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RANDOM DIRICHLET TYPE FUNCTIONS ON THE UNIT BALL OF $\mathbf{C}^n$

HU Pengyan* SHI Jihuai**

Abstract

The random Dirichlet type functions on the unit ball of $\mathbf{C}^n$ are studied. Sufficient conditions of the multipliers of $\mathcal{D}_\mu$ for $0 < \mu \le 1$, if $n = 1$ or $0 < \mu < 2$ if $n > 1$ are given. The smoothness of random Dirichlet type functions is discussed.

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§1. Introduction

Let B be the open unit ball in $\mathbf{C}^n (n \ge 1)$ with boundary S, ν the Lebesgue measure on B normalized so that $\nu(B) = 1$ and σ the positive normalized rotation invariant measure on S, i.e. $\sigma(S) = 1$. The class of all holomorphic functions with domain B will be denoted by $H(B)$.

Let f be in $H(B)$ with Taylor expansion $f(z) = \sum\limits_{\alpha \ge 0} a_\alpha z^\alpha$, f is said to be in the Dirichlet type space $\mathcal{D}_\mu (\mu \in \mathbf{R})$ provided that

$$\|f\|_\mu^2 = \sum_{\alpha \ge 0} (|\alpha| + n)^\mu \omega_\alpha |a_\alpha|^2 < \infty.$$

Here[5]

$$\omega_\alpha = \int_S |\zeta^\alpha|^2 d\sigma(\zeta) = \frac{(n-1)!\alpha!}{(n+|\alpha|-1)!}.$$

Specially, the space $\mathcal{D}_1$ is called Dirichlet space. The spaces $\mathcal{D}_0$ and $\mathcal{D}_{-1}$ are just the Hardy space $H^2(B)$ and the Bergman space $L_a^2(B)$ respectively.

Let $\phi : B \to \mathbf{C}$. ϕ is said to be a pointwise multiplier (or multiplier briefly) of $\mathcal{D}_\mu$ if $\phi f \in \mathcal{D}_\mu$ for all $f \in \mathcal{D}_\mu$. The collection of all multipliers of $\mathcal{D}_\mu$ is denoted by $M(\mathcal{D}_\mu)$. Since the constant function $f(z) \equiv 1$ is in $\mathcal{D}_\mu$ for any $\mu \in \mathbf{R}$, we see that $\phi \in M(\mathcal{D}_\mu)$ implies $\phi \in \mathcal{D}_\mu$. This means that $\phi \in H(B)$ and $M(\mathcal{D}_\mu) \subset \mathcal{D}_\mu$.

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*Department of Mathematics, Normal College of Shenzhen University, Shenzhen 518060, China.
E-mail: szuss@szu.edu.cn

**Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.

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Let $\{\varepsilon_\alpha(\omega)\}$ be a Bernoulli sequence of random variables on a probability space $(\Omega, \mathcal{A}, P)$. This means that the sequence is independent, and each $\varepsilon_\alpha(\omega)$ takes the values $+1$ and -1 with probability $\frac{1}{2}$ each. For $f(z) = \sum\limits_{\alpha\geq 0} a_\alpha z^\alpha \in H(B)$,

$$f_\omega(z) = \sum_{\alpha\geq 0} \varepsilon_\alpha(\omega) a_\alpha z^\alpha \tag{1.1}$$

is called the randomization of $f(z)$. Obviously, $f_\omega \in \mathcal{D}_\mu$ for all $\omega \in \Omega$ if $f \in \mathcal{D}_\mu$.

More recently, Cochram, Shapiro and Ullrich[1] studied the random Dirichlet type functions on the unit disc U and obtained some interesting results.

In this paper, we will generalize those results to the Dirichlet type spaces $\mathcal{D}_\alpha$ in the unit ball of $\mathbf{C}^n$. We will essentially follow the approach of [1], but some special techniques are used.

To state our results, we introduce the space $\mathcal{D}_{\mu,p}$. Let $f(z) = \sum\limits_{\alpha\geq 0} a_\alpha z^\alpha \in H(B)$. We say that $f \in \mathcal{D}_{\mu,p}$, if

$$\sum_{\alpha\geq 0} (|\alpha| + n)^\mu |a_\alpha|^2 \xi_{\alpha,p}^2 < \infty,$$

where $\quad \xi_{\alpha,p} = \left(\int_S |\zeta^\alpha|^p d\sigma(\zeta)\right)^{\frac{1}{p}}$.

