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ON PSEUDO-HOLOMORPHIC CURVES IN COMPLEX GRASSMANNIANS**

SHEN Yibing* DONG Yuxing*

Abstract

Further geometry and topology for pseudo-holomorphic curves in complex Grassmannians $G_m(\mathbb{C}^N)$ are studied. Some curvature pinching theorems for pseudo-holomorphic curves with constant Kähler angles in $G_m(\mathbb{C}^N)$ are obtained, so that the corresponding results for pseudo-holomorphic curves in complex projective spaces are generalized.

Keywords Pseudo-holomorphic curve, Complex Grassmannian, Kähler angle, Curvature pinching

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§0. Introduction

As is well known, begining with a harmonic map $f : M \to G_m(\mathbb{C}^N)$ from a Riemann surface to a complex Grassmannian, one can obtain the harmonic sequences by using the ∂-transform and $\bar{\partial}$-transform.[1] When f is holomorphic, each element in the harmonic sequence is called a pseudo-holomorphic curve and the coresponding map is called a pseudo-holomorphic map.[2] The importance of such maps comes from the fact that any harmonic map $\phi : M \to \mathbb{C}P^n = G_1(\mathbb{C}^{n+1})$ is pseudo-holomorphic if $|\deg(\phi)| > n(g-1)$ where $\deg(\phi)$ is the degree of ϕ and g is the genus of M.[3,4]

In [5] J. Bolton et al. gave some curvature pinching theorems for pseudo-holomorphic curves in $\mathbb{C}P^n$. Then Y. Zheng[6] generalized these results to pseudo-holomorphic curves in $G_m(\mathbb{C}^N)$. The first named author of this paper has obtained further pinching theorems for pseudo-holomorphic curves in $\mathbb{C}P^n$ with constant Kähler angles.[7] Some similar results were shown by T. Ogata independently.[8] Very recently, Li[9] studied holomorphic S^2 in $G_m(\mathbb{C}^N)$ with constant curvature with respect to the induced metric.

The purpose of the present paper is to generalize the results of [7] to pseudo-holomorphic curves in $G_m(\mathbb{C}^N)$ so that some theorems of [6] are improved. In §1 some necessary preliminaries for this paper are given (see [1, 6] for details). Then a general curvature pinching (Theorem 2.1) for pseudo-holomorphic curves with constant Kähler angles in $G_m(\mathbb{C}^N)$ will be shown in §2. In §3 we study the pseudo-holomorphic curves that generate the Frenet

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*Department of Mathematics, West Brook Campus, Zhejiang University, Hangzhou 310028, China.
E-mail: ybshen@tiger.hzuniv.edu.cn

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harmonic sequences. For such curves, a topological restriction (Theorem 3.1) and a further curvature pinching (Theorem 3.2) will be given, so that the corresponding results of [7] are extended. Finally, some examples are constructed in §4.

§1. Preliminaries

Let $\mathbb{C}^N$ be endowed with the usual Hermitian inner product $\langle\ ,\ \rangle$. We shall use the followimg notations

$$z_{\bar{A}} = \bar{z}_A, \qquad \sigma_{\bar{A}B} = \bar{\sigma}_{A\bar{B}}, \qquad \text{etc.} \qquad (1 \le A, B, C \le N). \tag{1.1}$$

Let $Z = \{Z_A\}$ be a unitary frame in $\mathbb{C}^N$ so that $\langle Z_A, Z_B\rangle = \delta_{AB}$. The space of unitary frames may be identified with the unitary group $U(N)$. The Maurer-Cartan forms $\omega_{A\bar{B}}$ of $U(N)$ satisfy

$$d\omega_{A\bar{B}} = \sum_C \omega_{A\bar{C}} \wedge \omega_{C\bar{B}}, \qquad \omega_{A\bar{B}} + \omega_{\bar{B}A} = 0. \tag{1.2}$$

An element of a complex Grassmannian $G_m(\mathbb{C}^N)$ can be expressed by the multivector $Z_1 \wedge \cdots \wedge Z_m \neq 0$, defined up to a nonzero factor. Thus, $G_m(\mathbb{C}^N)$ may be realized as a homogeneous space $U(N)/U(m) \times U(N-m)$. The positive definite Hermitian form

$$ds_G^2 = \sum_{i,\alpha} \omega_{i\bar{\alpha}}\omega_{\bar{i}\alpha} \qquad (1 \le i \le m;\ m+1 \le \alpha \le N) \tag{1.3}$$

defines a Kähler metric on $G_m(\mathbb{C}^N)$; when $m = 1$ and $N = n + 1$, this is the Fubini-Study metric on $\mathbb{C}P^n$ with constant holomorphic sectional curvature 4.

