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PHANTOM MAPS AND SPACES OF THE SAME n-TYPE FOR ALL n**

I. DO*

Abstract

After reviewing the theory of phantom maps and SNT, the author gives several general results which relate the theory of phantom maps and SNT and which extend that of Harper and Roitberg.

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§1. Introduction

Let X be a connected CW complex and X_n denote its n-skeleton. A map $f : X \to Y$ is called a phantom map if its restriction to each skeleton X_n is null homotopic. It is easy to know that every phantom map from a finite CW complex or to a space with only finite nontrivial homotopy groups is necessarily trivial up to homotopy. Hence essential phantom map can occur only when the domain X is an infinite dimensional space or the target is a space with infinite nontrivial homotopy groups. Such maps appear to be null homotopic from a number of different points of view; e.g., they induce the trivial homomorphism on homotopy groups, in homology, and in cohomology. How then do we detect them? In what cases are they trivial? These and other questions are what phantom map theory concerns.

On the other hand, the phantom map is the obstruction to extend many homotopy results from finite complexes to infinite complexes. For example, one of the fundamental results in homotopy theory is

Theorem 1.1.[5] *Let W be a finite CW complex and Y be a nilpotent CW complex. Then the natural map*

$$[W, Y] \to \underset{p\in I}{\text{Pullback}}\{[W, Y_{(p)}] \to [W, Y_{(0)}]\}$$

is a bijection.

In general, this theorem is false for infinite complexes, but we have the following:

Theorem 1.2.[9] *Let W and Y be two CW complexes and $Ph(W, Y)$ denote the set of homotopy classes of phantom maps from W to Y. If $Ph(W, Y) = *$, then*

$$[W, Y] \to \underset{p\in I}{\text{Pullback}}\{[W, Y_{(p)}] \to [W, Y_{(0)}]\}$$

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*Topology and Geometry Research Center, Kyungpook National University, Taegu 702-701, Korea.
E-mail: d6107@chollian.net.
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is a bijection.

Many other results generalize to infinite complexes under the condition that the only phantom map is the trivial one. This is maybe the most interesting aspect of the theory. Another interesting topic in homotopy theory is whether two spaces have the same n-type for all n. Let X be a space and P_nX be the nth section of the Postnikov tower. Let

$$\mathrm{SNT}(X) = \{Y : P_nX \simeq P_nY\}/\simeq .$$

The following is one of the fundamental results in this field[11].

Theorem 1.3. *Let X be a connected CW complex. Then*

$$\mathrm{SNT}(X) \approx \varprojlim{}^1 \mathrm{Aut}(P_nX),$$

where $\mathrm{Aut}(P_nX)$ *is the group of homotopy classes of homotopy equivalences of* P_nX.

The computation of it remains hard since it is difficult to compute the $\mathrm{Aut}(P_nX)$ and $\varprojlim^1 G$. But in [4], J. Harper and J. Roitberg observed an interesting relation between $\mathrm{Ph}(-,-)$ and $\mathrm{SNT}(-)$. The object of this note is to extend their results to the general case. Throughout this paper, a space will be a pointed CW complex of finite type. $\mathrm{map}(X,Y)$ is the space of all maps from X to Y and $\mathrm{map}_*(X,Y)$ is the subspace of $\mathrm{map}(X,Y)$ which consists of pointed maps from X to Y. For any nilpotent space X, $l_{(p)} : X \to X_{(p)}$ is the p-localization of X and $\hat{e} : X \to \widehat{X}$ is the Sullivan profinite completion. Lastly, denote by P_nX the nth section of the Postnikov tower of X and $\pi_n : X \to P_nX$ is the canonical map.

§2. A Review of Phantom Map

In this section we will review the necessary notions and results related to phantom maps which are needed in the later part of the paper.

Definition 2.1. *A tower of groups is an inverse sequence of groups and homomorphisms* $G = \{G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \cdots\}$.

One gets such towers when one sets $G_n = [\Sigma X_n, Y]$ and takes $f_n : G_{n+1} \to G_n$ to be the homomorphism induced by the inclusion $X_n \to X_{n+1}$. For such a tower of groups, two natural notions appear.

