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REPRESENTATIONS OF SOLVABLE LIE GROUPS AND GEOMETRIC QUANTIZATION***

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Abstract

Representations of solvable Lie groups are realized and classified by geometric quantization of coadjoint orbits through positive polarizations.

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§1. Introduction

The orbit metheod presented by Kirillov and Kostant has greatly been developed recently to realize representations of groups (finite and infinite-dimensional groups). It also has close relationships with the classification of representations[1]. In fact, the deep origin of the orbit metheod is geometric quantization. The problem of geometric quantization is, starting from the geometry of a symplectic manifold (M, ω) which gives the model of a classical mechenical system, to construct a Hilbert space H and a set of operators on it which give the quantum analogue of this systems. When the symplectic manifold M is a Hamiltonian G-space for a Lie group G, we will get an irreducible unitary representation of G. According to Kostant, Hamiltonian G-spaces are coadjoint orbits or their coverings, so to quantize coadjoint orbits is a useful metheod to construct representations of Lie groups. In [2], Kostant declared the result that a coadjoint orbit is quantizable if and only if it is integral. So finally the problem to quantize a coadjoint orbit comes to the proper choice of a polarization. In the paper we give a direct metheod of quantization for Hamiltonian G-spaces.

For semisimple Lie groups, we can get spherical representations by quantization of hyperbolic coadjoint orbits through real polarizations and holomorphic discrete series of representations by quantization of elliptic coadjoint orbits through Kahlerian polarizations.[1] In particular, we may get Borel-Weil theorem by geometric quantization through Kahlerian polarization. When G is a nilpotent Lie group, all irreducible representations of G can be

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[1]Zhao Qiang, Representations of reductive Lie groups and geometric quantization, Preprint.

obtained through real polarizations. More generally, we will construct all the irreducible unitary representations of solvable Lie groups by geometric quantization through positive polarizations.

§2. Unitary Representations of Solvable Lie Groups

2.1. Auslander and Kostant's Results

In this section we will briefly recall Auslander and Kostant's results in [3] about unitary representations of solvable Lie groups.

Let g be an arbitrary Lie algebra over $\mathbf{R}$ and let G be a Lie group with the Lie algebra g. G acts on $g^* = \mathrm{Hom}_{\mathbf{R}}(g, \mathbf{R})$ by the coadjoint representation. For $f \in g^*$ we will denote by $G_f \subset G$ the isotropy group of G at f with respect to the coadjoint representation and the Lie algebra of G_f will be denoted by g_f. Then we have

$$g_f = \{X \in g \mid f([X, Y]) = 0, \ \ \forall Y \in g\}$$

and the dimension of g/g_f is even.

Definition 2.1. *Let $g^{\mathbf{C}} = g + ig$ be the complexification of g and consider $f \in g^*$ as a complex-valued linear functional on $g^{\mathbf{C}}$. If $z = x+iy \in g^{\mathbf{C}}$, where $x, y \in g$, we let $\bar{z} = x - iy$. A polarization at f is a complex subalgebra $h \subset g^{\mathbf{C}}$ such that*

(1) *$g_f \subset h$ and h is stable under AdG_f;* (2) *$f([h, h]) = 0$;*

(3) *$\dim_{\mathbf{C}}(g^{\mathbf{C}}/h) = \frac{1}{2}\dim_{\mathbf{R}}(g/g_f)$;* (4) *$h + \bar{h}$ is a Lie algebra of $g^{\mathbf{C}}$.*

If h is a polarization at $f \in g^*$, let $d = h \cap \bar{h} \cap g$ and $e = (h + \bar{h}) \cap g$. Then it is easy to know that $d^{\mathbf{C}} = d + id = h \cap \bar{h}$, $e^{\mathbf{C}} = e + ie = h + \bar{h}$ and $(e/d)^{\mathbf{C}} = h/d^{\mathbf{C}} \oplus \bar{h}/d^{\mathbf{C}}$. Define $J \in \mathrm{End}(e/d)^{\mathbf{C}}$ by $J = i$ on $h/d^{\mathbf{C}}$ and $J = -i$ on $\bar{h}/d^{\mathbf{C}}$. We could see that J maps e/d onto itself and $J^2 = -id$ on e/d. Let S_f be the bilinear form on e/d defined by $S_f(u, v) = f([Jx, y])$, where $u = x + d, v = y + d \in e/d$. It is easy to be verified that S_f is well-defined and is a non-singular symmetric bilinear form on e/d.

Definition 2.2. *The polarization h at f is said to be positive if S_f is a positive definite bilinear form or in case $e/d = 0$.*

Let D_0 and E_0 be the connected Lie subgroups of G with the Lie algebras d and e respectively. Then $D = G_f D_0$ and $E = G_f E_0$ are subgroups of G and D is closed in G. If E is closed in G, we will say that h satisfies the Pukansky condition.

Assume that $f \in g^*$ is integral, that is, there exists a character $\eta_f : G_f \to S^1$ whose differential is $2\pi i f|g_f$. When the polarization h at f satisfies the Pukansky condition, we can extend η_f to a unique character $\Lambda_f : D \to S^1$ whose differential is $2\pi i f|d$. Next we recall the induced representation $\mathrm{Ind}_G(\eta_f, h) = \mathrm{Ind}_G(\mathrm{Ind}_E(\eta_f, h))$.