It is clear that if $p \geq 2$, then

$$\omega_\alpha = \int_S |\zeta^\alpha|^2 d\sigma(\zeta) \leq \left(\int_S |\zeta^\alpha|^p d\sigma(\zeta)\right)^{\frac{2}{p}} = \xi_{\alpha,p}^2.$$

This means that $\mathcal{D}_{\mu,p} \subset \mathcal{D}_\mu$, if $p \geq 2$.

Our main results are the following theorems.

Theorem 1.1. *Let $0 < \mu \leq 1$ if $n = 1$ or $0 < \mu < 2$ if $n \geq 2$, $f(z) = \sum\limits_{\alpha\geq 0} a_\alpha z^\alpha$, and $f_\omega(z) = \sum\limits_{\alpha\geq 0} \varepsilon_\alpha(\omega) a_\alpha z^\alpha$ be the randomization of f. If $f \in \mathcal{D}_{\mu,p}$ for some $p > \frac{2n}{\mu}$, then $f_\omega \in M(\mathcal{D}_\mu)$ for almost every $\omega \in \Omega$ (this can be written briefly as $f_\omega \in M(\mathcal{D}_\mu)$, a.s.).*

Since $\xi_{\alpha,p} = 1$ when $n = 1$, we have $\mathcal{D}_{\mu,p} = \mathcal{D}_\mu$. Therefore this theorem is the generalization of Theorem 2 in [1]. When $n > 1$, the above theorem says that the randomizations of all functions in the subsets $\mathcal{D}_{\mu,p}$ of $\mathcal{D}_\mu$ for some $p > \frac{2n}{\mu}$ are in $M(\mathcal{D}_\mu)$ almost surely. In [4], we gave the characterizations of $M(\mathcal{D}_\mu)$ except the case $0 < \mu \leq n$. So the above theorem gives a sufficient condition for $M(\mathcal{D}_\mu)$. We conjecture that the result is true for $f \in \mathcal{D}_\mu$.

The following theorems concern the smoothness of random Dirichlet type functions, which is characterized by Lipschitz conditions. If $0 < \tau \leq 1$, and f is a complex valued function on B, we say that f satisfies a Lipschitz condition of order τ (briefly: $f \in \mathrm{Lip}_\tau$) provided that

$$|f(z) - f(w)| \leq M|z - w|^\tau \tag{1.2}$$

for some constant M and all z, w in B. If, in addition, $\tau < 1$, and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(w)| \leq \varepsilon |z - w|^\tau$$

whenever $|z - w| < \delta$, then we say that f satisfies a little Lipschitz condition of order τ (briefly: $f \in \mathrm{lip}_\tau$).

Theorem 1.2. *Let* $0 \le \tau < 2, f_\omega(z) = \sum\limits_{\alpha \ge 0} \varepsilon_\alpha(\omega) a_\alpha z^\alpha$. *If*

$$\sum_{|\alpha|>0} (|\alpha| + n)^\tau |a_\alpha|^2 \eta_\alpha^2 \log |\alpha| < \infty, \tag{1.3}$$

then

$$\int_0^1 M_\infty^2(Rf_\omega, r)(1-r)^{1-\tau} dr < \infty, \tag{1.4}$$

a.s. Moreover, if $\tau > 0$, *then* $f_\omega \in \mathrm{Lip}_{\frac{\tau}{2}}$ *a.s., where* $\eta_\alpha = \max\limits_{\zeta \in S} |\zeta^\alpha|$.

This theorem implies that the randomization of the function in $\mathcal{D}_\tau$ is in lip_μ a.s. for $\mu < \frac{\tau}{2}$. Next theorem gives two functions, one shows that the conclusion of Theorem 1.2 can not extend to $\mu = \frac{\tau}{2}$, the other shows that there is a function in $\mathcal{D}_\tau (0 < \tau < 2)$, but all of its randomizations do not belong to $\mathrm{Lip}_{\tau/2}$.