Let M be a Riemann surface and $f : M \to G_m(\mathbb{C}^N)$ be a smooth map. The Riemannian metric of M may be written as $ds_M^2 = \varphi\bar{\varphi}$, where φ is a complex-valued 1-form defined up to a factor of norm one. For the smooth map f, we choose a local field of unitary frames $\{Z_A\}$ along f such that $\{Z_i\}$ span f. Then we have[1]

$$[f^*\omega_{A\bar{B}}] = \Phi = \begin{bmatrix} \Phi_{11} & A\varphi + B\bar{\varphi} \\ -{}^t\bar{A}\bar{\varphi} - {}^t\bar{B}\varphi & \Phi_{22} \end{bmatrix}, \tag{1.4}$$

where $A, B \in \mathfrak{M}_{m\times(N-m)}$, $\Phi_{11} \in u(m) \otimes T^*M$ and $\Phi_{22} \in u(N-m) \otimes T^*M$. Here and from now on, we denote by $\mathfrak{M}_{m_1\times m_2}$ the set of all locally defined $m_1 \times m_2$-matrix-valued smooth functions and by $u(n) \otimes T^*M$ the set of all locally defined $u(n)$-valued smooth 1-forms, where $u(n)$ is the Lie algebra of $U(n)$.

Thus, f is an isometric immersion if and only if

$$\mathrm{tr}(A^t\bar{B}) = 0, \qquad |\ A\ |^2 + |\ B\ |^2 = 1, \tag{1.5}$$

where the norm of a matrix Q is defined as $|\ Q\ |^2 = \mathrm{tr}(Q^t\bar{Q})$ in a standard manner. In such a case we can introduce the Kähler angle α_f of f which is defined to be the angle between $Jdf(e_1)$ and $df(e_2)$ for $0 \le \alpha_f \le \pi$, where $\{e_1, e_2\}$ is an orthonormal basis on M and J stands for the natural complex structure of $G_m(\mathbb{C}^N)$. It was pointed out in [1] that the Kähler angle α_f is a smooth real function on M except for at most isolated points. By the geometric interpretation of α_f, we have from (1.4)

$$\cos\alpha_f \varphi \wedge \bar{\varphi} = \sum (f^*\omega_{i\bar{\alpha}}) \wedge (f^*\omega_{\bar{i}\alpha}) = (|\ A\ |^2 - |\ B\ |^2)\varphi \wedge \bar{\varphi},$$

from which it follows that

$$|\ A\ |^2 - |\ B\ |^2 = \cos\alpha_f. \tag{1.6}$$

By (1.5) and (1.6) we obtain

$$| A |^2 = \frac{1}{2}(1+\cos\alpha_f), \qquad | B |^2 = \frac{1}{2}(1-\cos\alpha_f). \tag{1.7}$$

For a harmonic map $f : M \to G_m(\mathbb{C}^N)$, by making use of the ∂-transform (resp. $\bar\partial$-transform), Chern-Wolfson[1] gave the new harmonic map $\partial f : M \to G_{m_1}(\mathbb{C}^N)$ (reps. $\bar\partial f : M \to G_{m_{-1}}(\mathbb{C}^N)$), where m_1 (resp. m_{-1}) is the rank of ∂f (resp. $\bar\partial f$). By successive applications of the ∂-transform, we can obtain a sequence of harmonic maps

$$f \equiv f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \xrightarrow{\partial} \cdots , \tag{1.8}$$

which is called the harmonic sequence generated by f, where $f_j \equiv \partial f_{j-1} : M \to G_{m_j}(\mathbb{C}^N)$ with rank m_j for $j \geq 1$. There is a similar statement for the $\bar\partial$-transform. If for all $x \in M$, $f_j(x) \perp f_k(x)$ as linear subspaces in $\mathbb{C}^N$ with respect to the Hermitian inner product, then we say f_j and f_k are orthogonal. The sequence (1.8) is said to be orthogonal if $f_j \perp f_k$ for all $j \neq k$, so that the length of (1.8) is finite. When f is holomorphic, the harmonic sequence (1.8) must be orthogonal and hence be finite. In such a case each element f_j $(j \geq 1)$ in (1.8) is called a pseudo-holomorphic curve with position j generated by the directrix f_0.