Let $C^\infty(G)$ denote the space of all smooth complex-valued functions on G. We may consider $C^\infty(G)$ as a right $g^{\mathbf{C}}$-module as follows: Let $z = x + iy \in g^{\mathbf{C}}$ with $x, y \in g$ and let $\phi \in C^\infty(G)$. We define $\phi \cdot z = \phi \cdot x + i\phi \cdot y$ where we define $\phi \cdot x$ for $x \in g$ by

$$(\phi \cdot x)(a) = \frac{d}{dt}\Big|_{t=0} \phi(a\exp(-tx)), \quad a \in G.$$

Let $Y = E/D$. There exists a strongly quasi-invariant measure μ on Y, that is, there is a positive function $\lambda \in C^\infty(E \times E)$ such that $\frac{d\mu_x(yD)}{\mu(yD)} = \lambda(x, y)$ for any $x, y \in E$, where

μ_x is defined by $\mu_x(F) = \mu(xF)$ for any measurable subset F of Y. Let $H(E, \eta_f, h)$ be the closure of the set

$$\Big\{\phi \in C^\infty(E)\Big| \int_Y |\phi|^2 d\mu < \infty, \phi(x\xi) = \Lambda_f(\xi^{-1})\phi(x),$$
$$\phi \cdot z = 2\pi i f(z)\phi, x \in E, \xi \in D, z \in \bar{h}\Big\}.$$

Then the representation $\mathrm{Ind}_E(\eta_f, h)$ of E is defined by

$$(\mathrm{Ind}_E(\eta_f, h)(a)(\phi))(b) = \sqrt{\lambda(a,b)}\phi(a^{-1}b), \quad a, b \in E, \phi \in H(E, \eta_f, h).$$

Since E is closed in G, we could get the induced representation

$$\mathrm{Ind}_G(\eta_f, h) = \mathrm{Ind}_G(\mathrm{Ind}_E(\eta_f, h)).$$

When G is a connected solvable Lie group, we use n to denote the nil-radical of g . For an integral $f \in g^*$, let $g = f|n$. Since n is an ideal in g , we may consider n^* as a G-module. Thus if G_g is the isotropy group of G at g, then $G_f \subset G_g$. A polarization at f is said to be strongly admissible if $h \cap n^{\mathbf{C}}$ is stable under the action of G_g and a subspace a containing $h \cap n^{\mathbf{C}}$ of $n^{\mathbf{C}}$ satisfying $g([x, y]) = 0$ for any $x, y \in a$ implies $a = n^{\mathbf{C}} \cap h$.

Theorem 2.1. *Let G be a connected, simply connected solvable Lie group. Then there exists a strongly admissible positive polarization h at every $f \in g^*$ which satisfies the Pukansky condition so that if f is integral and η_f is a character satisfying $d\eta_f = 2\pi i f|g_f$ of G_f the unitary representation $\mathrm{Ind}_G(\eta_f, h)$ may be formed. Furthermore, the representation $\mathrm{Ind}_G(\eta_f, h)$ is irreducible and is independent of the choice of h at f. If $\mathrm{Ind}_G(\eta_{f_i}, h_i)$, $i = 1, 2$, are two representations of this form, then they are equivalent if and only if f_1 and f_2 lie on the same G-orbit and η_{f_1} corresponds to η_{f_2} under the isomorphism $G_{f_1} \to G_{f_2}$ defined by any element $a \in G$ such that $a \cdot f_1 = f_2$. Moreover, if G is of Type I then every irreducible unitary representation is equivalent to a representation of the form $\mathrm{Ind}_G(\eta_f, h)$.*

About the proof of the theorem, see Proposition II.2.3, Theorem II.3.2, III.4.1, IV.5.6 and V.3.3 in [3].

2.2. An Equivalent Definition of $\mathrm{Ind}_G(\eta_f, h)$

Let $f \in g^*$ be integral, h a strongly admissible poloarization at f and $\eta_f, e, d, E, D, \Lambda_f$ as before. We denote the modular functions of G and D by Δ_G and Δ_D respectively, and let $q : G \to G/D$ be the natural quotient map. Fix a left Haar measure dx on G. According to Section 2.6 in [4], we can choose strongly quasi-invanriant measures μ_E on E/D, μ_G on G/E and μ on G/D whose corresponding rho-functions are respectively $\rho_E \in C^\infty(E)$, $\rho_G \in C^\infty(G)$ and $\rho \in C^\infty(G)$ which satisfy

$$\rho_G(x\xi) = \frac{\Delta_E(\xi)}{\Delta_G(\xi)}\rho_G(x), \quad \rho(x\xi) = \rho_G(x\xi)\rho_E(\xi) \qquad x \in G, \xi \in E.$$

Define

$$\mathcal{F}^0 = \Big\{\phi \in C^\infty(G)|q(\mathrm{supp}\phi) \text{ is compact},$$
$$\phi(x\xi) = \sqrt{\frac{\Delta_D(\xi)}{\Delta_G(\xi)}}\Lambda_f(\xi^{-1})\phi(x),$$
$$\phi \cdot z = \Big[2\pi i f(z) + \frac{1}{2}\rho^{-1}(\rho \cdot z)\Big]\phi, \;\; x \in G, \xi \in D, z \in \bar{h}\Big\},$$

where $\text{supp}\phi = \{a \in G|\phi(a) \neq 0\}$ is the support of ϕ.

Let $P : C_c(G) \to C_c(G/D)$ be defined by

$$(P\psi)(xD) = \int_D \psi(x\xi)d\xi, \ \ x \in G, \psi \in C_c(G).$$

This is a surjective map. It is easy to verify that for any $\phi \in \mathcal{F}^0$,

$$P\psi \to \int_G \psi(x)||\phi(x)||^2 dx(\psi \in C_c(G))$$

is a well-defined positive linear functional on $C_c(G/D)$. Hence there is a Radon measure μ_ϕ on G/D such that

$$\int_{G/D} P\psi d\mu_\phi = \int_G \psi|\phi|^2 dx, \quad \psi \in C_c(G)$$

and $\mu_\phi(G/D) < \infty$. By polarization it now follows that if $\phi, \psi \in \mathcal{F}^0$, there is a complex Radon measure

$$\mu_{\phi,\psi} = \frac{1}{4}(\mu_{\phi+\psi} + i\mu_{\phi+\psi} - \mu_{\phi-\psi} - i\mu_{\phi-\psi})$$

on G/H. We define

$$(\phi, \psi) = \mu_{\phi,\psi}(G/D), \quad \phi, \psi \in \mathcal{F}^0.$$

It is easy to verify that this is an inner product on $\mathcal{F}^0$. We denote by $\mathcal{F}$ the Hilbert space completion of $\mathcal{F}^0$. For $x \in G$ we define the operator $\pi(\eta_f, h)(x)$ on $\mathcal{F}^0$ by $(\pi(\eta_f, h)(x)\phi)(y) = \phi(x^{-1}y), y \in G$ which extends to a unitary operator on $\mathcal{F}$, and we obtain a representation $\pi(\eta_f, h)$ of G. This is an equivalent definition of $\text{Ind}_G(\eta_f, h)$.

