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S^1xS^1-INDEX THEORY ON PRODUCT SPACE***

ZHONG Chengkui* ZHAO Peihao* SHI Ke**

Abstract

A new index is constructed by use of the canonical representation of $S^1 \times S^1$ group over a product space. This index satisfies the general properties of the usual index but does not satifsy the dimension property. As an application, two abstract critical point theorems are given.

Keywords Index, Equivariant mapping, Critical point, Critical value
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§1. Introduction

Let G be a compact Lie group and let $\{T(g)\}_{g\in G}$ be an isometric representation of G over a Banach space X. An index (for $\{T(g)\}_{g\in G}$) is a mapping from the closed invariant subset of X into $N \cup \{+\infty\}$ such that

(i) $\mathrm{ind}\, A = 0$ if and only if $A = \emptyset$;

(ii) if $R{:}A \to B$ is equivariant and continuous, then $\mathrm{ind}\, A \le \mathrm{ind}\, B$;

(iii) if A is compact invariant, there is a closed invariant neighborhood N of A such that $\mathrm{ind}\, N = \mathrm{ind}\, A$;

(iv) $\mathrm{ind}\,(A \cup B) \le \mathrm{ind}\, A + \mathrm{ind}\, B$ for all invariant subsets A and B.

It is well known that index theories, especially Z_2-index theory and S^1-index theory, are very useful in the study of the existence of the multiple critical points of the functional which is invariant with respect to the representation of some compact Lie group G (see [1–5]).

As we know, almost every index theory constructed so far satisfies the dimension property, that means there is a positive integer d such that $\mathrm{ind}\,(V^{dk} \cap S_1) = k$ for all dk-dimensional invariant subspace V^{dk} of X such that $V^{dk} \cap \mathrm{Fix}T(G) = \{0\}$, where $S_1 = \{x \in X|\ \|x\| = 1\}$ and $\mathrm{Fix}T(G) = \{x \in X | T(g)x = x\ \forall g \in G\}$.

In this paper, we shall construct a new index for product space which does not satisfy the dimension property.

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*Department of Mathematics, Lanzhou University, Lanzhou 730000, China.

**Department of Mathematics, Xinjiang University, Wulumuji 830046, China.

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§2. S^1xS^1-Index on Product Space

Let X, Y be two complex Banach spaces and let $Z = X \times Y$ be their product space. The element of Z is denoted by z or (x, y). The isometric representation of $S^1 \times S^1$ over $X \times Y$ is given by

$$T(\theta_1, \theta_2)(x, y) = (T_1(\theta_1)x, T_2(\theta_2)y),$$

where $\{T_1(\theta_1)\}_{\theta_1 \in S^1}$ and $\{T_2(\theta_2)\}_{\theta_2 \in S^1}$ are isometric representations of S^1 over X and Y respectively.

Definition 2.1. *For every closed $S^1 \times S^1$-invariant subset A of $X \times Y$, define* ind$A$ *as the smallest integer k such that there exist an integer $n \in N \setminus \{0\}$ and a continuous mapping $\phi = (\phi_1; \phi_2) : A \to C^k \times C^k$ satisfying the following properties:*

(A_1) *$\phi(T(\theta_1, \theta_2)z) = (e^{in\theta_1}\phi_1(z); e^{in\theta_2}\phi_2(z))$ for all $(\theta_1, \theta_2) \in S^1 \times S^1$ and all $z = (x, y) \in A$;*

(A_2) *$\phi_1(z) \neq 0$ and $\phi_2(z) \neq 0$ for all $z \in A$;*

(A_3) *for each $z \in A$, there exists some $j, 1 \leq j \leq k$, such that $\phi(z) = (\phi_1(z); \phi_2(z)) \in C^j \times C^{k+1-j}$;*

where C^j is taken as a subspace of C^k for which the last $k - l$ components vanish.

If such a mapping ϕ does not exist, we define $\operatorname{ind} A = \infty$.