Definition 2.2. *Given a tower of groups,* $G = \{G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \cdots\}$, *define*

$$\varprojlim G = \{(a_1, a_2, \cdots) \mid a_i = f_i(a_{i+1})\},$$
$$\varprojlim{}^1 G = \prod_n G_n / \prod_n G_n,$$

where $\prod_n G_n / \prod_n G_n$ *is the quotient space of the action of the group* $\prod_n G_n$ *acting on the set* $\prod_n G_n$ *by*

$$\{g_n\} \cdot \{x_n\} = \{g_n x_n (f_n(g_{n+1}))^{-1}\}.$$

In their book, Bousfield and Kan[2] showed

Theorem 2.1. *For any pointed complexes X and Y, there is a short exact sequence of pointed sets*

$$* \to \varprojlim{}^1 [\Sigma X_n, Y] \to [X,Y] \to \varprojlim [X_n, Y] \to *.$$

Remark 2.1. For a sequence of pointed sets $S_1 \xrightarrow{i} S_2 \xrightarrow{j} S_3$, one says that the sequence is exact at S_2 if every $f \in S_2$ which maps under j to the given distinguised element in S_3 is in the image of i.

Corollary 2.1. *For any complexes X and Y, we have* $\mathrm{Ph}(X,Y) \approx \varprojlim{}^1[\Sigma X_n, Y]$.

The first published account of an essential phantom map was due to J. F. Adams and G. Walker[1]. It is a map from ΣCP^∞ to $\bigvee\limits_1^\infty S^4$. But most important advances in this area appear after the Sullivan Conjecture was verified by H. Miller[10]. The two fundamental results in the theory of phantom maps are the following[6]:

Theorem 2.2. *Let X and Y be connected nilpotent complexes of finite type. A map $f: X \to Y$ is a phantom map if and only if*

(1) *the composition $X \xrightarrow{f} Y \xrightarrow{\hat{e}} \widehat{Y}$ is null homotopic, or*

(2) *there is a diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle l_{(0)}}\downarrow & & \downarrow{\scriptstyle id} \\ X_{(0)} & \xrightarrow{\tilde{f}} & Y \end{array}$$

which commutes up to homotopy.

Theorem 2.3. *Let X and Y be 1-connected CW complexes of finite type. If the mapping space* $\mathrm{map}_*(X,\widehat{Y})$ *is weakly contractible, then for every $k \geq 0$,*

$$[X, \Omega^k Y] = \mathrm{Ph}(X, \Omega^k Y) \approx [X_{(0)}, \Omega^k Y].$$

Thus the problem in this case reduces to the computation of $\mathrm{map}_*(X,\widehat{Y})$. For this we have the following:

Theorem 2.4. *Let Y be a 1-connected finite complex. If*

(1) *X is a 1-connected Postnikov space (i.e., $\pi_n(X) = 0$ for n sufficiently large) of finite type, or*

(2) *$X = BG$, where G is a connected Lie group, or*

(3) *X is a 1-connected infinite loop space,*

then the $\mathrm{map}_*(X,\widehat{Y})$ *is weakly contractible.*

The first part of this theorem is due to Zabrodsky[12], the second part is a special case of a result of Friedlander and Mislin[3] and the last part is a special case of a result of McGibbon[7]. All these results are consequences of H. Miller's celebrated theorem[10]:

Theorem 2.5. *If G is a locally finite group and Y is a finite dimensional complex, then* $\mathrm{map}_*(BG,Y)$ *is weakly contractible.*

§3. Phantom Maps and SNT

For a map $f: X \to Y$ between 1-connected spaces X and Y, let F_f denote the homotopy fiber of f and C_f the mapping cone of f. As X and Y are 1-connected, F_f is path connected and C_f is 1-connected. Then we have the following:

Theorem 3.1. *There are (pointed) functions*

$$F: \mathrm{Ph}(X,Y) \to \mathrm{SNT}(X \times \Omega Y), \quad C: \mathrm{Ph}(X,Y) \to \mathrm{SNT}(Y \vee \Sigma X),$$

defined by $F(f) = F_f$, $C(f) = C_f$, *where "pointed" means* $F(*) = X \times \Omega Y$, $C(*) = Y \vee \Sigma X$.

Proof. For the proof of the theorem, it is sufficient to prove the following: for all n,

$$P_n F_f \simeq P_n X \times P_n \Omega Y = P_n(X \times \Omega Y), \quad P_n C_f = P_n(Y \vee \Sigma X).$$

We will only prove the first homotopy equivalence, the proof of the other is similar. Note a well-known fact that a map $f : X \to Y$ is a phantom map if and only if its nth Postnikov approximation $P_n f : P_n X \to P_n Y$ is trivial for all n.