Theorem 2.2. *The representation* $\pi(\eta_f, h)$ *is equivalent to* $\text{Ind}_G(\eta_f, h)$.

Proof. By Section 6.1 of [4] we know that the representation

$$\text{Ind}_G(\eta_f, h) = \text{Ind}_G(\text{Ind}_E(\eta_f, h))$$

has the following equivalent definition: Let

$$\begin{aligned}\mathcal{F}_E^0 = \Big\{ & p \in C^\infty(E)|q_E(\text{supp}(p)) \text{ is compact}, \\ & p(x\xi) = \sqrt{\frac{\Delta_D(\xi)}{\Delta_E(\xi)}}\Lambda_f(\xi^{-1})p(x), \\ & p \cdot z = [2\pi i f(z) + \frac{1}{2}\rho_E^{-1}(\rho_E \cdot z)]p, \ \ x \in E, \xi \in D, z \in \bar{h}\Big\},\end{aligned}$$

where $q_E : E \to E/D$ is the natural quotient map.

With the inner pruduct similiar to the one on $\mathcal{F}^0$ the completion $\mathcal{F}_E$ of $\mathcal{F}_E^0$ is a Hilbert space and E acts on $\mathcal{F}_E^0$ by $\pi_E(\eta_f, h)$:

$$(\pi_E(\eta_f, h)(x)p)(y) = p(x^{-1}y), \quad x, y \in E, \ p \in \mathcal{F}_E^0.$$

This is a representation equivalent to $\text{Ind}_E(\eta_f, h)$ (see [4]). Next let

$$\begin{aligned}\mathcal{F}_G^0 = \Big\{ & \phi \in C^\infty(G, \mathcal{F}_E)|q_G(\text{supp}\phi) \text{ is compact}, \\ & \phi(x\xi) = \sqrt{\frac{\Delta_E(\xi)}{\Delta_G(\xi)}}\pi_E(\eta_f, h)(\xi^{-1})\phi(x), \ \ x \in G, \xi \in E\Big\}\end{aligned}$$

and define

$$||\phi||^2 = \int_G ||\phi(x)||^2_{\mathcal{F}_E} dx.$$

Here $C^\infty(G, \mathcal{F}_E)$ is the set of smooth maps from G to $\mathcal{F}_E$ and $q_G : G \to G/E$ is the natural quotient map. Then we get the Hilbert space $\mathcal{F}_G$, the completion of $\mathcal{F}_G^0$ and a unitary representation $\pi'(\eta_f, h)$ which is equivalent to $\mathrm{Ind}_G(\eta_f, h)$. Below we prove the equivalence of $\pi(\eta_f, h)$ and $\pi'(\eta_f, h)$.

Define the linear map $A : \mathcal{F}_G^0 \to C^\infty(G)$ as

$$(A\phi)(a) = \phi(a)(1), \quad \phi \in \mathcal{F}_G,\ a \in G.$$

A is an injective map. In fact, if $A\phi = A\psi$ for $\phi, \psi \in \mathcal{F}_G^0$, then $\phi(x)(1) = \psi(x)(1)$ for any $x \in G$, and for any $\xi \in E$,

$$\begin{aligned}\phi(x)(\xi) &= [(\pi_E(\eta_f, h)(\xi^{-1}))\phi(x)](1) = \Big(\sqrt{\frac{\Delta_G(\xi)}{\Delta_E(\xi)}}\phi(x\xi)\Big)(1)\\ &= \Big(\sqrt{\frac{\Delta_G(\xi)}{\Delta_E(\xi)}}\psi(x\xi)\Big)(1) = \psi(x)(\xi),\end{aligned}$$

For $\phi \in \mathcal{F}_G, x \in G, \xi \in D$ and $z = z_1 + iz_2 \in \bar{h}$ with $z_1, z_2 \in e$, we have

$$\begin{aligned}(A\phi)(x\xi) = \phi(x\xi)(1) &= \sqrt{\frac{\Delta_E(\xi)}{\Delta_G(\xi)}}\phi(x)(\xi)\\ &= \sqrt{\frac{\Delta_E(\xi)}{\Delta_G(\xi)}}\sqrt{\frac{\Delta_D(\xi)}{\Delta_E(\xi)}}\Lambda_f(\xi^{-1})\phi(x)(1)\\ &= \sqrt{\frac{\Delta_D(\xi)}{\Delta_G(\xi)}}\Lambda_f(\xi^{-1})(A\phi)(x),\end{aligned}$$

$$\begin{aligned}[(A\phi)\cdot z](x) &= \frac{d}{dt}\Big|_{t=0}(A\phi)(x\exp(-tz_1)) + i\frac{d}{dt}\Big|_{t=0}(A\phi)(x\exp(-tz_2))\\ &= \frac{d}{dt}\Big|_{t=0}\phi(x\exp(-tz_1))(1) + i\frac{d}{dt}|_{t=0}\phi(x\exp(-tz_2))(1).\end{aligned}$$