Remark 2.1. The requirement (A_1) of ϕ in the definition is reasonable. It is similar to S^1-index. The requirement (A_2) is equivalent to

$$\phi(A) \cap (\{0\} \times C^k \cup C^k \times \{0\}) = \emptyset,$$

where $\{0\}$ is the origin of C^k. However, the restriction (A_3) of ϕ is not obvious. In fact, it is equivalent to

$$\phi(A) \subset \bigcup_{j=1}^{k} C^j \times C^{k+1-j}.$$

Combining with (A_2), it relates to the homology groups of the product space $(\overline{X} \setminus \mathrm{Fix}T_1(S^1)) \times (\overline{Y} \setminus \mathrm{Fix}T_2(S^1))$, where $\overline{X} = X/T_1(S^1)$ and $\overline{Y} = Y/T_2(S^1)$.

Remark 2.2. In the same way, one can define $Z_p \times Z_p$-index for product space.

Theorem 2.2. *The index defined as Definition* 2.1 *is indeed an index.*

Proof. We first notice that if $\phi = (\phi_1; \phi_2) \in C(A, C^k \times C^k)$ satisfies (A_1), then there exists a continuous extension ψ of ϕ over $X \times Y$ satisfying (A_1). It suffices to use first Tietze theorem to obtain a continuous extension $\bar{\phi} = (\bar{\phi_1}; \bar{\phi_2})$ of ϕ over $X \times Y$ and then the mapping $\psi = (\psi_1; \psi_2)$ defined by

$$\psi(z) = (\psi_1(z); \psi_2(z)) = \frac{1}{2\pi}\left(\int_0^{2\pi} e^{-in\theta_1}\bar{\phi_1}(T_1(\theta_1)x)d\theta_1; \int_0^{2\pi} e^{-in\theta_2}\bar{\phi_2}(T_2(\theta_2)y)d\theta_2\right)$$

will satisfy the requirement.

We now check that the properties(i) to (iv) are satisfied.

(i) Follows directly from the definition, and (ii) is trivial.

(iii) Let N be any closed $S^1 \times S^1$-invariant neighborhood of the invariant subset A. Property (ii) implies that $\operatorname{ind} A \leq \operatorname{ind} N$ since the inclusion mapping is equivariant. Thus, the result is trivial if $\operatorname{ind} A = \infty$. If $\operatorname{ind} A = k < \infty$, there is a $\phi = (\phi_1; \phi_2) \in C(A, C^k \times C^k)$

satisfying (A_1), (A_2) and (A_3), and hence, a continuous extension $\psi = (\psi_1; \psi_2)$ of ϕ over $X \times Y$ satisfying (A_1). Since $0 \notin \phi_i(A) = \psi_i(A)$, and $\phi_i(A)$ is compact, $i = 1, 2$, there exist a $\delta > 0$ and a closed $S^1 \times S^1$-invariant neighborhood $N(A)$ such that

$$\min_{z \in N(A)} \{\|\psi_1(z)\|, \|\psi_2(z)\|\} \geq \delta \tag{2.1}$$

and for each $z \in N(A)$ there is a $z_1 \in A$ satisfying

$$\|\psi(z) - \psi(z_1)\| < \frac{\delta}{4}. \tag{2.2}$$

Let $P_j : C^k \to C^j, j = 1, 2, \cdots, k$, be the canonical projections, and let $\chi : R^+ \to [0, 1]$ be a continuous function such that

$$\chi(t) = \begin{cases} 0, & t \geq \frac{\delta}{4}, \\ 1, & t \leq \frac{\delta}{8}. \end{cases}$$

Define $\eta = (\eta_1; \eta_2) : N(A) \to C^k \times C^k$ by

$$\begin{aligned} \eta_1 &= (\psi_{1,1}, \chi(\|P_1\psi_1(\cdot)\|)\psi_{1,2}, \cdots, \chi(\|P_{k-1}\psi_1(\cdot)\|)\psi_{1,k}), \\ \eta_2 &= (\psi_{2,1}, \chi(\|P_1\psi_2(\cdot)\|)\psi_{2,2}, \cdots, \chi(\|P_{k-1}\psi_2(\cdot)\|)\psi_{2,k}), \end{aligned}$$

where $(\psi_{1,1}, \cdots, \psi_{1,k}) = \psi_1, (\psi_{2,1}, \cdots, \psi_{2,k}) = \psi_2$. Then η satisfies the requirements (A_1), (A_2) and (A_3). Obviously, (A_1) is satisfied. So we need to show that (A_2) and (A_3) are satisfied.