When we concentrate on a special pseudo-holomorphic curve $f : M \to G_m(\mathbb{C}^N)$ with position r, we may rewrite the sequence (1.8) as

$$f_{-r} \xrightarrow{\partial} \cdots \xrightarrow{\partial} f_{-1} \xrightarrow{\partial} f_0 \equiv f \xrightarrow{\partial} f_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} f_s \tag{1.9}$$

for some nonnegative integer s, where the directrix f_{-r} is holomorphic and f_s is anti-holomorphic so that $m_{s+1} = 0$. In addition, if m_j's $(-r \leq j \leq s)$ are equal to some m, then we have

$$f_{-r} \rightleftharpoons \cdots \rightleftharpoons f_{-1} \rightleftharpoons f_0 \equiv f \rightleftharpoons f_1 \cdots \rightleftharpoons f_s, \tag{1.10}$$

which is called a Frenet harmonic sequence of rank m.

§2. Curvature Pinchings for Pseudo-Holomorphic Curves with Constant Kähler Angles

Let M be a compact Riemann surface and $f : M \to G_m(\mathbb{C}^N)$ be a pseudo-holomorphic curve with position r. Then, f generates an orthogonal harmonic sequence (1.9). Denote by α_f and K_f the Kähler angle and the Gauss curvature of f relative to the metric induced by f, respectively. We begine with the following

Proposition 2.1. *If f is not anti-holomorphic and $K_f \geq 2(1+\cos\alpha_f)/(\mathrm{rank}\partial f)$ on M pointwisely, then $K_f = 2(1+\cos\alpha_f)/(\mathrm{rank}\partial f)$. Similarly, if f is not holomorphic and $K_f \geq 2(1-\cos\alpha_f)/(\mathrm{rank}\bar\partial f)$ on M pointwisely, then $K_f = 2(1-\cos\alpha_f)/(\mathrm{rank}\bar\partial f)$.*

Proof. Assume that f is not anti-holomorphic so that $\mathrm{rank}\partial f = k_1 \neq 0$. Choose a local unitary frame $Z = \{Z_1, \cdots, Z_N\}$ along f such that $Z_1, \cdots, Z_m$ span f, $Z_{m+1}, \cdots, Z_{m+k_1}$ span ∂f and $Z_{k_1+1}, \cdots, Z_m$ span the kernel of the ∂-transform. The pull back of the Maurer-Cartan forms of $U(N)$ by Z is then (1.4) with

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$A_{11}, B_{11} \in \mathfrak{M}_{k_1 \times k_1}, \quad B_{12} \in \mathfrak{M}_{k_1 \times (N-m-k_1)},$$
$$B_{21} \in \mathfrak{M}_{(m-k_1) \times k_1}, \quad B_{22} \in \mathfrak{M}_{(m-k_1) \times (N-m-k_1)}.$$

It follows that

$$| A |^2 = | A_{11} |^2, \qquad | B |^2 = | B_{11} |^2 + | B_{12} |^2 + | B_{21} |^2 + | B_{22} |^2 . \tag{2.1}$$

By making use of Proposition 1.1 in [6], the harmonicity condition of f gives (see the proof of Theorem 2.1 of [6] for details)

$$\begin{aligned} \triangle \log | \det A_{11} | &\geq k_1 K_f + 2(2 | B_{11} |^2 + | B_{12} |^2 + | B_{21} |^2 - 2 | A_{11} |^2) \\ &\geq k_1 K_f - 4 | A_{11} |^2 . \end{aligned} \tag{2.2}$$

From (1.7), (2.1) and (2.2) we have

$$\triangle \log | \det A_{11} | \geq k_1 K_f - 2(1 + \cos\alpha_f). \tag{2.3}$$

As is pointed out in [6], $| \det A_{11} |$ is a globally defined nonnegative invariant on M. Since $K_f \geq 2(1 + \cos\alpha_f)/k_1$ everywhere, we have $\triangle \log | \det A_{11} | \geq 0$, which holds on the compact surface M except for at most finitely many points. By the maximum principle of subharmonic functions, $| \det A_{11} |$ must be constant on M. Thus, by (2.3) and the continuity of α_f, $K_f = 2(1 + \cos\alpha_f)/k_1$. The first claim of the Proposition is proved.

The proof of the second case in the proposition is similar and we omit it here.

For a pseudo-holomorphic curve $f : M \to G_m(\mathbb{C}^N)$ with position r, which generates the harmonic sequence (1.9), we write

$$\operatorname{rank}(f_{j_0}) = m_0 \text{ for } 0 \leq j_0 \leq k_0; \ \operatorname{rank}(f_{j_t}) = m_t \text{ for } \sum_{i=0}^{t-1} k_i + 1 \leq j_t \leq \sum_{i=0}^{t} k_i,$$

where $1 \leq t \leq p$ and $\sum\limits_{i=0}^{p} k_i = s$.