But

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\phi(x\exp(-tz_1))(1) &= \frac{d}{dt}\Big|_{t=0}\sqrt{\frac{\Delta_E(exp(-tz_1))}{\Delta_G(\exp(-tz_1))}}\phi(x)(\exp(-tz_1))\\ &= \frac{d}{dt}\Big|_{t=0}\sqrt{\frac{\rho_G(x\exp(-tz_1))}{\rho_G(x)}}\phi(x)(\exp(-tz_1))\\ &= \frac{1}{2}\rho_G^{-1}(x)(\rho_G\cdot z_1)(x)\phi(x)(1)\\ &\quad + 2\pi i f(z_1)(\phi(x)(1)) + \frac{1}{2}\rho_E^{-1}(1)(\rho_E\cdot z_1)(1)\phi(x)(1)\\ &= 2\pi i f(z_1)(\phi(x)(1)) + \frac{1}{2}\rho^{-1}(x)(\rho\cdot z_1)(x)\phi(x)(1)\\ &= [(2\pi i f(z_1) + \frac{1}{2}\rho^{-1}(\rho\cdot z_1))(A\phi)](x).\end{aligned}$$

Therefore $[(A\phi)\cdot z](x) = [(2\pi i f(z) + \frac{1}{2}\rho^{-1}(\rho\cdot z))(A\phi)](x)$ and $A\phi \in \mathcal{F}^0$. Moreover for any

$\Phi \in \mathcal{F}^0$ define $\phi : G \to \mathcal{F}_E$ as

$$\phi(x)(\xi) = \sqrt{\frac{\Delta_G(\xi)}{\Delta_E(\xi)}}\Phi(x\xi).$$

It is easy to verify that $\phi \in \mathcal{F}$ and $A\phi = \Phi$. So $A : \mathcal{F}_G^0 \to \mathcal{F}^0$ is a linear isomorphism. It is easy to prove that A preserves the norms of $\mathcal{F}_G^0$ and $\mathcal{F}^0$ so that it could be extended to a unitary operator from $\mathcal{F}_G$ to $\mathcal{F}$.

Since for $x, y \in G$, $\phi \in \mathcal{F}_G$,

$$\begin{aligned}[A(\pi'_G(\eta_f, h)(x)\phi)](y) &= [\pi'_G(\eta_f, h)(x)\phi](y)(1) = \phi(x^{-1}y)(1) \\ &= (A\phi)(x^{-1}y) = [(\pi(\eta_f, h)(x))(A\phi)](y),\end{aligned}$$

we see that A is an equivalent map between the representations $\pi(\eta_f, h)$ and $\pi'_G(\eta_f, h)$.

§3. Geometric Quantization

Suppose that (M, ω) is a symplectic manifold. We denote the set of all smooth real functions on M by $C^\infty(M)$ and the set of all smooth real vector fields on M by $\mathcal{X}(M)$. For $f \in C^\infty(M)$, let X_f be the vector field determined by $i(X_f)\omega + df = 0$, where by $i(X_f)\omega$ we denote the contraction of X_f with ω. X_f is called the Hamiltonian vector field generated by f. The Poisson bracket $[f, g] \in C^\infty(M)$ of $f, g \in C^\infty(M)$ is defined by $[f, g] = X_f(g) = \omega(X_f, X_g)$. It is well-known that the Poisson bracket makes $C^\infty(M)$ into a Lie algebra—the Poisson algebra, and the map $f \to X_f$ is a Lie algebra homomorphism from $C^\infty(M)$ into $V^H(M)$, the set of Hamiltonian vector fields. A subset $F_1, F_2, \cdots, F_m$ of $C^\infty(M)$ is called complete if the conditons $[F_i, F] = 0$ $(i = 1, 2, \cdots, m)$ for $F \in C^\infty(M)$ imply F =constant.

Definition 3.1. *A quantization of (M, ω) is a linear mapping $F \to \check{F}$ of the Poisson algebra $C^\infty(M)$ (or some subalgebra of it) into the set* End(H) *of operators on some Hilbert space H, having the properties:*

(1) $\check{1} = 1$;

(2) $[F_1, F_2]^\vee = [\check{F}_1, \check{F}_2]_h = \frac{2\pi i}{h}(\check{F}_1\check{F}_2 - \check{F}_2\check{F}_1)$;

(3) $(\bar{F})^\vee = (\check{F})^*$;

(4) *for some complete set $F_1, \cdots, F_m$ of functions the operators $\check{F}_1, \cdots, \check{F}_m$ act irreducibly on H;*

where $F, F_1, \cdots, F_m \in C^\infty(M)$, and h is the Plank's constant. A linear mapping which possesses the first three properties is called a prequantization.

A symplectic manifold (M, ω) is called quantizable whenever the form $h^{-1}\omega \in H^2(M, R)$ is integral. If (M, ω) is quantizable, following Sourian-Kostant, there is a complex line bundle L over M, a Hermitian structure (,) and a connection ∇ with the curvature $h^{-1}\omega$ on L which are compatible:

$$\xi(s_1, s_2) = (\nabla_\xi s_1, s_2) + (s_1, \nabla_\xi s_2), \quad \xi \in \mathcal{X}(M), \ \ s_1, s_2 \in \Gamma^\infty(L).$$

Here $\Gamma^\infty(L)$ is the set of all smooth sections of the line bundle L. The triple $(L, \nabla, (,))$ is called the prequantum bundle. By H we denote the space of square-integrable sections with the inner product

$$\langle s, s' \rangle = \int_M (s, s') \frac{\omega^n}{n!}, \quad \text{if } \dim M = 2n.$$

Theorem 3.1.[2] *If the manifold (M, ω) is quantizable, the Sourian-Kostant formula*

$$f \longrightarrow \check{f} = \frac{h}{2\pi i} \nabla_{X_f} + f \in \mathrm{End}(H), \qquad f \in C^\infty(M)$$

gives a prequantzation of (M, ω).

Remark. In general, the Sourian-Kostant prequantization is not a quantization. In order to get quantization, we should use polarizations.

Let (M, ω) be a symplectic manifold, and $T^C(M)$ the complexification of the tangent bundle over M.