Theorem 1.3. (i) *Let* $0 \le \tau < 2$. *Given a positive sequence* $\{c_\alpha\}$ *decreasing in the sense: when* $|\alpha| < |\beta|$, *then* $c_\alpha > c_\beta$ *and* $c_\alpha \to 0$, *as* $|\alpha| \to \infty$. *Then there is a sequence* $\{a_\alpha\}$ *such that*

$$\sum_{\alpha>0} c_\alpha (|\alpha| + n)^\tau |a_\alpha|^2 \log |\alpha| < \infty,$$

but

$$M_\infty(\mathrm{R}f_\omega, r) \ne O\Big(\Big(\frac{1}{1-r}\Big)^{1-\frac{\tau}{2}}\Big) \quad a.s.$$

Moreover, if $\tau > 0$, *then* $f_\omega \overline{\in} \mathrm{Lip}_{\frac{\tau}{2}}$ *a.s.*

(ii) *Let* $0 < \tau < 2$. *Then there is a function* $f(z) = \sum\limits_{\alpha \ge 0} a_\alpha z^\alpha \in \mathcal{D}_\tau$, *such that* $\sum\limits_{\alpha \ge 0} \pm a_\alpha z^\alpha \overline{\in} \mathrm{Lip}_{\tau/2}$ *for every choice of signs.*

In the following C denotes a positive constant which may be different from one place to the next.

§2. Multipliers

In this section, we will give the proof of Theorem 1.1. Some lemmas are needed.

Lemma 2.1. *Let* $\tau < 2$. *Then* $f \in \mathcal{D}_\tau$ *if and only if*

$$\int_B |\mathrm{R}f(z)|^2 (1-|z|^2)^{1-\tau} d\nu(z) < \infty. \tag{2.1}$$

Here $\mathrm{R}f(z) = \sum\limits_{j=1}^n z_j \frac{\partial f(z)}{\partial z_j}$ *is the radial derivative of* f.

Proof. Let $f(z) = \sum\limits_{\alpha \ge 0} a_\alpha z^\alpha$. With integration in polar coordinates, we have

$$\begin{aligned}\int_B |\mathrm{R}f(z)|^2 (1-|z|^2)^{1-\tau} d\nu(z) &= n \sum_{\alpha \ge 0} |\alpha|^2 |a_\alpha|^2 \omega_\alpha \int_0^1 r^{n+|\alpha|-1}(1-r)^{1-\tau} dr \\ &= n \sum_{\alpha \ge 0} B(n+|\alpha|, 2-\tau) |\alpha|^2 |a_\alpha|^2 \omega_\alpha \\ &\sim \sum_{\alpha \ge 0} (|\alpha|+n)^\tau |a_\alpha|^2 \omega_\alpha. \end{aligned} \tag{2.2}$$

Here $B(\cdot,\cdot)$ is the classical Beta function and the symbol "$\sim$" means comparable. In the above, the orthogonality of $\{\zeta^\alpha\}$ on S and Stirling formula are used. The proof is obtained by the definition of $\mathcal{D}_\tau$ and (2.2).

Lemma 2.2.[3] *For $0<\tau<n$, let K_τ be the kernel on $S\times S$ given by*

$$K_\tau(\zeta,\eta)=\int_0^1(1-r)^{\tau-1}\mathcal{P}(r\zeta,\eta)dr.$$

Then the mapping

$$f\to\int_S K_\tau(\cdot,\eta)f(\eta)d\sigma(\eta) \tag{2.3}$$

sends $L^p(\sigma)$ to $L^q(\sigma)$, where $\mathcal{P}(\cdot,\cdot)$ is the Poisson-Szegö kernel,

$$\frac{1}{q}=\frac{1}{p}-\frac{\tau}{n}\quad and\quad 1<p<q<\infty.$$

Using Lemma 2.2, we can prove the following result, which is of independent interest.