Theorem 2.1. *Let M be a compact Riemann surface and $f : M \to G_m(\mathbb{C}^N)$ be the pseudo-holomorphic curve as above. If the Kähler angle α_f is constant and the Gauss curvature K_f is not less than the constant*

$$c_f = \frac{2 \prod\limits_{t=1}^{p} (k_t + 1) (1 + (2k_0 + 1) \cos\alpha_f)}{\sum\limits_{t=0}^{p} \prod\limits_{j=t}^{p} (k_j + 1) k_t m_t}, \tag{2.4}$$

then $K_f = c_f$.

Proof. Choose suitably a smooth unitary frame $Z = \{Z_A\}$ such that the pull back Φ of the Maurer-Cartan forms by Z is

$$\begin{bmatrix} \Phi_0 & A_0\varphi & & & 0 & & B_1\bar{\varphi} & B_2\bar{\varphi} \\ -{}^t\bar{A}_0\bar{\varphi} & \Phi_1 & A_1\varphi & & & & & \\ & 0 & & & & & 0 & \\ & & & -{}^t\bar{A}_{j-1}\bar{\varphi} & \Phi_j & A_j\varphi & & \\ -{}^t\bar{B}_1\varphi & & & & & -{}^t\bar{A}_{s-1}\bar{\varphi} & \Phi_s & 0 \\ -{}^t\bar{B}_2\varphi & & & & 0 & & 0 & \Phi_{s+1} \end{bmatrix},$$

where $B_1 \in \mathfrak{M}_{m_0\times m_t}$; $B_2 \in \mathfrak{M}_{m_0\times(N-\sum_{i=0}^{p} k_i m_i - m_0)}$; $A_j \in \mathfrak{M}_{m_t\times m_t}$ for $\sum_{i=-1}^{t-1} k_i < j < \sum_{i=-1}^{t} k_i$ and $0 \le t \le p$; $A_s \in \mathfrak{M}_{m_s\times(N-\sum_{i=0}^{p} k_i m_i - m_0)}$; $\Phi_0 \in u(m_0)\otimes T^*M$; $\Phi_j \in u(m_t)\otimes T^*M$ for $\sum_{i=-1}^{t-1} k_i < j < \sum_{i=-1}^{t} k_i$ and $0 \le t \le p$; $\Phi_{s+1} \in u(N-\sum_{i=0}^{p} k_i m_i - m_0)\otimes T^*M$. Here we set $k_{-1}=0$ for notational convenience.

Now (1.7) becomes

$$| A_0 |^2 = \frac{1}{2}(1+\cos\alpha_f), \quad | B_1 |^2 + | B_2 |^2 = \frac{1}{2}(1-\cos\alpha_f). \tag{2.5}$$

By the same manner as in the proof of Proposition 2.1, we have[6]

$$\triangle \log | \det A_0 | = m_0 K_f + 2(| A_1 |^2 - 2 | A_0 |^2 + | B_1 |^2 + | B_2 |^2) \tag{2.6}$$

for $k_0 > 0$;

$$\triangle \log | \det A_j | = m_t K_f + 2(| A_{j-1} |^2 - 2 | A_j |^2 + | A_{j+1} |^2) \tag{2.7}$$

for $\sum_{i=-1}^{t-1} k_i < j < \sum_{i=-1}^{t} k_i$, $0 \le t \le p$, and $j \ne s-1$; and

$$\triangle \log | \det A_{s-1} | = m_p K_f + 2(| A_{s-2} |^2 - 2 | A_{s-1} |^2 + | A_s |^2 + | B_1 |^2). \tag{2.8}$$

Moreover, we can require

$$A_{k_0+\cdots+k_t} = \begin{bmatrix} 0 \\ \tilde{A}_{k_0+\cdots+k_t} \end{bmatrix} \qquad \text{for} \quad 0 \le t < p,$$

where $\tilde{A}_{k_0+\cdots+k_t} \in \mathfrak{M}_{m_{t+1}\times m_{t+1}}$. Thus, we may write

$$\det A_{k_0+\cdots k_t} = \det \tilde{A}_{k_0+\cdots+k_t},$$

even though the first one does not make sense. By the same computation as in [6], we have

$$\triangle \log | \det A_{k_0+\cdots+k_t} | \ge m_{t+1} K_f + 2(| A_{k_0+\cdots+k_t+1} |^2 - 2 | A_{k_0+\cdots+k_t} |^2) \tag{2.9}$$

for $0 \le t \le p-1$ and $k_0 > 0$, where it is used that $| \tilde{A}_{k_0+\cdots+k_t} |^2 = | A_{k_0+\cdots+k_t} |^2$. If $k_0 = 0$, then

$$\triangle \log | \det \tilde{A}_0 | \ge m_1 K_f + 2(| A_1 |^2 - 2 | A_0 |^2). \tag{2.10}$$