Indeed, if there exists some $z \in N(A)$ such that $\eta_1(z) = 0$ or $\eta_2(z) = 0$, without loss of generality, we assume that $\eta_1(z) = 0$, hence we first obtain $\psi_{1,1}(z) = 0$. This implies $\chi(\|P_1\psi_1(z)\|) = 1$, so $\psi_{1,2}(z) = 0$. Step by step, we finally get $\psi_1(z) = 0$, which contradicts (2.1).

Now, we show that η satisfies (A_3). In fact, for any $z \in N(A)$, we will find some $z_1 \in A$ such that (2.2) holds. Since $\phi = (\phi_1; \phi_2)$ satisfies (A_3), there exists some $j, 1 \leq j \leq k$, such that

$$\psi(z_1) = (\psi_1(z_1); \psi_2(z_1)) = (\phi_1(z_1); \phi_2(z_1)) \in C^j \times C^{k+1-j}. \tag{2.3}$$

We claim that $\eta(z) = (\eta_1(z); \eta_2(z))$ is also in $C^j \times C^{k+1-j}$. Indeed, if $\eta_1(z) \notin C^j$, then there exists unique $i \geq 1$ such that $\eta_1(z) \in C^{j+i}$, but $\eta_1(z) \notin C^{j+i-1}$. This implies

$$\chi(\|P_{j+i-1}\psi_1(z)\|) \neq 0.$$

By the definition of χ, we have

$$\|P_{j+i-1}\psi_1(z)\| < \frac{\delta}{4}. \tag{2.4}$$

Combining with (2.1) and (2.3), we have

$$\begin{aligned} \|\psi_1(z) - \psi_1(z_1)\| &\geq \|P_{j+i-1}(\psi_1(z) - \psi_1(z_1))\| \\ &= \|P_{j+i-1}\psi_1(z) - \psi_1(z_1)\| \geq \delta - \frac{\delta}{4} = \frac{3}{4}\delta, \end{aligned}$$

which contradicts (2.2). Thus we obtain $\operatorname{ind} N(A) \leq k = \operatorname{ind} A$.

(iv) If $\operatorname{ind} A$ or $\operatorname{ind} B$ is infinite, the result is trivial. So, assume that $\operatorname{ind} A = k_1$, $\operatorname{ind} B =$

k_2 are both finite. Then there exist two integers n, m and two continuous mappings

$$\phi = (\phi_1; \phi_2) : A \to \left(\bigcup_{j=1}^{k_1} C^j \times C^{k_1+1-j} \right) \Big\backslash \left(\{0\} \times C^{k_1} \cup C^{k_1} \times \{0\} \right),$$

$$\psi = (\psi_1; \psi_2) : B \to \left(\bigcup_{j=1}^{k_2} C^j \times C^{k_2+1-j} \right) \Big\backslash \left(\{0\} \times C^{k_2} \cup C^{k_2} \times \{0\} \right)$$

satisfying respectively

$$\phi(T(\theta_1, \theta_2)z) = (e^{in\theta_1}\phi_1(z); e^{in\theta_2}\phi_2(z)), \tag{2.5}$$

$$\psi(T(\theta_1, \theta_2)z) = (e^{im\theta_1}\psi_1(z); e^{im\theta_2}\psi_2(z)). \tag{2.6}$$

By the statement at beginning of the proof, we can assume that ϕ and ψ are extended to $A \cup B$ such that (2.5) and (2.6) hold.