Lemma 2.3. *If $0<\tau<n$, then $\mathcal{D}_\tau\subset H^{\frac{2n}{n-\tau}}(B)$.*

Proof. Let $f(z)=\sum\limits_{\alpha\ge0}a_\alpha z^\alpha\in\mathcal{D}_\tau$. Then $f\in H^2(B)$ since $\tau>0$. So

$$\lim_{r\to1}f(r\zeta)=f^*(\zeta)$$

exists for almost every $\zeta\in S$ and

$$f(z)=\int_S\frac{f^*(\zeta)}{(1-<z,\zeta>)^n}d\sigma(\zeta). \tag{2.4}$$

Denote

$$f^{[\frac{\tau}{2}]}(z)=\sum_{\alpha\ge0}\frac{\Gamma(|\alpha|+\frac{\tau}{2}+1)}{\Gamma(|\alpha|+1)}a_\alpha z^\alpha. \tag{2.5}$$

Then $f\in\mathcal{D}_\tau$ implies $f^{[\frac{\tau}{2}]}\in H^2(B)$. Hence

$$\lim_{r\to1}f^{[\frac{\tau}{2}]}(r\zeta)=f^{[\frac{\tau}{2}]*}(\zeta)$$

exists for almost every $\zeta\in S$, and $f^{[\frac{\tau}{2}]*}\in L^2(\sigma)$. A direct computation gives

$$\begin{aligned}\int_S K_{\frac{\tau}{2}}(\zeta,\eta)f^{[\frac{\tau}{2}]}(r\eta)d\sigma(\eta)&=\int_0^1(1-\rho)^{\frac{\tau}{2}-1}d\rho\int_S\mathcal{P}(\rho\zeta,\eta)f^{[\frac{\tau}{2}]}(r\eta)d\sigma(\eta)\\&=\int_0^1(1-\rho)^{\frac{\tau}{2}-1}f^{[\frac{\tau}{2}]}(r\rho\zeta)d\rho\\&=\sum_{\alpha\ge0}\frac{\Gamma(|\alpha|+\frac{\tau}{2}+1)}{\Gamma(|\alpha|+1)}a_\alpha\zeta^\alpha r^{|\alpha|}\int_0^1(1-\rho)^{\frac{\tau}{2}-1}\rho^{|\alpha|}d\rho\\&=\Gamma\Big(\frac{\tau}{2}\Big)\sum_{\alpha\ge0}a_\alpha\zeta^\alpha r^{|\alpha|}=\Gamma\Big(\frac{\tau}{2}\Big)f(r\zeta).\end{aligned}$$

Letting $r\to1$ in the above equality yields

$$\Gamma\Big(\frac{\tau}{2}\Big)f^*(\zeta)=\int_S K_{\frac{\tau}{2}}(\zeta,\eta)f^{[\frac{\tau}{2}]*}(\eta)d\sigma(\eta). \tag{2.6}$$

Lemma 2.2 and the fact $f^{[\frac{\tau}{2}]*}\in L^2(\sigma)$ give $f^*\in L^{\frac{2n}{n-\tau}}(\sigma)$. Now $f\in H^{\frac{2n}{n-\tau}}(B)$ follows from (2.4). This completes the proof.

Note that this lemma is not true if $\tau = n$. In fact, we have given a function f (see [4]), such that $f \in \mathcal{D}_n$, but $f \overline{\in} H^\infty(B)$. We conjecture that $\mathcal{D}_n \subset \mathrm{BMOA}(B)$, but we cannot prove it.

We will use the following corollary of Lemma 2.3.

Corollary 2.1. *If $0 < \tau \le n$, $\mathcal{D}_\tau \subset H^{\frac{2p}{p-2}}(B)$ for any $p > \frac{2n}{\tau}$.*

Proof. Suppose $0 < \tau < n$. Since $\frac{2n}{n-\tau} > \frac{2p}{p-2}$, then Lemma 2.3 gives the desired result. When $\tau = n$, we can choose τ' such that $n > \tau' > \frac{2n}{p}$ since $p > \frac{2n}{\tau}$. The fact $\mathcal{D}_n \subset \mathcal{D}_{\tau'}$ and the result of the case $0 < \tau < n$ complete the proof.

Lemma 2.4. *Let $\phi \in H(B), 0 < \mu \le 1$ if $n = 1$ or $0 < \mu < 2$ if $n > 1$. If*

$$\int_0^1 M_p^2(\mathrm{R}\phi, r)(1-r)^{1-\mu} dr < \infty \tag{2.7}$$

for some $p > \frac{2n}{\mu}$, then $\phi \in M(\mathcal{D}_\mu)$.