For $0 \le t \le p$, we put

$$F_t = \sum_{j=0}^{k_t-1} \left(\sum_{i=k_{-1}+k_0+\cdots+k_{t-1}}^{k_{-1}+k_0+\cdots+k_{t-1}+j} \triangle log | \det A_i | \right),$$

$$F = F_p + \sum_{t=1}^{p} \left(\prod_{j=t}^{p} (k_j+1) \right) F_{t-1}.$$

Then, F_t as well as F is a well-defined continuous function on M.[6] By (2.5)–(2.10), a careful computation gives

$$F \ge \frac{1}{2} \sum_{t=0}^{p} \prod_{j=t}^{p} (k_j+1) k_t m_t K_f - \left(\prod_{t=1}^{p} (k_t+1) \right) [1 + (2k_0+1)\cos\alpha_f]. \tag{2.11}$$

It follows from (2.11) that $F \ge 0$ provided that $K_f \ge c_f$, where c_f is defined by (2.4). Now, an argument similar to that of [6] shows that $K_f = c_f$.

Remark 2.1. There is a similar statement for the harmonic sequence

$$\cdots \overset{\bar{\partial}}{\leftarrow} f_{-2} \overset{\bar{\partial}}{\leftarrow} f_{-1} \overset{\bar{\partial}}{\leftarrow} f_0 \equiv f$$

generated by f via $\bar{\partial}$-transforms (cf. Proposition 2.1). When $\cos\alpha_f = 1$, i.e., f is holomorphic, the pinching constant c_f was obtained by Y. Zheng.[6]

An interesting situation is when the pseudo-holomorphic curve f generates a Frenet harmonic sequence (1.10) of rank m. In such a case, we have obviously

$$m_0 = m, \quad k_0 = p = s, \quad k_1 = k_2 = \cdots = 0.$$

Noting that $\text{rank}\bar{\partial} f_j = \text{rank}\partial f_{j-1} = m$ for the Frenet sequence, from Theorem 2.1 and Remark 2.1 we have immediately

Corollary 2.1. *Let $f : M \to G_m(\mathbb{C}^N)$ be a pseudo-holomorphic curve with constant Kähler angle α_f, which generates a Frenet harmonic sequence of rank m. If the Gauss curvature K_f relative to the metric induced by f satisfies*

$$K_f \geq \frac{2[1 \pm (2s+1)\cos\alpha_f]}{s(s+1)m}, \tag{2.12}$$

then $K_f = 2[1 \pm (2s+1)\cos\alpha_f]/s(s+1)m$.

Remark 2.2. A further pinching theorem will be given in next section. The case that $m = 1$ has been given in [7].

§3. Frenet Harmonic Sequences

Let M be a compact Riemann surface with genus g and $f : M \to G_m(\mathbb{C}^N)$ a pseudo-holomorphic map with position r which generates a Frenet harmonic sequence (1.10) of rank m. Denote by V_i the vector bundle over M induced by f_i for $-r \leq i \leq s$. For each i there is a holomorphic bundle map $\partial_i : V_i \to V_{i+1} \otimes TM^{(1,0)}$, where $TM^{(1,0)}$ denotes the cotangent bundle on M of type (1,0). By taking the mth exterior power of each bundle, we obtain the holomorphic bundle map $\det\partial_i : \wedge^m V_i \to \wedge^m V_{i+1} \otimes (TM^{(1,0)})^m$. Then, from (1.10) we have the following sequence of line bundles (see [10] for details):

$$\wedge^m V_{-r} \overset{\det\partial_{-r}}{\longrightarrow} \cdots \overset{\det\partial_{-2}}{\longrightarrow} \wedge^m V_{-1} \overset{\det\partial_{-1}}{\longrightarrow} \wedge^m V_0 \overset{\det\partial_0}{\longrightarrow} \wedge^m V_1 \overset{\det\partial_1}{\longrightarrow} \cdots \overset{\det\partial_{s-1}}{\longrightarrow} \wedge^m V_s.$$

Moreover, denoting by $c_1(\wedge^m V_i)$ the first Chern number of the line bundle $\wedge^m V_i$, we have the following Plücker formula[10]

$$c_1(\wedge^m V_{i+1}) = c_1(\wedge^m V_i) + \delta_i - 2m(g-1), \tag{3.1}$$

where δ_i stands for the ramification index of $\det\partial_i$.