Definition 3.2. *A subbundle $P \subset T^C(M)$ is called a polarization if it fulfills the conditions:*

(1) *The fibre P_x is a Lagrangian subspace of $T_x^C(M)$ for each $x \in M$.*

(2) *The distribution $x \to P_x$ is integrable.*

It is clear that if the subbundle P is a polarization then the complex conjugate subbundle $\bar{P}$ is also a polarizaton and $\tilde{D} = (P \cap \bar{P}) \cap TM$ is integrable. If $\tilde{E} = (P + \bar{P}) \cap TM$ is integrable, we say that P is admissible. Fix $m \in M$, then ω_m projects onto a symplectic form ω'_m on $V_m = \tilde{E}_m / \tilde{D}_m$ and $P'_m = P_m / \tilde{D}_m^{\mathbf{C}}$ is a Lagrangian subspace of $V_m^{\mathbf{C}}$. Clearly $P'_m \cap \bar{P}'_m = \{0\}$. We define nondegenerate Hermitian form $b(\cdot,\cdot)$ on P'_m:

$$b(X + \tilde{D}_m^{\mathbf{C}}, Y + \tilde{D}_m^{\mathbf{C}}) = i\omega'_m(X, \bar{Y}), \quad X, Y \in P_m.$$

If $b(\cdot,\cdot)$ is positively definite or $P = \bar{P}$, P is said to be positive.

Let P be a polarization of the symplectic manifold (M, ω) and $(L, \nabla, (,))$ a prequantum bundle over M. We denote the space of all vector fields tangent to P on M by $V_P(M)$. A smooth section $s \in \Gamma^\infty(L)$ is said to be polarized if $\nabla_{\bar{X}} s = 0$ for every $X \in V_P(M)$. Our idea is to quantize M by replacing the Hilbert space H of prequantization by the completion H_P of the subspace Γ_P of square-integrable polarized sections of L. Then the operator $\hat{f}$ with $f \in C^\infty(M)$ maps local polarized sections into polarized sections if and only if $[X, X_f] \in V_P(M)$ whenever $X \in V_P(M)$. In this case, we say f is quantizable. We denote the space of quantizable functions by $C_P(M)$. It is a subalgebra of the Poisson algebra $C^\infty(M)$.

§4. Realization of Representations of Solvable Lie Groups by Geometric Quatization

Suppose that G is a connected and simply-connected Lie group with the Lie algebra g . For $f \in g^*$, the coadjoint orbit $O = Gf \cong G/G_f$ is a G-homogeneous space, and we have a homomorphism from g into $\mathcal{X}(O)$: $X \to \xi^X$, $X \in g$ defined by

$$(\xi^X \phi)(f') = \frac{d}{dt}\Big|_{t=0} \phi(\exp(-tX)f'), \qquad f' \in O, \phi \in C^\infty(O).$$

Now we attach a skew-symmetric bilinear form $\omega_{f'}$ on $T_{f'}O$ for any $f' \in O$ defined by

$$\omega_{f'}(\xi^X, \xi^Y) = f'([X, Y]), \qquad X, Y \in g.$$

It is easy to see that $\omega_{f'}$ is well-defined and the family of $\omega_{f'}$ defines a closed two-form ω, and therefore a G-invariant symplectic structure on O. For $Y \in g$, write $\phi^Y \in C^\infty(O)$ for the function given by

$$\phi^Y(f') = f'(Y), \qquad f' \in O.$$

Recall that a symplectic manifold (M, ω) is said to be a Hamiltonian G-space if the Lie group G acts transively on M by symplectic automorphisms and there is a Lie homomorphism $\lambda : g \longrightarrow C^\infty(M)$ such that $X_{\lambda(Y)}$ is the vector field generated by the one-parameter subgroup exptY of automorphisms on M for every $Y \in g$.

Theorem 4.1. *ξ^Y is the Hamiltonian vector field generated by ϕ^Y, that is, $i(\xi^Y)\omega + d\phi^Y = 0$. Moreover,*

$$\lambda : g \longrightarrow C^\infty(O), \quad Y \to \phi^Y$$

is a Lie algebra homomorphism and (O, ω, λ) is a Hamiltonian G-space. Write $C_g(O) \subset C^\infty(O)$ for the image of λ which is a subalgebra of $C^\infty(O)$.

Proof. It is a direct verification.

Let G_f^* be the character group of G_f and $G_f^h \subset G_f^*$ be the set of all characters $\Lambda : G_f \to S^1$ such that for any $X \in g_f$ one has

$$\frac{d}{dt}\Big|_{t=0} \Lambda(\exp tX) = \frac{2\pi i}{h} f(X).$$

Definition 4.1. *The coadjoint orbit $O = G_f \subset g^*$ is said to be h-integral if $G_f^h \neq \emptyset$. In particular, 1-integral coadjoint orbit is said to be integral.*

Theorem 4.2.[2] *The orbit (O, ω) is quantizable if and only if it is h-integral.*

Let O be a coadjoint orbit of G in g^*. For every $f \in O$, it is well-known that there is a "standard" isomorphism

$$\rho_f : G_f/(G_f)^0 \to \pi(O),$$

where $(G_f)^0$ is the connected component containing unit element of G_f. Here standard ρ_f means that

$$\rho_{f'} = \rho_f \circ \rho_a \text{ if } f' = af \in O, a \in G,$$

where $\rho_a : G_{f'}/(G_{f'})^0 \to G_f/(G_f)^0$ such that $\rho_a[g] = [a^{-1}ga] \in G_f/(G_f)^0$ for any $[g] \in G_{af}/(G_{af})^0$ since of $G_{af} = aG_f a^{-1}$.