Proof. By (2.7), there exists $r_0 > 0$, such that

$$\int_r^1 M_p^2(\mathrm{R}\phi, t)(1-t)^{1-\mu} dt < \varepsilon$$

whenever $1 > r > r_0$. So it follows that

$$M_p(\mathrm{R}\phi, r) = o\left((1-r)^{\frac{\mu}{2}-1}\right), \quad r \to 1^-$$

from the monotonicity of $M_p(\mathrm{R}\phi, r)$ in r, and

$$M_\infty(\mathrm{R}\phi, r) = o\left((1-r)^{-\frac{n}{p}+\frac{\mu}{2}-1}\right), \quad r \to 1^-.$$

Then we have $\phi \in H^\infty(B)$ by Theorem 6.4.10 of [5].

Let $f \in \mathcal{D}_\mu$,

$$\begin{aligned} &\int_B |\mathrm{R}(\phi f)(z)|^2 (1-|z|^2)^{1-\mu} d\nu(z) \\ &\le 2\Big(\int_B |f(z)|^2 |R\phi(z)|^2 (1-|z|^2)^{1-\mu} d\nu(z) + \int_B |\phi(z)|^2 |\mathrm{R}f(z)|^2 (1-|z|^2)^{1-\mu} d\nu(z)\Big) \\ &= 2(\mathrm{M}_1 + \mathrm{M}_2). \end{aligned} \tag{2.8}$$

By Lemma 2.1 and $\phi \in H^\infty(B)$,

$$\mathrm{M}_2 \le \|\phi\|_\infty^2 \int_B |\mathrm{R}f(z)|^2 (1-|z|)^{1-\mu} d\nu(z) < \infty. \tag{2.9}$$

The integration in polar coordinates, Hölder's inequality with conjugate indices $\frac{p}{2}$ and $\frac{p}{p-2}$ and Corollary 2.1 give

$$\mathrm{M}_1 \le C\|f\|_{\frac{2p}{p-2}}^2 \int_0^1 M_p^2(\mathrm{R}\phi, r)(1-r)^{1-\mu} dr < \infty. \tag{2.10}$$

By (2.8)–(2.10) and Lemma 2.1, we have $\phi f \in \mathcal{D}_\mu$, that is, $\phi \in M(\mathcal{D}_\mu)$.

Next we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $f(z) = \sum\limits_{\alpha \ge 0} a_\alpha z^\alpha$, and

$$f_\omega(z) = \sum_{\alpha \ge 0} \varepsilon_\alpha(\omega) a_\alpha z^\alpha$$

its randomization. Then by using Fubini's theorem, inequalities of Khintchine[8], Jensen and Minkowski,

$$
\begin{aligned}
&E\Big(\int_0^1 M_p^2(\,\mathrm{R}f_\omega, r)(1-r)^{1-\mu}dr\Big)\\
&=\int_0^1\int_\Omega\Big(\int_S|\,\mathrm{R}f_\omega(r\zeta)|^p d\sigma(\zeta)\Big)^{\frac{2}{p}}dP(1-r)^{1-\mu}dr\\
&\le\int_0^1\Big(\int_\Omega\int_S|\mathrm{R}f_\omega(r\zeta)|^p d\sigma(\zeta)dP\Big)^{\frac{2}{p}}(1-r)^{1-\mu}dr\\
&=\int_0^1\Big(\int_S\int_\Omega|\mathrm{R}f_\omega(r\zeta)|^p dPd\sigma(\zeta)\Big)^{\frac{2}{p}}(1-r)^{1-\mu}dr\\
&\le C_p\int_0^1\Big(\int_S\|\mathrm{R}f_\omega(r\zeta)\|^p_{L^2(\Omega)}d\sigma(\zeta)\Big)^{\frac{2}{p}}(1-r)^{1-\mu}dr\\
&=C_p\int_0^1\Big[\int_S\Big(\sum_{|\alpha|>0}|\alpha|^2|a_\alpha|^2|\zeta^\alpha|^2r^{2|\alpha|}\Big)^{\frac{p}{2}}d\sigma(\zeta)\Big]^{\frac{2}{p}}(1-r)^{1-\mu}dr\\
&\le C_p\int_0^1\sum_{|\alpha|>0}\Big(\int_S|\alpha|^p|a_\alpha|^p|\zeta^\alpha|^pr^{p|\alpha|}d\sigma(\zeta)\Big)^{\frac{2}{p}}(1-r)^{1-\mu}dr\\
&=C_p\sum_{|\alpha|>0}|\alpha|^2|a_\alpha|^2\xi^2_{\alpha,p}\int_0^1 r^{2|\alpha|}(1-r)^{1-\mu}dr\\
&\le C_p\sum_{|\alpha|>0}|\alpha|^\mu|a_\alpha|^2\xi^2_{\alpha,p}<\infty.
\end{aligned}
$$