Let $\chi_i \in C(A \cup B, R^1), i = 1, 2$ be two functions defined respectively by

$$\chi_1(z) = \mathrm{dist}([z], B), \quad \chi_2(z) = \mathrm{dist}([z], \overline{A \setminus B}),$$

where $[\,z]$ denotes the $T(S^1 \times S^1)$-orbit of z and dist denotes the distance function. Clearly, χ_1 and χ_2 are both $T(S^1 \times S^1)$-invariant continuous functions. Now define $\xi : A \cup B \to C^{k_1+k_2} \times C^{k_1+k_2}$ by $\xi(z) = (\xi_1(z); \xi_2(z))$, where

$$\xi_1(z) = \left(\phi_{1,1}^m(z), \cdots, \phi_{1,k_1}^m(z), \chi_2(z)\psi_{1,1}^n(z), \cdots, \chi_2(z)\psi_{1,k_2}^n(z) \right),$$
$$\xi_2(z) = \left(\psi_{2,1}^n(z), \cdots, \psi_{2,k_2}^n(z), \chi_1(z)\phi_{2,1}^m(z), \cdots, \chi_1(z)\phi_{2,k_1}^m(z) \right),$$

where $\phi_i = (\phi_{i,1}, \cdots, \phi_{i,k_1}), \psi_i = (\psi_{i,1}, \cdots, \psi_{i,k_1}), \ \ i = 1, 2$.

Clearly, $\xi(T(\theta_1, \theta_2)z) = \left(e^{inm\theta_1}\xi_1(z), e^{inm\theta_2}\xi_2(z) \right)$. Hence ξ satisfies (A_1). Now it remains to show that ξ satisfies (A_2) and (A_3) .

(A_2) For each $z \in A \cup B$, if $z \in \overline{A \setminus B}$, then $\xi_1(z) \neq 0$ since $\phi_1(z) \neq 0$; if $z \notin \overline{A \setminus B}$, then $z \in B$. By the definitions of χ_2 and ψ, we have

$$\chi_2(z)(\psi_{1,1}^n(z), \cdots, \psi_{1,k_2}^n(z)) \neq 0,$$

which means that the last k_2 components of $\xi_1(z)$ are not all zero. Similarly, we can obtain $\xi_2(z) \neq 0$ for any $z \in A \cup B$.

(A_3) For each $z \in A \cup B$, we have $\chi_1(z) = 0$ or $\chi_2(z) = 0$. Here we only consider $\chi_1(z) = 0$. In fact, if $\chi_1(z) = 0$, then $z \in B$ and

$$\begin{aligned}\xi(z) &= (\xi_1(z); \xi_2(z)) \\ &= (\phi_{1,1}^m(z), \cdots, \phi_{1,k_1}^m(z), \chi_2(z)\psi_{1,1}^n(z), \cdots, \chi_2(z)\psi_{1,k_2}^n(z); \psi_{2,1}^n(z), \cdots, \psi_{2,k_2}^n(z), 0, \cdots, 0) \\ &\in \bigcup_{j=1}^{k_2} (C^{k_1} \times C^j) \times (C^{k_2+1-j} \times \{0\}) \subset \bigcup_{j=1}^{k_1+k_2} C^j \times C^{k_1+k_2+1-j}.\end{aligned}$$

Thus, we prove that ξ satisfies (A_1), (A_2) and (A_3) so that $\mathrm{ind}\,(A \cup B) \leq k_1 + k_2$. The proof is completed.

Proposition 2.1. *The index defined as Definition* 2.1 *is normal, i.e., for each orbit* $[z] = [(x, y)]$ *such that* $x \notin \mathrm{Fix}T_1(S^1)$ *and* $y \notin \mathrm{Fix}T_2(S^1)$, *we have* $\mathrm{ind}\,[z] = 1$.

Proof. It follows directly from the definition.

Proposition 2.2. *Let V be a $(k+1)$-dimensional $T_1(S^1)$-invariant subspace of X such that $V \cap \mathrm{Fix}T_1(S^1) = \{0\}$, and let W be an $(l+1)$-dimensional $T_2(S^1)$-invariant subspace of Y such that $W \cap \mathrm{Fix}T_2(S^1) = \{0\}$. Then*

$$\mathrm{ind}\,((V \cap S_X) \times (W \cap S_Y)) = k + l + 1,$$

where $S_X = \{x \in X|\, \|x\| = 1\}$ and $S_Y = \{y \in Y|\, \|y\| = 1\}$.

In order to prove this proposition, we need the following so-called $S^1 \times S^1$-Borsuk-Ulam theorem on product space (see [6]).

