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A CLOSEDNESS CRITERION FOR THE DIFFERENCE OF TWO CLOSED CONVEX SETS IN GENERAL BANACH SPACES***

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Abstract

The authors give some sufficient conditions for the difference of two closed convex sets to be closed in general Banach spaces, not necessarily reflexive.

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§1. Introduction

In this paper we will give some suitable sufficient conditions which assure the closedness property of the difference of two closed convex sets in general Banach spaces, not necessarily reflexive. The closedness problem we are concerned with is frequently encountered in different branches of applied mathematics; for example, the solvability of linear systems over cone and the duality theory of abstract mathematical programming. Our motivation comes from a Baiocchi-Gastaldi-Tomarelli's result[3] concerned with the case of Hilbert space. Furthermore, Köthe[6] showed that in a Banach space the sum of two bounded closed convex sets is closed if and only if the space is reflexive. Let A be a non-empty, closed convex subset of a Banach space E. We denote by $\mathrm{ca}A = \bigcap\limits_{\lambda>0} \lambda(A-a)$ the asymptotic cone of A which is the greatest cone included in A if A contains o, the origin of E. We refer to [7] for more detailed properties of $\mathrm{ca}A$. Our main result is the following

Throrem 1.1. *Let A and B be two closed convex subsets of a Banach space E which satisfy*

$$A \text{ is included in a finite dimensional subspace of } E, \tag{1.1}$$

$$\mathrm{ca}A \cap \mathrm{ca}B \text{ is a linear subspace of } E. \tag{1.2}$$

Then $A-B$ is a closed convex subset of E.

The proof of Theorem 1.1 is related to the following

Proposition 1.1. *Let A and B be two closed convex subsets of E which satisfy* (1.1) *and*

$$D\Phi_A^* + D\Phi_B^* \text{ is a closed subspace of } E' \text{ for the strong topology.} \tag{1.3}$$

Then we conclude that $A-B$ is a closed convex subset of E.

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Recall here $D\Phi_A^*$ (resp. $D\Phi_B^*$) is given by $D\Phi_A^* = \{y \in E'; \sup_{x\in A}\langle y, x\rangle_{E',E} < +\infty\}$. Note that in a non reflexive Banach space, we can find (see [5]) that for two closed convex subsets A and B satisfying the condition (1.3) and not satisfying the condition (1.1), $A - B$ is not closed. But in a reflexive Banach space, this will not occur. Indeed, it is proved in [2] (see also [9]) that in this case, condition (1.3) suffices to guarantee the closedness of the convex set $A - B$. We notice that in the reflexive Banach space case, Theorem 1.1 was shown in a simple way (see [9]).

§2. Proof of Proposition 1.1

Let Φ be a convex function defined on E. We denote by Φ^* its conjugate convex function defined on E'. For two convex functions Φ and Ψ defined on E, we define the inf-convolution of Φ and Ψ by $\Phi\Box\Psi(x) = \inf_{y\in E}\{\Phi(x-y) + \Psi(y)\}$ for all $x \in E$. Remark that we have $\Phi_A\Box\Phi_{-B} = \Phi_{A-B}$ where Φ_A (resp. Φ_{-B}, Φ_{A-B}) is the indicator function of A (resp. $-B$, $A-B$). One deduces that $A - B$ is closed in E if and only if the function $\Phi_A\Box\Phi_{-B}$ is lower semi-continuous.

Proof of Proposition 1.1. Set $G = D\Phi_A^* + D\Phi_B^*$. It is easy to check that

$$G = D\Phi_A^* - D\Phi_{-B}^*. \tag{2.1}$$

Let j be the canonical injection of G into E' and j^* the dual surjection of j. Let $\Psi_A = \Phi_A^* \circ j$ and $\Psi_{-B} = \Phi_{-B}^* \circ j$ be two convex functions defined on G. Then we have, for all $x \in E$, considered as a subspace of E'':

$$\Psi_A^*(j^*(x)) = \Phi_A(x) \quad \text{and} \quad \Psi_{-B}^*(j^*(x)) = \Phi_{-B}(x). \tag{2.2}$$

We claim that $\Phi_A\Box\Phi_{-B}$ is lower semi-continuous in E.

Indeed, let $\{x_n\}$ be a sequence in E, which converges to x in E, such that there exists a constant C satisfying

$$\Phi_A\Box\Phi_{-B}(x_n) \le C. \tag{2.3}$$

Therefore there exist a sequence $\{u_n\}$ in E and a sequence of positive numbers $\{\varepsilon_n\}$ tending to 0 such that $\Phi_A(u_n) + \Phi_{-B}(x_n - u_n) \le C + \varepsilon_n$. By (2.2), we deduce that

$$\Psi_A^*(j^*(u_n)) + \Psi_{-B}^*(j^*(x_n - u_n)) \le C + \varepsilon_n. \tag{2.4}$$