Consider now a phantom map $f : X \to Y$, the maps $P_n f : P_n X \to P_n Y$ are trivial for all n. It follows that $\pi_n \circ f \simeq P_n f \circ \pi_n \simeq *$ for all n, and the following diagram commutes up to homotopy

$$\begin{tikzcd} P_n F_f \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ P_n X \times \Omega P_{n+1} Y \arrow[r] & P_n X \arrow[r, "*"] & P_{n+1} Y. \end{tikzcd}$$

It follows from the homotopy exact sequence of the homotopy fibration sequence that the map $P_n F_f \to P_n X \times \Omega P_{n+1} Y$ is a homotopy equivalence.

As noted by Harper and Roitberg[4], the functions F and C are far from being surjective. Our aim here is to find conditions on X and Y ensuring that F and C have nontrivial images, which extends Harper and Roitberg's results.

3.1. Results on F

Let $Y = \Omega^k W$, where W is a 1-connected finite CW complex and X be a CW complex satisfying the following:

(1) $\mathrm{Ph}(X, \Omega Y) = [X, \Omega Y]$,

(2) Any map $t : X \to X$ is either trivial or rational homotopy equivalence.

Then we have the following:

Theorem 3.2. *If* $f : X \to Y$ *is a nontrivial phantom map, then* $F_f \neq X \times \Omega Y$.

Proof. It is enough to prove that any map between F_f and $X \times \Omega Y$ cannot be a homotopy equivalence. Let $h : X \times \Omega Y \to F_f$ be any map and $j : X \to X \times \Omega Y$ the embedding into the first factor and consider the following diagram

$$\begin{tikzcd} \cdots \arrow[r] & \Omega Y \arrow[r, "k"] & F_f \arrow[r, "i"] & X \arrow[r, "f"] & Y \\ & & X \times \Omega Y \arrow[u, "h"] & & \\ & & X \arrow[u, "j"] & & \end{tikzcd}$$

Denote $d = i \circ h \circ j$. We have $f \circ d = f \circ i \circ h \circ j \simeq *$, since $f \circ i \simeq *$. By Theorem 2.3 and Theorem 2.4, we have $f \simeq \tilde{f} \circ l_{(0)}$, where $\tilde{f} : X_{(0)} \to Y$ is uniquely determined by f. There is a commutative diagram up to homotopy

$$\begin{tikzcd} X \arrow[r, "l_{(0)}"] \arrow[d, "d"] & X_{(0)} \arrow[d, "d_{(0)}"] & \\ X \arrow[r, "l_{(0)}"] & X_{(0)} \arrow[r, "\tilde{f}"] & Y \end{tikzcd}$$

It follows from this diagram that

$$0 = f \circ d = \tilde{f} \circ l_{(0)} \circ d = \tilde{f} \circ d_{(0)} \circ l_{(0)}.$$

Then Theorem 2.3 and Theorem 2.4 imply that $\tilde{f} \circ d_{(0)} = 0$. By the condition (2), d is either trivial or rational homotopy equivalence. If $d \neq 0$, then $d_{(0)}$ is a homotopy equivalence and the equation above implies that $\tilde{f} = 0$ and thus $f = 0$, which is a contradiction.

We conclude therefore that $d = 0$, i.e., $i \circ h \circ j = 0$. Thus there is a map $g : X \to \Omega Y$ such that $h \circ j = k \circ g$. By condition (1), g is a phantom map and thus g induces the trivial homomorphism on homotopy groups. It follows that the homomorphism

$$\pi_*(h) : \pi_*(X \times \Omega Y) \to \pi_*(\Omega Y)$$

must be trivial on the first factor $\pi_*(X)$ of $\pi_*(X \times \Omega Y) = \pi_*(X) \oplus \pi_*(\Omega Y)$ and hence h cannot be a homotopy equivalence.

Corollary 3.1. *Let $Y = \Omega^k W$ where $k \geq 0$ and W is a 1-connected finite CW complex. Let $X = K(\mathbb{Z}, n)$ or BG where G is a 1-connected compact Lie group. Then the conclusion of Theorem 3.2 is true.*

Remark 3.1. Corollary 3.1 extends the corresponding result of Harper and Roitberg[4].