Lemma 2.1. *Let S^{2k+1} and S^{2l+1} denote the unit sphere of C^{k+1} and C^{l+1} respectively. Let T_1 and T_2 be representations of S^1 over C^{k+1} and C^{l+1} respectively such that $\mathrm{Fix}T_1(S^1) \cap C^{k+1} = \{0\}$ and $FixT_2(S^1) \cap C^{l+1} = \{0\}$. Suppose that*

$$\phi = (\phi_1, \phi_2) : S^{2k+1} \times S^{2l+1} \to C^k \times C^l$$

is a continuous mapping satisfying

$$(\phi_1(T_1(\theta_1)x, T_2(\theta_2)y), \phi_2(T_1(\theta_1)x, T_2(\theta_2)y) = (e^{in\theta_1}\phi_1(x, y), e^{in\theta_2}\phi_2(x, y))$$

for all $(\theta_1, \theta_2) \in S^1 \times S^1$ and $(x, y) \in S^{2k+1} \times S^{2l+1}$. Then there exists at least one orbit $[(x, y)]$ such that $\phi(x, y) = 0$.

Proof of Proposition 2.2. It is easy to know that

$$\mathrm{ind}\,((V \cap S_X) \times (W \cap S_Y)) \le k + l + 1. \tag{2.7}$$

Now, if we assume that

$$\mathrm{ind}\,((V \cap S_X) \times (W \cap S_Y)) = m < k + l + 1, \tag{2.8}$$

then there exist an integer n and a continuous mapping $\phi = (\phi_1; \phi_2) : (V \cap S_X) \times (W \cap S_Y) \to C^m \times C^m$ such that (A_1), (A_2) and (A_3) hold. Since $\phi_1 \ne 0, \phi_2 \ne 0$, by the usual S^1-Borsuk-Ulam theorem, it is easy to know that $m \ge 1 + \max\{k, l\}$.

Let $P_k : C^m \to C^k, P_l : C^m \to C^l$ be the canonical projections, and let $P = (P_1; P_2)$. Then

$$\psi = P \cdot \phi = (P_k\phi_1; P_l\phi_2) : (V \cap S_X) \times (W \cap S_Y) \to C^k \times C^l$$

satisfies (A_1). Using Lemma 2.1, we see that there exists a point $z_0 \in (V \cap S_X) \times (W \cap S_Y)$ such that $\psi(z_0) = (\psi_1(z_0); \psi_2(z_0)) = 0$. By ($\mathrm{A}_3$), there exists some $j, 1 \le j \le m$, such that

$$\begin{aligned}\phi(z_0) &= (\phi_{1,1}(z_0), \cdots, \phi_{1,j}(z_0), 0, \cdots, 0;\ \phi_{2,1}(z_0), \cdots, \phi_{2,m+1-j}(z_0), 0, \cdots, 0) \\ &\in C^j \times C^{m+1-j}.\end{aligned}$$

If $1 \le j \le k$, then $\psi_1(z_0) = P_k\phi_1(z_0) = \phi_1(z_0) = 0$, which contradicts ($\mathrm{A}_2$). Hence, $k < j \le m$. Combining with (2.8), we have $m + 1 - j \le k + l + 1 - j \le l$. Therefore

$$\psi_2(z_0) = P_l\phi_2(z_0) = \phi_2(z_0) = 0,$$

which again contradicts (A_2). Thus, we get

$$\mathrm{ind}\,((V \cap S_X) \times (W \cap S_Y)) \ge k + l + 1.$$

Combining with (2.7) , we finally obtain

$$\mathrm{ind}\,((V \cap S_X) \times (W \cap S_Y)) = k + l + 1.$$

The proof is completed.

§3. Some Abstract Critical Point Theorems

In this section, we apply the index theory on product space to obtain two abstract critical point theorems. One is for the constrained case and the other is for the free case. These theorems can be applied to consider nonlinear eigenvalue problems with two parameters and the existence of nontrivial solutions of differential equation systems.

Let X, Y be two Banach spaces and T_1, T_2 be two isometric representations of S^1 over X and Y respectively, and let $T = (T_1, T_2)$ be the representation of $S^1 \times S^1$ over $Z = X \times Y$.