Fix $f \in O$. For a character η of G_f, we have a line bunde $L_\eta = G \times_\eta \mathbf{C} = G \times \mathbf{C}/\sim$ associated with η, where $(g, z) \sim (g', z')$ for $g, g' \in G, z, z' \in \mathbf{C}$ if and only if $g' = g\xi, z' = \eta(\xi^{-1})z$ for $\xi \in G_f$. We denote the equivalence class containing (g, z) as $[g, z]$. Then every section $s \in \Gamma^\infty(L_\eta)$ can be identified with a function $F_s \in C^\infty(G)$ satisfying

$$F_s(ga) = \eta(a^{-1})F_s(g), \qquad g \in G, a \in G_f$$

such that

$$s(gf) = [g, F_s(g)], \quad g \in G.$$

Now if O is h-integral, we have the isomorphism

$$\rho_f^* : \pi(O)^* \to (G_f/(G_f)^0)^* \cong G_f^h.$$

Fix $\sigma \in \pi(O)^*$. Then $\eta_f = \rho_f^*(\sigma)$ is a character of G_f such that

$$\frac{d}{dt}\Big|_{t=0} \eta_f(\exp tX) = \frac{2\pi i}{h} f(X), \quad X \in g_f.$$

So we have the associated line bundle $L = G \times_{G_f} \mathbf{C}$ corresponding to the character η_f : $G_f \to \mathrm{Aut}(C) = C$. In fact L is a Hermitian line bundle with the Hermitian structure (,) on L such that for $g \in G$ and $c \in C^*$, $([g,c],[g,c]) = |c|^2$. Meanwhile G acts on L by $a[g,c] = [ag,c]$ for $a, g \in G, c \in \mathbf{C}$. Let

$$H = \Big\{ s \in \Gamma(L) \Big| \int_O (s,s)\frac{\omega^n}{n!} < \infty, \text{ if } \dim O = 2n \Big\}.$$

G has a unitary representation π_1 on H:

$$(\pi_1(g)s)(f') = gs(g^{-1}f'), \quad g \in G, \quad f' \in O, \ s \in H.$$

The set of smooth vectors of π_1 (see [6]) is $\Gamma^\infty(L)$, so we have a representation $d\pi_1$ of the Lie algebra g on $\Gamma^\infty(L)$. For $X \in g, s \in \Gamma^\infty(L)$, define

$$\nabla_{\xi^X} s = d\pi_1(X)s - \frac{2\pi i}{h}\phi^X s.$$

Theorem 4.3.[2] *∇ defines a connection on L which is compatible with (,) and has the curvature $h^{-1}\omega$. Therefore*

$$\phi^X \longrightarrow \frac{h}{2\pi i} d\pi_1(X), \quad \phi^X \in C_g(O)$$

is a prequantization of (O,ω).

Set $\xi^{X+iY} = \xi^X + i\xi^Y$ for $X, Y \in g$. Then $X \to \xi^X, X \in g^C$ is a Lie homomorphism from g^C into $\mathcal{X}(O)^{\mathbf{C}} = \mathcal{X}(O) + i\mathcal{X}(O)$.

Theorem 4.4. *For $f \in g^*$ a polarization h at f determines a polarization P of the symplectic manifold $(O = Gf, \omega)$ which is G-invariant, that is, $g_* P_f = P_{gf}$ for any $g \in G$. And $C_g(O) \subset C_P(O)$, that is, $\phi^X, \forall X \in g$ is quantizable. Moreover P is admissible, and P is positive if and only if h is positive.*

Proof. Define $P_f = \mathrm{span}\{\xi_f^X | X \in h\}$ and

$$P_{gf} = g_* P_f = \mathrm{span}\{\xi_{gf}^{gX} \ | \ X \in h\}$$

for any $g \in G$. P_{gf} is well-defined since $G_f h \subset h$. It is also an isotropic subspace of $T_{gf}^C O$ because

$$\omega_{gf}(\xi^{gX}, \xi^{gY}) = (gf)([gX, gY]) = f([X,Y]) = 0$$

for any $g \in G$ and $X, Y \in h$. By (3) of Definition 2.1, P_{gf} is a Lagrangian subspace of $T_{gf}^C O$. And the distribution $gf \to P_{gf}, gf \in O$ is integrable since h is a Lie subalgebra. Therefore P is a G-invariant polarization of O.

That P is G-invariant implies $[\xi^X, \eta] \in V_P(O), \forall \eta \in V_P(O), X \in g$, i.e, $C_g(O) \subset C_P(O)$. The others are obvious.

In the case of $P = \bar{P}$, [5] gives a metheod of quantization. Below we extend the metheod to the case of general positive polarizations for Hamiltonian G-space.

Suppose that G is a solvable Lie group and $f \in g^*$ is h-integral. There exists a strongly admissible positive polarization $h \in g^{\mathbf{C}}$ at f. Let d, e, D, E and η_f as in Section 2.1. By Theorem 4.4, h induces a G-invariant admissible positive polarization P on $O = Gf$. Let

$$\tilde{D} = (P \cap \bar{P}) \cap TO, \quad \tilde{E} = (P + \bar{P}) \cap TO.$$

Since P is G-invariant, so are $\tilde{D}$ and $\tilde{E}$.

Lemma 4.1. *A G-homogeneous space M is orientable.*

Proof. Fix $m \in M$. Then $M \cong G/G_m$ and there exists a strongly quasi-invariant measure μ_M on M. Assume $\rho_M \in C^\infty(G)$ is the rho-function corresponding to μ_M. Suppose that $\dim M = n$. We may identify the complex line bundle $\Lambda^n_{\mathbf{C}}(M)$ with the associated line bundle $G \times_{G_m} \mathbf{C}$ corresponding to the character

$$\Delta_G/\Delta_{G_m} : G_m \to S^1.$$

There exists a non-zero section s of $\Lambda^n(M)$ which satisfies

$$s(gm) = [g, \rho_m(g)], \quad g \in G.$$

s is not zero anywhere. Therefore $\Lambda^n_{\mathbf{C}}(M)$ is trivial and M is orientable.

