{\lambda,x_0}) = P(\varphi_{\lambda,x_0}) = p(\lambda)x_0$ (by (3.3)) with $p(\lambda) < 1, p(\lambda) \to 1$ as $\lambda \to 1$, so $Q(u_\lambda) = x_0$ and $d(u_\lambda) = 1 - p(\lambda)$, denoted by ε_λ, which positively tends to zero as $\lambda \to 1$. Set $v_\lambda = u_\lambda + c_\lambda$ where $c_\lambda =$

$-\log\left(\frac{1}{4\pi}\int_{S^2} e^{u_\lambda}dA\right)$ so that $\int_{S^2} e^{v_\lambda}dA = 4\pi$. Clearly $Q(v_\lambda) = Q(u_\lambda) = x_0, d(v_\lambda) = d(u_\lambda) = \varepsilon_\lambda$, i.e. $v_\lambda \in C_{x_0,\varepsilon_\lambda}$. A direct computation yields

$$I(v_\lambda) = J(u_\lambda) + 8\pi\log\int_{S^2} Re^{v_\lambda}dA,$$

$$J(\psi_{\lambda,x_0}) = -8\pi\log\int_{S^2} Re^{\varphi_{\lambda,x_0}}dA,$$

while Lemma 2.1 implies

$$\int_{S^2} Re^{v_\lambda}dA = [R(x_0) + O(\varepsilon_\lambda^{\frac{1}{3}})]\int_{S^2} e^{v_\lambda}dA = 4\pi R(x_0) + O(\varepsilon_\lambda^{1/3}),$$

$$\int_{S^2} Re^{\varphi_{\lambda,x_0}}dA = [R(x_0) + O(\varepsilon_\lambda^{\frac{1}{3}})]\int_{S^2} e^{\varphi_{\lambda,x_0}}dA = 4\pi R(x_0) + O(\varepsilon_\lambda^{1/3})$$

(by recalling (3.1)). Thus

$$\begin{aligned} I(v_\lambda) &\le J(\psi_{\lambda,x_0}) + 8\pi\log\int_{S^2} Re^{v_\lambda}dA \\ &= 8\pi\log\left\{\int_{S^2} Re^{v_\lambda}dA \Big/ \int_{S^2} Re^{\varphi_{\lambda,x_0}}dA\right\} = O(\varepsilon_\lambda^{1/3}). \end{aligned}$$

Let λ be so close to 1 that ε_λ is sufficiently small to meet the condition of Lemma 2.3. Using Lemma 2.3 with $u = v_\lambda$, we obtain

$$\frac{1}{4\pi}\int_{S^2} Re^{v_\lambda}dA = R(x_0) + \frac{1}{2}\triangle R(x_0)\varepsilon_\lambda + O\left(\frac{\varepsilon_\lambda}{-\log\varepsilon_\lambda}\right) < R(x_0) \tag{4.2}$$

in accordance with $\triangle R(x_0) < 0$. On the other hand, the inequality[7]

$$\int_{S^2} e^u dA \le 4\pi\exp\left(\frac{1}{16\pi}\|u\|^2\right), \quad \forall u \in H_0,$$

implies $I(v_\lambda) = \frac{1}{2}\|u_\lambda\|^2 + 8\pi c_\lambda \ge 0$. From this and (4.2) we obtain

$$J(u_\lambda) = I(v_\lambda) - 8\pi\log\int_{S^2} Re^{v_\lambda}dA > -8\pi\log 4\pi R(x_0),$$

and arrive at (4.1).

The proof is finished.

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