So

$$P\Big\{\omega:\int_0^1 M_p^2(\mathrm{R}f_\omega, r)(1-r)^{1-\mu}dr<\infty\Big\}=1.$$

This means $\int_0^1 M_p^2(\mathrm{R}f_\omega, r)(1-r)^{1-\mu}dr<\infty$ a.s. Then Lemma 2.4 gives the desired result.

§3. Smoothness

In this section, we will give the proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. For $f_\omega(z)=\sum\limits_{\alpha\ge0}\varepsilon_\alpha(\omega)a_\alpha z^\alpha$, let

$$Q_m(z)=\sum_{k=1}^m\sum_{|\alpha|=k}\varepsilon_\alpha(\omega)|\alpha|a_\alpha z^\alpha,\qquad A_m^2=\sum_{k=1}^m k^2\sum_{|\alpha|=k}|a_\alpha|^2\eta_\alpha^2.$$

Then

$$
\begin{aligned}
\mathrm{R}f_\omega(r\zeta)&=\sum_{m=1}^\infty\sum_{|\alpha|=m}|\alpha|\varepsilon_\alpha(\omega)a_\alpha\zeta^\alpha r^m\\
&=\sum_{m=1}^\infty(Q_m(\zeta)-Q_{m-1}(\zeta))r^m=(1-r)\sum_{m=1}^\infty Q_m(\zeta)r^m. \qquad (3.1)
\end{aligned}
$$

By Lemma 2 in [7],

$$Q_m(\zeta)=O(A_m(\log m)^{\frac{1}{2}}) \qquad (3.2)$$

holds uniformly for $\zeta \in S$ a.s. Therefore, (3.1) and (3.2) give

$$|\mathrm{R}f_\omega(r\zeta)| \le C(1-r)\sum_{m=1}^{\infty} A_m(\log m)^{\frac{1}{2}} r^m.$$

Thus

$$\int_0^1 M_\infty^2(\mathrm{R}f_\omega, r)(1-r)^{1-\tau}dr \le C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} A_j\sqrt{\log j}A_k\sqrt{\log k}\frac{1}{(j+k+1)^{4-\tau}}$$

$$\le C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{A_j\sqrt{\log j}}{j^{\frac{3-\tau}{2}}}\cdot\frac{A_k\sqrt{\log k}}{k^{\frac{3-\tau}{2}}}\cdot\frac{1}{j+k} \le \ C\sum_{j=1}^{\infty}\frac{A_j^2\log j}{j^{3-\tau}}$$

$$= C\sum_{j=1}^{\infty}\frac{\log j}{j^{3-\tau}}\sum_{k=1}^{j}k^2\sum_{|\alpha|=k}|a_\alpha|^2\eta_\alpha^2 = C\sum_{k=1}^{\infty}k^2\sum_{|\alpha|=k}|a_\alpha|^2\eta_\alpha^2\sum_{j=k}^{\infty}\frac{\log j}{j^{3-\tau}}$$

$$\le C\sum_{k=1}^{\infty}\sum_{|\alpha|=k}|\alpha|^\tau|a_\alpha|^2\eta_\alpha^2\log|\alpha| < \infty, \quad \text{a.s.}$$

In the above, Hilbert's inequality[2] is used. This proves (1.4).

By the monotonicity of $M_\infty(\mathrm{R}f_\omega, r)$ of r and (1.4) we have

$$M_\infty(\mathrm{R}f_\omega, r) = o((1-r)^{\frac{\tau}{2}-1}) \ \ (r \to 1^-), \quad \text{a.s.}$$

This gives $f_\omega \in \mathrm{Lip}_{\tau/2}$ a.s., and the proof is complete.