We prove now that

$$\sup_{n\in\mathbb{N}} \|j^*(u_n)\|_{G'} < +\infty. \tag{2.5}$$

In virtue of Banach-Steinhauss Theorem, it suffices to show that for any y in G there is a constant $C(y)$ such that, for all n,

$$|\langle j^*(u_n), y\rangle_{G',G}| \le C(y). \tag{2.6}$$

By (2.1), for any y in G, there exists $(a, b) \in D\Phi_A^* \times D\Phi_{-B}^*$ such that $y = a - b$. Thus we have

$$\langle j^*(u_n), y\rangle_{G',G} = \langle j^*(u_n), a\rangle_{G',G} + \langle j^*(x_n - u_n), b\rangle_{G',G} - \langle j^*(x_n), b\rangle_{G',G}.$$

According to the definition of Ψ_A^* and Ψ_{-B}^*, we get

$$\begin{aligned}\langle j^*(u_n), y\rangle_{G',G} &\le \Psi_A^*(j^*(u_n)) + \Psi_A(a) + \Psi_{-B}^*(j^*(x_n - u_n)) \\ &\quad + \Psi_{-B}(b) - \langle j^*(x_n), b\rangle_{G',G} \\ &\le C + \varepsilon_n + \Psi_A(a) + \Psi_{-B}(b) - \langle j^*(x_n), b\rangle_{G',G}.\end{aligned}$$

The last term in the above inequality is less than a constant $C_1(y)$. In the same way, there is another constant $C_2(y)$ such that $\langle j^*(u_n), -y\rangle_{G',G} \le C_2(y)$, which proves (2.6), and so (2.5).

Now let F be a finite dimensional subspace containing the convex set A. Note that $\Phi_A(u_n)$ is finite for all n, which deduces that u_n belongs to A for all n. The sequence $\{j^*(u_n)\}$ is contained in the vector space $j^*(F)$, which is a finite dimensional subspace of G'. On the other hand, (2.5) shows that $\{j^*(u_n)\}$ is bounded in G'. There exists hence an element $\bar{u}$ of F such that a subsequence of $\{j^*(u_n)\}$ converges to $j^*(\bar{u})$ of $j^*(F)$. By the lower semi-continuity of the convex functions Ψ_A^* and Ψ_{-B}^* together with (2.4), we have $\Psi_A^*(j^*(\bar{u})) + \Psi_{-B}^*(j^*(x-\bar{u})) \le C$. By (2.2), this means exactly that

$$\Phi_A(\bar{u}) + \Phi_{-B}(x-\bar{u}) \le C.$$

We conclude that $\Phi_A \Box \Phi_{-B}$ is lower semi-continuous in E and so the proof of Proposition 1.1 is completed.

§3. Proof of Theorem 1.1

It suffices to prove that under the hypothsis (1.1) and (1.2), we have the property (1.3). Then Theorem 1.1 follows from Proposition 1.1. According to [8], we use the pointview of the paired spaces, which is possible since the weak* topology on E' is compatible with the duality (E', E).

Recall that for any non-empty convex cone C of E, the polar cone C° of C is defined by

$$C^\circ = \{\varphi \in E'; \quad \langle \varphi, x\rangle_{E',E} \le 0, \quad \forall x \in C\}.$$

The polar C° of C is closed with respect to the weak* topology of E'. If C is a vector subspace of E, then C° coincides with the orthogonal of C, denoted by $C^\perp$. We define the polar D° of a convex cone D in E' in the same way, i.e. $D^\circ = \{x \in E; \quad \langle \varphi, x\rangle_{E',E} \le 0, \quad \forall \varphi \in D\}$. Using Hahn-Banach Theorem, it is easy to prove that, for all closed convex set M which contains o of E, we have

$$(D\Phi_M^*)^\circ \subset M. \tag{3.1}$$

We first prove the following

Proposition 3.1. *Let A and B be two closed convex sets in a Banach space E. Then we have* $\mathrm{cl}(D\Phi_A^* + D\Phi_B^*) = (\mathrm{ca}A \cap \mathrm{ca}B)^\circ$, *where* cl *designates the closure for the weak* topology of E'.*

Proof. Since the sets $\mathrm{ca}A$ and $D\Phi_A^*$ are invariant under translations, we may assume that o belongs to $A\cap B$. We claim that

$$\mathrm{cl}(D\Phi_A^* + D\Phi_B^*) = \mathrm{cl}D\Phi_{A\cap B}^*. \tag{3.2}$$

In fact, it is easy to see that $\mathrm{cl}(D\Phi_A^* + D\Phi_B^*) \subset \mathrm{cl}D\Phi_{A\cap B}^*$. On the other hand, assume that there exists an element φ of $D\Phi_{A\cap B}^*$ such that φ does not belong to $\mathrm{cl}(D\Phi_A^* + D\Phi_B^*)$. Since $\varphi \in D\Phi_{A\cap B}^*$, there is a constant m such that $\langle \varphi, x\rangle_{E',E} \le m$, for all $x \in A\cap B$. By Hahn-Banach Theorem, there exists a vector x_0 of E such that $\langle \varphi, x_0\rangle_{E',E} > m$ and $\langle y, x_0\rangle_{E',E} \le m$ for all $y \in \mathrm{cl}(D\Phi_A^* + D\Phi_B^*)$. Since $\mathrm{cl}(D\Phi_A^* + D\Phi_B^*)$ is a convex cone, we get

$$\langle y, x_0\rangle_{E',E} \le 0 \quad \text{for all} \quad y \in \mathrm{cl}(D\Phi_A^* + D\Phi_B^*).$$