A pseudo-holomorphic map $f : M \to G_m(\mathbb{C}^N)$ is said to be totally unramified if all $\delta_i \equiv 0$ for $-r \leq i \leq s$. We have the following

Theorem 3.1. *Let M be a compact Riemann surface with genus g and $f : M \to G_m(\mathbb{C}^N)$ a pseudo-holomorphic curve which generates a Frenet harmonic sequence. If f is totally unramified, then $g = 0$, namely, M is homeomorphic to S^2.*

Proof. Denote by $E(\partial_i)$ (resp. $E(\bar{\partial}_i)$) the ∂-energy (resp. $\bar{\partial}$-energy) of the map $M \to G_m(\mathbb{C}^N)$ determined by V_i. We have[10]

$$c_1(\wedge^m V_i) = \frac{1}{\pi}E(\partial_{i-1}) - \frac{1}{\pi}E(\partial_i). \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\delta_i = \frac{1}{\pi}\{-E(\partial_{i+1}) + 2E(\partial_i) - E(\partial_{i-1})\} - 2m(1-g) \tag{3.3}$$

for $-r \le i \le s-1$. Applying (3.3) inductively gives

$$\frac{1}{\pi}E(\partial_i) = (i+r+1)[\frac{1}{\pi}E(\partial_{-r}) - m(i+r)(1-g)] - \sum_{j=-r}^{i-1}(i-j)\delta_j,$$

$$\sum_{j=-r}^{s-1}(s-j)\delta_j = \frac{1}{\pi}(r+s+1)E(\partial_{-r}) - m(r+s)(r+s+1)(1-g).$$

From these we deduce

$$\frac{1}{\pi}E(\partial_i) = m(s-i)(r+1+i)(1-g) + \frac{s-i}{r+s+1}\sum_{j=-r}^{i-1}(j+r+1)\delta_j + \frac{r+1+i}{r+s+1}\sum_{j=i}^{s-1}(s-j)\delta_j.$$

The assumption of Theorem 3.1 implies that $E(\partial_i) = \pi m(s-i)(r+1+i)(1-g)$, so that

$$E(\partial_{-1}) + E(\partial_0) = \pi m(2rs + r + s)(1-g). \tag{3.4}$$

On the other hand, by the definition of the energy of V_i, we have

$$E(\partial_{-1}) + E(\partial_0) = E(\bar{\partial}_0) + E(\partial_0) = E(V_0) > 0,$$

which together with (3.4) yields $g = 0$. This completes the proof.

Noting that the degree of $\det\partial_i$ is $\deg(\det\partial_i\ ) = -c_1(\wedge^m V_i)$, we have directly

Corollary 3.1.[10] *If a pseudo-holomorphic curve $f : T^2 \to G_m(\mathbb{C}^N)$ from a torus T^2 generates a Frenet harmonic sequence, then there exists at least an element with nonzero degree in the sequence of corresponding line bundles.*

Now, we consider the further curvature pinching for the pseudo-holomorphic curve $f : S^2 \to G_m(\mathbb{C}^N)$ with position r which generates a Frenet harmonic sequence of rank m. Without loss of generality, assume that f is *full*, i.e., the image $f(S^2)$ does not lie in any linear subspace of $\mathbb{C}^N$. In such a case, we can choose suitably a smooth unitary frame $Z = \{Z_A\}$ such that the pull back Φ of the Maurer-Cartan forms by Z is[2]

$$\begin{bmatrix} \Phi_{-r} & A_{-r}\varphi & & & & & 0 \\ & & & & & & \\ & & -{}^t\bar{A}_{-1}\bar{\varphi} & \Phi_0 & A_0\varphi & & 0 \\ & 0 & & -{}^t\bar{A}_0\bar{\varphi} & \Phi_1 & A_1\varphi & \\ & & & & & & \\ 0 & & & & & -{}^t\bar{A}_{s-1}\bar{\varphi} & \Phi_s \end{bmatrix},$$

where $\Phi_j \in u(m)\otimes T^*S^2$ for $-r \le j \le s$; $A_j \in \mathfrak{M}_{m\times m}$ for $-r \le j \le s-1$. Similar to (2.10), we then have

$$\triangle \log \mid \det A_j \mid = mK_f + 2(\mid A_{j-1}\mid^2 - 2\mid A_j\mid^2 + \mid A_{j+1}\mid^2) \tag{3.5}$$

with $A_{-r-1} = A_s = 0$. Moreover, (1.7) reduces

$$\mid A_0\mid^2 = \frac{1}{2}(1+\cos\alpha_f), \quad \mid A_{-1}\mid^2 = \frac{1}{2}(1-\cos\alpha_f). \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$\triangle \log \mid \det A_0 \mid = mK_f - 2\cos\alpha_f + 2(\mid A_1\mid^2 - \mid A_0\mid^2), \tag{3.7}$$

$$\triangle \log \mid \det A_{-1} \mid = mK_f + 2\cos\alpha_f + 2(\mid A_{-2}\mid^2 - \mid A_{-1}\mid^2). \tag{3.8}$$