Proof of Theorem 1.3. (i) Suppose $\alpha = (k, 0, \cdots, 0)$, $k = 1, 2, \cdots$, and let $c_k = c_\alpha$. Then $\{c_k\}$ is a strictly decreasing positive sequence with $\lim\limits_{k\to\infty} c_k = 0$. By Theorem 3(b) in [1], there is a positive sequence $\{b_k\}$ such that

$$\sum c_k k^\tau |b_k|^2 \log k < \infty,$$

but the randomizations of $g(\lambda) = \sum b_k\lambda^k$, $\lambda \in U$, satisfying

$$M_\infty(g'_\omega, r) \ne O\Big(\Big(\frac{1}{1-r}\Big)^{1-\frac{\tau}{2}}\Big), \quad \text{a.s.}$$

So there exists a sequence $\{\lambda_k\}$ satisfying $|\lambda_k| < 1, |\lambda_k| \to 1$ as $k \to \infty$, and

$$|g'_\omega(\lambda_k)|(1-|\lambda_k|)^{1-\frac{\tau}{2}} \to \infty \quad (k \to \infty), \quad \text{a.s.}$$

Let

$$a_\alpha = \begin{cases} b_k, & \alpha = (k, 0, \cdots, 0), \\ 0, & \alpha \ne (k, 0, \cdots, 0) \end{cases}$$

and

$$f(z) = \sum_{\alpha\ge 0} a_\alpha z^\alpha.$$

Then

$$\sum_{|\alpha|>0} c_\alpha(|\alpha|+n)^\tau|a_\alpha|^2\log|\alpha| = \sum_k c_k(k+n)^\tau|b_k|^2\log k < \infty,$$

$$f_\omega(z) = \sum_{k=0}^{\infty}\sum_{|\alpha|=k}\varepsilon_\alpha(\omega)a_\alpha z^\alpha = \sum_{k=0}^{\infty}\varepsilon_k(\omega)b_k z_1^k = g_\omega(z_1).$$

So

$$\frac{\partial f_\omega}{\partial z_1}(z) = g'_\omega(z_1), \quad \frac{\partial f_\omega}{\partial z_k}(z) = 0, \quad k = 2, \cdots, n.$$

Denote $e_1 = (1, 0, \cdots, 0)$. Then

$$\sup_{z\in B}(1-|z|)^{1-\frac{\tau}{2}}|(Rf_\omega)(z)| \geq (1-|\lambda_k e_1|)^{1-\frac{\tau}{2}}|(Rf_\omega)(\lambda_k e_1)|$$
$$= (1-|\lambda_k|)^{1-\frac{\tau}{2}}|\lambda_k||g'_\omega(\lambda_k)| \to \infty.$$

That is

$$M_\infty(\mathrm{R}f_\omega, r) \neq O\Big(\Big(\frac{1}{1-r}\Big)^{1-\frac{\tau}{2}}\Big),$$

and $f_\omega \overline{\in} \mathrm{Lip}_{\frac{\tau}{2}}$.

(ii) By Theorem 4 in [1], there exists a function

$$g(\lambda) = \sum_{k=0}^{\infty} b_k \lambda^k \in \mathcal{D}_\tau(U), \quad 0 < \tau < 2,$$

and for every choice of signs $\sum\limits_{k=0}^{\infty} \pm b_k \lambda^k \overline{\in} \mathrm{Lip}_{\frac{\tau}{2}}$. Let

$$a_\alpha = \begin{cases} b_k, & \alpha = (k, 0, \cdots, 0), \\ 0, & \alpha \neq (k, 0, \cdots, 0) \end{cases}$$

and $f(z) = \sum\limits_{\alpha\geq 0} a_\alpha z^\alpha$. Then

$$\sum_{|\alpha|\geq 0}(|\alpha|+n)^\tau |a_\alpha|^2 \omega_\alpha = \sum_{k=0}^{\infty}(k+n)^\tau |b_k|^2 \frac{(n-1)!k!}{(n+k-1)!}$$
$$\leq C\sum_{k=0}^{\infty} k^\tau |b_k|^2 < \infty.$$

So $f \in \mathcal{D}_\tau$. The proof that $\sum\limits_{\alpha\geq 0} \pm a_\alpha z^\alpha \overline{\in} \mathrm{Lip}_{\frac{\tau}{2}}$ for every choice of signs is the same as (i). The theorem is proved.

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