Hence x_0 belongs to $(D\Phi_A^*)^\circ \cap (D\Phi_B^*)^\circ$ which is included in $A\cap B$ by (3.1). This proves (3.2). Next we will prove

$$\mathrm{cl}D\Phi_{A\cap B}^* = (\mathrm{ca}(A\cap B))^\circ. \tag{3.3}$$

Clearly we have $\mathrm{cl}D\Phi_{A\cap B}^* \subset (\mathrm{ca}(A\cap B))^\circ$. Suppose now that there is an element φ of $(\mathrm{ca}(A\cap B))^\circ$ which does not belong to $D\Phi_{A\cap B}^*$. Again by Hahn-Banach Theorem, there exists a vector x_0 of E such that $\langle \varphi, x_0\rangle_{E',E} > 1$ and $\langle y, x_0\rangle_{E',E} \le 1$ for all $y \in D\Phi_{A\cap B}^*$. Since $D\Phi_{A\cap B}^*$ is a convex cone, one deduces that, for any λ positive, $\langle y, \lambda x_0\rangle_{E',E} \le 0$ for all $y \in D\Phi_{A\cap B}^*$. Therefore the vector λx_0 belongs to $(D\Phi_{A\cap B}^*)^\circ$, which is included in $A\cap B$ by

(3.1). This proves that x_0 belongs to $\mathrm{ca}(A\cap B)$ and (3.3) follows. Since $A\cap B$ is non-empty, we have $\mathrm{ca}(A\cap B)=\mathrm{ca}A\cap\mathrm{ca}B$ which completes the proof of Proposition 3.1.

Proof of Theorem 1.1 completed. By using condition (1.2) and Proposition 3.1, we see that

$$\mathrm{cl}(D\Phi_A^*+D\Phi_B^*)=(\mathrm{ca}A\cap\mathrm{ca}B)^\perp. \tag{3.4}$$

Recall that A is included in a finite dimensional subspace F of E. We denote by H the topological complementary subspace of $F^\perp$ in E'. H is also a finite dimensional subspace and more precisely we have $E'=F^\perp+H$ and

$$E=F+H^\perp. \tag{3.5}$$

Since $F^\perp$ is included in $D\Phi_A^*+D\Phi_B^*$, we have

$$D\Phi_A^*+D\Phi_B^*=F^\perp+H\cap(D\Phi_A^*+D\Phi_B^*). \tag{3.6}$$

At the same time, $F^\perp$ is included in $(\mathrm{ca}A\cap\mathrm{ca}B)^\perp$, and we then get

$$(\mathrm{ca}A\cap\mathrm{ca}B)^\perp=F^\perp+H\cap(\mathrm{ca}A\cap\mathrm{ca}B)^\perp. \tag{3.7}$$

We should use (3.4), (3.5) and (3.6) to prove that the closure of the convex cone $H\cap(D\Phi_A^*+D\Phi_B^*)$, in the subspace $H\cap(\mathrm{ca}A\cap\mathrm{ca}B)^\perp$ with the inherited topology of E', is just $H\cap(\mathrm{ca}A\cap\mathrm{ca}B)^\perp$. In fact, let h be an arbitrary element of $H\cap(\mathrm{ca}A\cap\mathrm{ca}B)^\perp$. Then by (3.4), it belongs to all closed half-space with respect to the weak* topology containing $D\Phi_A^*+D\Phi_B^*$. We now prove that it belongs to all closed half-space for the weak* topology which contains $H\cap(D\Phi_A^*+D\Phi_B^*)$. So we pick such a half-space L, it is defined by a vector x of E and a real number α such that $\langle y,x\rangle_{E',E}\le\alpha$, for all $y\in H\cap(D\Phi_A^*+D\Phi_B^*)$. Then by (3.5), we can suppose that x belongs to F, and consequently by (3.6), we see that

$$\langle y,x\rangle_{E',E}\le\alpha \quad \text{for all} \quad y\in D\Phi_A^*+D\Phi_B^*.$$

Hence the vector h is contained in L, which means that h belongs to the closure of $H\cap(D\Phi_A^*+D\Phi_B^*)$ for the weak* topology. On the other hand, since $H\cap(\mathrm{ca}A\cap\mathrm{ca}B)^\perp$ is a finite dimensional subspace, every convex subset which is dense in this space equals to it (see [7, Corollary 6.3.1]). Therefore we get $H\cap(D\Phi_A^*+D\Phi_B^*)=H\cap(\mathrm{ca}A\cap\mathrm{ca}B)^\perp$. Using this equality together with (3.6) and (3.7), we obtain $D\Phi_A^*+D\Phi_B^*=(\mathrm{ca}A\cap\mathrm{ca}B)^\perp$. Finally, Proposition 1.1 applies and yields the closedness of the convex set $A-B$. This proves Theorem 1.1.

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