The space $Q = O/\tilde{D}$ of leaves of $\tilde{D}$ is the homogeneous space G/D which are orientable. Denote by $\Delta_Q \to Q$ the line bundle $\Lambda^r_{\mathbf{C}}(Q)$ which is in fact the line bundle associated to the character $\Delta_G/\Delta_D : G_f \to S^1$, with $r = \dim Q$, and

$$V_{\tilde{D}}(O) = \{\xi \in \mathcal{X}(O) | \xi_m \in \tilde{D}_m, m \in O\}.$$

Let $K_{\tilde{D}} \subset \Lambda^r_{\mathbf{C}}(O)$ be the canonical line bundle whose fibre at $m \in O$ is the one-dimensional subspace of $\Lambda^r T^*_{m,\mathbf{C}} O$ of forms α such that $i(\xi)\alpha = 0$ for every $\xi \in \tilde{D}_m$. It is obvious that $K_{\tilde{D}} = pr^* \Delta_Q$, where $pr : O \to Q = O/\tilde{D}$ is the natural quotient map. Since Q is oriented, the transition functions of Δ_Q and $K_{\tilde{D}}$ can all be made real and positive. So we can take their square roots $\sqrt{\Delta_Q}$ and $\sqrt{K_{\tilde{D}}}$, squares ${\Delta_Q}^2$ and ${K_{\tilde{D}}}^2$ and inverses ${\Delta_Q}^{-1}$ and ${K_{\tilde{D}}}^{-1}$ by taking the square roots, squares and inverses of the transition functions, respectively.

Since $\tilde{D}$ is G-invariant, the Lie derivative $L_{\xi^X}\beta$ is still a section of $K_{\tilde{D}}$ for any section β of $K_{\tilde{D}}$ and $X \in g$. But $T_m O = \mathrm{span}\{(\xi^X)_m \mid X \in g\}$ for every $m \in O$. So L_ξ maps sections of $K_{\tilde{D}}$ to sections of $K_{\tilde{D}}$ for any $\xi \in \mathcal{X}(O)^{\mathbf{C}}$.

Assume that $\rho \in C^\infty(G)$ is the rho-function of the measure on G/D as in Section 2.2. We have a section γ' of Δ_Q such that $\gamma'(gf) = [g, \rho(g)]$ for $g \in G$. Set $\gamma = pr^*(\gamma') \in \Gamma^\infty(K_{\tilde{D}})$.

The covariant derivative $\bigtriangledown$ on $K_{\tilde{D}}$ is defined for $\xi \in \mathcal{X}(O)^{\mathbf{C}}$ by

$$\bigtriangledown_\xi \beta = L_\xi \beta - \gamma^{-1}(L_\xi \gamma)\beta, \quad \beta \in \Gamma^\infty(K_{\tilde{D}}).$$

The sections of $K_{\tilde{D}}$ which are covariantly constant along $\tilde{D}$ are the pull-backs of r-forms on Q. The $\bigtriangledown$ and L can pass to the bundle $\delta_{\tilde{D}} = \sqrt{K_{\tilde{D}}}$ where they are determined by

$$2(\bigtriangledown_\xi \tau)\tau = \bigtriangledown_\xi \tau^2, \quad 2(L_\xi \tau)\tau = L_\xi \tau^2.$$

Here τ is a section of $\delta_{\tilde{D}}$ and $\xi \in \mathcal{X}(O)^{\mathbf{C}}$.

Let $\sigma \in \pi(O)^*, \eta_f = \rho_f^*(\sigma)$. Then the associated line bundle $L = G \times_{\eta_f} \mathbf{C}$ corresponding to η_f is a complex line bundle over O with a hermitean metric (,) and a compatible hermitean linear connection $\bigtriangledown$ with the curvature $h^{-1}\omega$ described as before. Set $L_P = L \otimes \delta_{\tilde{D}}$. Define

$$V_P = \{\tilde{s} = s\tau \in \Gamma^\infty(L_P) | \bigtriangledown_\xi \tilde{s} = (\bigtriangledown_\xi s)\tau + s \bigtriangledown_\xi \tau = 0, \ \ \xi \in V_P(O)\}.$$

If $\tilde{s} = s\tau$ and $\tilde{s}' = s'\tau' \in V_P$, then $(\tilde{s}, \tilde{s}') = (s, s')\tau\tau' \in \Gamma^\infty(K_P)$ and for $\xi \in V_{\tilde{D}}(O) \subset V_P(O)$,

$$\bigtriangledown_\xi(\tilde{s}, \tilde{s}') = (\bigtriangledown_\xi \tilde{s}, \tilde{s}') + (\tilde{s}, \bigtriangledown_\xi \tilde{s}') = 0.$$

Hence we can identify $(\tilde{s}, \tilde{s}')$ with an r-form on Q and define an inner product on V_P by

$$\langle \tilde{s}, \tilde{s}' \rangle = \int_Q (\tilde{s}, \tilde{s}').$$

The completion of $\{\tilde{s} \in V_P | \langle \tilde{s}, \tilde{s} \rangle < \infty\}$ is our quantum space H_P.

For $F \in C_P(O)$, the possible choice of the corresponding quantum observable is the operator $\hat{F}$ that acts on V_P by

$$\hat{F}\tilde{s} = \check{F}(s)\tau + csL_{X_F}\tau,$$

where $\tilde{s} = s\tau \in V_P, c \in \mathbf{C}$. It is easy to check that $\hat{F}$ is well-defined iff $c = -ih$. So

$$\hat{F}\tilde{s} = \check{F}(s)\tau - ihsL_{X_F}\tau,$$

where $\tilde{s} = s\tau \in V_P$.

Theorem 4.5. *The Hilbert space H_P and the mapping $F \to \hat{F}, F \in g_P$ define a geometric quantization of O.*

This is a direct verification.

Remark. The metheod of quantization above is efficient for any Hamiltonian G-space.