Theorem 3.1. *Let $M \times N$ be a smooth $T(S^1 \times S^1)$-invariant submanifold of $X \times Y$ such that $M \cap \mathrm{Fix}T_1(S^1) = \emptyset$ and $N \cap \mathrm{Fix}T_2(S^1) = \emptyset$. Suppose that $f : M \times N \to R^1$ is a $T(S^1 \times S^1)$-invariant C^1-functional satisfying P.S. condition. Define*

$$\Gamma_m = \{A \subset M \times N | A \textit{ is closed } T(S^1 \times S^1)\textit{-invariant and } \operatorname{ind} A \geq m\},$$

$$c_m = \inf_{A\in\Gamma_m} \sup_{z\in A} f(z), \quad m = 1, 2, \cdots .$$

If c_m is finite, then c_m is a critical value of f. Moreover, if $c_m = c_{m+j}$ for some integer $j \geq 1$, then $\operatorname{ind} K_{c_m} \geq j + 1$, where $K_{c_m} = \{z \in M \times N | f(z) = c_m, \langle f'(z), u\rangle = 0, \forall u \in T_z(M \times N)\}$ and $T_z(M \times N)$ is the tangent space of $M \times N$ at z.

Theorem 3.2. *Let $f : X \times Y \to R^1$ be a $T(S^1 \times S^1)$-invariant C^1-functional satisfying P.S. condition. Set*

$$c_0 = \inf_{z\in(\mathrm{Fix}T_1(S^1)\times Y)\cup(X\times \mathrm{Fix}T_2(S^1))} f(z), \quad c_m = \inf_{A\in\Gamma_m} \sup_{z\in A} f(z), \quad m = 1, 2, \cdots ,$$

where

$$\Gamma_m = \{A \subset X \times Y | A \textit{ is closed } T(S^1 \times S^1)\textit{-invariant and } \operatorname{ind} A \geq m\}.$$

If $-\infty < c_m < c_0$, then c_m is a critical value of f, and the $S^1 \times S^1$-orbit of critical points of f corresponding to c_m is outside of $\mathrm{Fix}T_1(S^1) \times Y \cup X \times \mathrm{Fix}T_2(S^1)$. Moreover, if $-\infty < c_m = c_{m+j} < c_0$ for some integer $j \geq 1$, then there exist at least $j + 1$ distinct $T(S^1 \times S^1)$-orbits of critical points of f outside of $(\mathrm{Fix}T_1(S^1) \times Y) \cup (X \times \mathrm{Fix}T_2(S^1))$.

The proof of these two theorems is standard in critical point theory, so we omit it here (see [4] for details).

References

[1] Benci, V., A geometrical index for the group S^1 and some applications to the study of periodic solutions of ordinary differential equations, *Comm. Pure Appl. Math.,* **34**(1981), 393–432.

[2] Benci, V., On critical point theory for infinite functionals in the presence of symmetries, *Trans. Amer. Math. Soc.,* **274**(1982), 533–572.

[3] Fadell, E. R., Husseini, S. Y. & Rabinowitz, P. H., Borsuk-Ulam theorems for arbitrary S^1-actions and applications, *Trans. Amer. Math. Soc.,* **274**(1982), 345–360.

[4] Mawhin, J. & Willem, M., Critical point theory and Hamiltonian systems, Springer-Verleg, New York, 1989.

[5] Rabinowitz, P. H., Variational methods for nonlinear eigenvalue problems, Nonlinear Eigenvalue Problems, (G. Prodi, editor), Edizione Cremonese, Roma, 1974, 141–195.

[6] Zhong, C. K., $S^1 \times S^1$-Borsuk-Ulam theorem on product spaces, *Northeastern Math. J.* (*Changchun, China*), **12:**2(1996), 144–152.

[7] Zhong, C. K., On Ekeland's variational principle and a minimax theorem , *J. Math. Anal. Appl.,* (1996).

[8] Zhong, C. K., The Borsuk-Ulam theorems on product space, Proceedings of the second interational conference on "Fixed Point Theory and Applications (K. K. Tan, editor), World Scientific, 1992, 362–372.