It is pointed out in [2] that all $|\det A_j|$ are globally defined invariants of analytic type on S^2 vanishing only at isolated points. By (3.5), (3.7) and (3.8), a direct computation gives

$$\frac{1}{2}\triangle\log\prod_{k=1}^{p-1}\prod_{j=0}^{k-1}|\det A_j| = \frac{1}{4}mp(p+1)K_f - \frac{1}{2}[1+(2p+1)\cos\alpha_f] + |A_p|^2 \tag{3.9}$$

for $1 \le p \le s$, and

$$\frac{1}{2}\triangle\log\prod_{k=-1}^{-p}\prod_{j=-1}^{k}|\det A_j| = \frac{1}{4}mp(p+1)K_f - \frac{1}{2}[1-(2p+1)\cos\alpha_f] + |A_{-p-1}|^2 \tag{3.10}$$

for $1 \le p \le r$.

Theorem 3.2. *Let $f: S^2 \to G_m(\mathbb{C}^N)$ be a full pseudo-holomorphic curve with constant Kähler angle α_f, which is not ±holomorphic and generates the Frenet harmonic sequence of rank m. If the Gauss cutvature K_f of f satisfies*

$$\frac{2[1\pm(2p+1)\cos\alpha_f]}{mp(p+1)} \le K_f \le \frac{2[1\pm(2p-1)\cos\alpha_f]}{mp(p-1)} \tag{3.11}$$

for some integer $p \ge 2$, and if the well-defined invariants $|\det A_j| \ne 0$ for $j = 0, \cdots, p-2$ (resp. $|\det A_{-j}| \ne 0$ for $j = 1, \cdots, p-1$), then K_f is a constant with the end-values of (3.11).

Proof. We consider two cases separately.

Case (i) $|A_{p-1}|^2 \equiv 0$. By the definition of the ramification index δ_i in (3.1) and Lemma 4.1 of [11], it is easy to see that

$$\int_{S^2}\triangle\log|\det A_j|^2 * 1 = -2\pi\delta_j.$$

Thus, by integrating (3.9), we obtain

$$0 \ge \int_{S^2}\{mp(p+1)K_f - 2[1+(2p+1)\cos\alpha_f]\} * 1,$$

which together with the left hand side of (3.11) yields $K_f = 2[1+(2p+1)\cos\alpha_f]/mp(p+1)$.

Case (ii) $|A_{p-1}|^2 \equiv 0$. By a formula similar to (3.9), we have

$$\triangle\log\prod_{k=1}^{p-2}\prod_{j=0}^{k-1}|\det A_j| = \frac{1}{2}mp(p-1)K_f - [1+(2p-1)\cos\alpha_f],$$

from which it follows that

$$0 = \int_{S^2}\{mp(p-1)K_f - 2[1+(2p-1)\cos\alpha_f]\} * 1 \tag{3.12}$$

because, by the assumption, $|\det A_j| > 0$ for $j = 0, \cdots, p-2$. Combining (3.12) with the right hand side of (3.11) gives $K_f = 2[1+(2p-1)\cos\alpha_f]/mp(p-1)$.

In the same way, the other curvature pinching case of the theorem can follow from (3.10). Hence, the theorem is proved completely.

Corollary 3.2. *Under the same hypothesis as in Theorem* 3.2, *if*

$$K_f \ge 2[1\pm(2p+1)\cos\alpha_f]/mp(p+1) \tag{3.13}$$

for some integer $p \ge 2$ and $|A_{p-1}|^2 \equiv 0$ (resp. $|A_{-p}|^2 \equiv 0$), then $K_f = 2[1\pm(2p+1)\cos\alpha_f]/mp(p+1)$.

Remark 3.1. When $m = 1$, the results in this section have been given in [7].

§4. Some Examples

Example 4.1. As is well known, a harmonic two-sphere $\phi : S^2 \to \mathbb{C}P^n = G_1(\mathbb{C}^{n+1})$ is pseudo-holomorphic. In particular, if the Gauss curvature of the metric induced by ϕ is constant, then, up to a holomorphic isometry of $\mathbb{C}P^n$, ϕ generates the Veronese harmonic sequence.[5]