The previous result shows that a nontrivial phantom map corresponds to a nontrivial element under the map F. Actually we can say something more as in [4] as follows:

Theorem 3.3. *If X and Y are as in Theorem 3.2 but $k \geq 1$, then the image of $F : \mathrm{Ph}(X, Y) \to \mathrm{SNT}(X \times \Omega Y)$ is an uncountable subset of $\mathrm{SNT}(X \times \Omega Y)$.*

3.2. Results on C

In the following we turn to the study of map C which is in certain sense dual to the map F.

Let $X = \Sigma^k W$ where W is a CW complex and $k \geq 1$. Let Y be a CW complex satisfying the following:

(1) $\mathrm{Ph}(\Sigma X, Y) = [\Sigma X, Y] = [\Sigma X_{(0)}, Y]$,

(2) Any map $t : Y \to Y$ is either trivial or rational homotopy equivalence.

Then we have

Theorem 3.4. *If $f : X \to Y$ is a nontrivial phantom map, then $C_f \neq Y \vee \Sigma X$.*

Proof. Let $h : C_f \to Y \vee \Sigma X$ be any map and $j' : Y \vee \Sigma X \to Y$ the canonical map. Let $d = j' \circ h \circ i$, then $d \circ f = j' \circ h \circ i \circ f = 0$ since $i \circ f = 0$. Now by Theorem 2.2, there is a map $\tilde{f} : X_{(0)} \to Y$ such that $f = \tilde{f} \circ l_{(0)}$ since f is a phantom map. It follows that $d \circ \tilde{f} \circ l_{(0)} = 0$ and thus $d \circ \tilde{f} = 0$ by the condition (1).

Let $j : Y \to \bar{Y}$ be an integral approximation of Y. This means that the homotopy groups of $\bar{Y}$ are torsion free and finitely generated, the loop space $\Omega\bar{Y}$ is homotopy equivalent to a product of Eilenberg-MacLane spaces and the map j is a rational equivalence. Such approximation exists by [12]. Since the fiber of j has finite groups, it follows that j induces a bijection

$$j_* : [X_{(0)}, Y] \to [X_{(0)}, \bar{Y}].$$

If $d \neq 0$, then $d_{(0)}$ is a homotopy equivalence by the condition (2). Since $\Omega\bar{Y}$ is a product of Eilenberg-MacLane spaces, there is a map $\bar{d} : \Omega\bar{Y} \to \Omega\bar{Y}$ such that the following diagram commutes up to homotopy

$$\begin{array}{ccc} \Omega Y & \xrightarrow{j} & \Omega \bar{Y} \\ {\scriptstyle \Omega d}\downarrow & & {\scriptstyle \bar{d}}\downarrow \\ \Omega Y & \longrightarrow & \Omega \bar{Y} \end{array}$$

Since d and thus $d_{(0)}$ is a rational homotopy equivalence, it follows that $\bar{d}$ is a rational homotopy equivalence.

From the commutative diagram

$$\begin{array}{ccccc} [\Sigma^{(k-1)}W_{(0)}, \Omega Y] & \xrightarrow{j_*} & [\Sigma^{(k-1)}W_{(0)}, \Omega \bar{Y}] & \longrightarrow & [\Sigma^{(k-1)}W_{(0)}, \Omega \bar{Y}_{(0)}] \\ {\scriptstyle \Omega d_*}\downarrow & & {\scriptstyle \bar{d}_*}\downarrow & & {\scriptstyle \bar{d}_{(0)*}}\downarrow \\ [\Sigma^{(k-1)}W_{(0)}, \Omega Y] & \xrightarrow{j_*} & [\Sigma^{(k-1)}W_{(0)}, \Omega \bar{Y}] & \longrightarrow & [\Sigma^{(k-1)}W_{(0)}, \Omega \bar{Y}_{(0)}] \end{array}$$

It follows from $d \circ \tilde{f} = 0$ that $\tilde{f} = 0$ and thus $f = \tilde{f} \circ l_{(0)} = 0$, which is a contradiction. Thus we have $d = 0$ and the same argument as in Theorem 3.2 shows that f cannot be a homotopy equivalence.

Remark 3.2. Our proof is dual to that of Theorem 3.2, while in [4] Harper and Roitberg have to use a different argument which does not extend to the general case.

Corollary 3.2. *If X is as in Theorem* 3.4 *and*

(1) $Y = S^n$ *or*

(2) $Y = BG$ *where G is a* 1*-connected compact Lie group,*

then the conclusion of Theorem 3.4 *remains true.*

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