On the other hand, G acts on the associated line bundles L and $\delta_{\tilde{D}}$. The actions of G on L, $\delta_{\tilde{D}}$ and O induce the actions π_1 on $\Gamma^\infty(L)$ and π_2 on $\Gamma^\infty(\delta_P)$ of G:

$$(\pi_1(g)s)(f') = gs(g^{-1}f'), \quad g \in G, f' \in O, s \in \Gamma^\infty(L),$$
$$(\pi_2(g)\tau)(f') = g\tau(g^{-1}f'), \quad g \in G, f' \in O, \tau \in \Gamma^\infty(\delta_{\tilde{D}}).$$

So G acts on $\Gamma^\infty(L_P)$ by $\pi(\sigma, f, h) = \pi_1 \otimes \pi_2$ and $\pi(\sigma, f, h)(G)H_P \subset H_P$ since P is G-invariant polarization.

Theorem 4.6. *The map $\pi(\sigma, f, h): \quad G \longrightarrow \mathrm{Aut}(H_P)$ is a unitary representation of G. Its differential is*

$$d\pi(X)\tilde{s} = (\nabla_{\xi^X}s + 2\pi i h^{-1}\phi^X s)\nu + sL_{\xi^X}\nu, \ \ \forall X \in g, \tilde{s} = s\nu \in V_P.$$

Proof. It is easy to prove $d\pi_2(X) = L_{\xi^X}$ by $\delta_{\tilde{D}} = \sqrt{K_{\tilde{D}}}$ and $K_{\tilde{D}} = pr^*\Lambda^r_{\mathbf{C}}Q$. According to Theorem 4.3, $d\pi_1(X) = \nabla_{\xi^X} + 2\pi i h^{-1}\phi^X$. The conclusion is now obvious.

Corollary 4.1. *The map $\phi^X \longrightarrow \frac{h}{2\pi i}d\pi(X), \ \ X \in g$ defines a geometric quantization of O. In other words, the representation $\pi(\sigma, f, h)$ is obtained by the geometric quantization of O.*

Theorem 4.7. *Fix the Plank constant $h = 1$. Suppose O is integral. Then the representation $\pi(\sigma, f, h)$ is equivalent to the representation $\pi(\eta_f, h)$ constructed in Section* 2.2.

Proof. Fix a section $\tilde{s} = s\tau \in V_P$. The Plank constant $h = 1$. There are functions $F_s, F_\tau \in C^\infty(G)$ such that

$$s(gf) = [g, F_s(g)], \quad \tau(gf) = [g, F_\tau(g)], \ \ g \in G.$$

Moreover

$$F_s(ga) = \eta_f(a^{-1})F_s(g),$$
$$F_\tau(ga) = \sqrt{\frac{\Delta_D(a)}{\Delta_G(a)}}F_\tau(g)$$

for $a \in G_f$ and $g \in G$. So $F_{\tilde{s}} = F_sF_\tau$ satisfies

$$F_{\tilde{s}}(ga) = \sqrt{\frac{\Delta_D(a)}{\Delta_G(a)}}\eta_f(a^{-1})F_{\tilde{s}}(g)$$

for $a \in G_f$ and $g \in G$. But

$$\nabla_{\xi_{gf}^{gX}} s = d\pi_1(gX)s(gf) - 2\pi i(\phi^{gX}s)(gf),$$

so

$$F_{\nabla_{\xi^{gX}} s}(g) = (F_s \cdot X)(g) - 2\pi i f(X)F_s(g), \quad g \in G, X \in \bar{h}.$$

Similarly we have

$$(F_{\nabla_{\xi^{gX}} \tau})(g) = [F_\tau \cdot X - \frac{1}{2}\rho^{-1}(\rho \cdot X)F_\tau](g).$$

Therefore

$$\begin{aligned}(F_{\nabla_{\xi^{gX}} \tilde{s}})(g) &= (F_{\nabla_{\xi^{gX}} s}F_\tau + F_s F_{\nabla_{\xi^{gX}} \tau})(g)\\ &= ((F_s \cdot X - 2\pi i f(X)F_s)F_\tau)(g) + (F_s(F_\tau \cdot X - \frac{1}{2}\rho^{-1}(\rho \cdot X)F_\tau))(g)\\ &= 0, \quad X \in \bar{h},\end{aligned}$$

since $(\nabla_{\xi^{gX}} \tilde{s})(g) = 0$. So

$$F_{\tilde{s}} \cdot X = [2\pi i f(X) + \frac{1}{2}\rho^{-1}(\rho \cdot X)]F_{\tilde{s}}, X \in \bar{h}.$$

Since $D = G_f D_0$, we know $F_{\tilde{s}} \in \mathcal{F}$. We define a linear map $T : V_P \to \mathcal{F}$ by $T\tilde{s} = F_{\tilde{s}}$ which is an isomorphism and preserves the norms. Extending T we get a linear unitary operator from H_P into $\mathcal{F}$. It is easy to see that T is G-equivariant.

Combining the results of Theorem 2.1, Theorem 2.2 and Theorem 4.7 we have the following theorem.

Theorem 4.8. *Suppose that G is a connected and simply connected solvable Lie group. For every integral coadjoint orbit O and every $\sigma \in \pi(O)^*$, the unitary representation $\pi(\sigma, \eta_f, \mathrm{h})$ may be formed by geometric quantization through choosing $f \in O$ and a polarization h at f. Furthurmore, the representation $\pi(\sigma, \eta_f, \mathrm{h})$ is irreducible and is independent of the choice of $f \in O$ and h at f up to the equivalence of representations, so we denote it by $\pi(O, \sigma)$. If $\pi(O_i, \sigma_i)$, $i = 1, 2$, are two representations of this form, then they are equivalent if and only if $O_1 = O_2$ and $\sigma_1 = \sigma_2$. If G is of type I, every irreducible unitary representation of G is equivalent to a representation of the form $\pi(O, \sigma)$.*

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