Let

$$\phi_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \phi_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} \phi_n \tag{4.1}$$

be the Veronese sequence in $\mathbb{C}P^n$. We have[5]

$$\mid \partial\phi_p \mid^2 = \mid \bar\partial\phi_{p+1} \mid^2 = \frac{(p+1)(n-p)}{(1+\mid z \mid^2)^2} \tag{4.2}$$

for $0 \le p \le n$, where $z = z_1/z_0$ and $[z_0, z_1] \in \mathbb{C}P^1 = S^2$. The metric induced by ϕ_p is

$$ds_p^2 = \left(\mid \partial\phi_p \mid^2 + \mid \bar\partial\phi_p \mid^2\right) \mid dz \mid^2 = \frac{n+2p(n-p)}{(1+\mid z \mid^2)^2} \mid dz \mid^2,$$

whose curvature is

$$K_p = \frac{4}{n+2p(n-p)} = \frac{2[1-(1+2p)\cos\alpha_p]}{p(p+1)},$$

where α_p is the Kähler angle of ϕ_p satisfying $\cos\alpha_p = (n-2p)/[n+2p(n-p)]$.

By [12], we can construct a harmonic map $f_{p,m} : S^2 \to G_m(\mathbb{C}^{n+1})$ by $f_{p,m} = \phi_p \perp \cdots \perp \phi_{p+m-1}$ for $m \le n+1-p$, where $\perp$ denotes the orthogonal direct sum. It is clear that $\partial f_{p,m} = \partial\phi_{p+m-1}$ and $\bar\partial f_{p,m} = \bar\partial\phi_p$. Thus, by (4.2), we have

$$ds_f^2 = \left(\mid \partial f_{p,m} \mid^2 + \mid \bar\partial f_{p,m} \mid^2\right) = \frac{(2p+m)(n+1-p)-m(p+m)}{(1+\mid z \mid^2)^2} \mid dz \mid^2 . \tag{4.3}$$

The constant Kähler angle α_f of $f_{p,m}$ is determined by

$$\cos\alpha_f = \frac{\mid \partial f_{p,m} \mid^2 - \mid \bar\partial f_{p,m} \mid^2}{\mid \partial f_{p,m} \mid^2 + \mid \bar\partial f_{p,m} \mid^2} = \frac{m(n+1-2p-m)}{(2p+m)(n+1-p)-m(p+m)}. \tag{4.4}$$

Then the curvature K_f of the metric (4.3) is

$$K_f = \frac{4}{(2p+m)(n+1-p)-m(p+m)} = \frac{2[m-(2p+m)\cos\alpha_f]}{pm(p+m)}. \tag{4.5}$$

Hence, $f_{p,m} : S^2 \to G_m(\mathbb{C}^{n+1})$ is a pseudo-holomorphic curve with constant Kähler angle and constant curvature in $G_m(\mathbb{C}^{n+1})$.

For examples of holomorphic 2-spheres with constant curvature in $G_m(\mathbb{C}^N)$, see [9].

Example 4.2. For any positive integer m, consider m copies of the Veronese cequence in $\mathbb{C}P^n$:

$$\phi_0^{(i)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \phi_p^{(i)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \phi_n^{(i)}$$

for $1 \le i \le m$. We now construct a harmonic map $F_p : S^2 \to G_m(\mathbb{C}^{m(n+1)})$ defined by $F_p = \phi_p^{(1)} \perp \cdots \perp \phi_p^{(m)}$, which generates the following harmonic sequence

$$F_0 \rightleftharpoons \cdots \rightleftharpoons F_p \rightleftharpoons \cdots \rightleftharpoons F_n.$$

Clearly, it is a Frenet harmonic sequence of rank m. By [12], it is easy to see that

$$|\partial F_p|^2 = \sum_{i=1}^{m} |\partial \phi_p^{(i)}|^2 = \frac{m(p+1)(n-p)}{(1+|z|^2)^2},$$

$$|\bar{\partial} F_p|^2 = \sum_{i=1}^{m} |\bar{\partial} \phi_p^{(i)}|^2 = \frac{mp(n+1-p)}{(1+|z|^2)^2}.$$

Thus, the metric induced by F_p is

$$ds_F^2 = \frac{m[n+2p(n-p)]}{(1+|z|^2)^2} |dz|^2 . \tag{4.6}$$

The constant Kähler angle α_F of F_p is determined by

$$\cos\alpha_F = \frac{n-2p}{m[n+2p(n-p)]}. \tag{4.7}$$

Then the curvature K_F of the metric (4.6) is

$$K_F = \frac{4}{m\{n+2p(n-p)]} = \frac{2[1-(2p+1)\cos\alpha_F]}{mp(p+1)}. \tag{4.8}$$

Hence, $F_p : S^2 \to G_m(\mathbb{C}^{m(n+1)})$ is a pseudo-holomorphic curve with constant Kähler angle and constant curvature in $G_m(\mathbb{C}^{m(n+1)})$, which generates the Frenet harmonic sequence of rank m.

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