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RAPID EXACT CONTROLLABILITY OF THE SEMILINEAR WAVE EQUATION**

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Abstract

In this paper the semilinear wave equation with homogeneous Dirichlet boundary condition having a locally distributed controller is considered, and the rapid exact controllability of this system is obtained by changing the shape and/or the location of the controller under proper conditions. For this purpose, the author derives an (rapid) observability inequality for wave equations with linear time-variant potential by means of the energy estimate. The main difference of the method from the previous ones is that any unique continuation property of the corresponding linear time-variant wave equations is not needed.

Keywords Rapid exact controllability, Semilinear wave equation, Changing controller, Energy estimate
1991 MR Subject Classification 93B05, 35L05
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§1. Introduction

Consider the following semilinear wave equation with a locally distributed controller:

$$\begin{cases} y'' - \Delta y = f(y) + \chi_{G(t)}(x)u(t,x), & \text{in } (0,\infty)\times\Omega, \\ y = 0, & \text{on } (0,\infty)\times\partial\Omega, \\ y(0,x) = y_0(x), \quad y'(0,x) = y_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $' = \frac{\partial}{\partial t}$, $'' = \frac{\partial^2}{\partial t^2}$, $\Omega \subset \mathbf{R}^n$ is a bounded domain with a boundary $\partial\Omega \in C^1$, and for each $t \in [0,\infty)$, $G(t)$ is a subdomain of Ω. In the above, $y(t,x)$ is the state and $\chi_{G(t)}(x)u(t,x)$ is the control. Thus, $u(t,x)$ is the intensity of the control action and $G(t)$ is the location and the shape of the controller. We will allow the location and the shape of the controller to change. Let $\mathcal{U} = L^2_{\text{loc}}(0,\infty; L^2(\Omega))$ and let $\mathcal{G}$ be a family of set-valued functions $G(t)$ defined on $[0,\infty)$ taking subdomains of Ω as its values. In what follows, $u(\cdot,\cdot) \in \mathcal{U}$ and $G(\cdot) \in \mathcal{G}$. The conditions on $\mathcal{G}$ will be given in Section 2.

The purpose of this paper is to study the rapid exact controllability for System (1.1), i.e., for any prescribed time duration $T > 0$, try to find a controller $G(\cdot) \in \mathcal{G}$ such that System (1.1) with this controller is exactly controllable in the usual sense[8−10]. One can find an interesting physical background for this problem[12].

Linear system with fixed controller (i.e., $G(t) \equiv G$ is a fixed subdomain of Ω) is now well understood for both abstract and concrete cases[8−11,15,17]. For the exact controllability of semilinear systems with *fixed* controllers, there are many existing results which can be found

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in the literature (see [4–6, 13, 16, 18] and references cited therein). Linear system with changing controllers was also studied by some authors. For example, Butkovskii[2,3] studied the mobile control for distributed parameter systems; McLaughlin and Slemrod[14] discussed the scanning control for vibrating string; Khapalov[7] considered the exact controllability of the wave equation with moving point control by means of some duality method; Recently, Liu and Yong[12] obtained the rapid exact controllability of the linear wave equation (i.e., $f(\cdot) \equiv 0$ in System (1.1)) with homogeneous Dirichlet boundary condition having a locally distributed controller.

We are now, naturally, in a position to ask such a question: Does the rapid exact controllability remain true for some nonlinear system with changing controller? However, to the author's knowledge, there are no papers concerning this problem so far. The main object of this paper is to show that under some reasonable conditions for semilinear wave equations the answer to this question is "YES".

Just as the analysis in [12], we see that if $\mathcal{G}$ consists of only one constant set-valued function, i.e., $\mathcal{G} = \{G\}$, the answer to the rapid exact controllability problem is either trivially true ($G = \Omega$, see Remark 2.1 in Section 2) or negative ($G \neq \Omega$) (even for the linear wave equation, i.e., $f(\cdot) \equiv 0$ in System (1.1)). In fact, in the case $G \neq \Omega$ it is well-known that if the location and the shape of the controller is fixed, then even if the system is exactly controllable, the length of the time that is needed to steer any given initial state to the zero state has a positive lower bound; and in many cases, the exact controllability may even be lost if the location and/or the shape of the controller is not properly chosen. Consequently, we will allow the controller to change, i.e., the class $\mathcal{G}$ contains non-constant set-valued functions (the assumption on the class $\mathcal{G}$ is the same as that of Liu and Yong's[12]). Under proper conditions, we show that by doing that one can obtain the rapid exact controllability for the semilinear wave equation with a locally supported distributed controller.

It is well-known that the main problem in the theory of exact controllability is how to derive the observability inequality of the linear system. We would like to point out that it seems that the method in [12] does not apply to our present problem because our linearized problem is time-variant. The main contribution of this paper is to derive the rapid observability inequality of the linear time-varaint wave equations by means of the usual energy estimate (see Theorem 3.1 in Section 3). The main difference of our method from the previous ones is that we do not need any unique continuation property of the corresponding linear time-variant wave equations.

§2. Statement of the Main Results

Without loss of generality, we assume the following

$$\begin{cases} \inf\{x_1 \in \mathbf{R};\ \exists x' \in \mathbf{R}^{n-1},\ \text{such that } (x_1, x') \in \Omega\} = 0, \\ \sup\{x_1 \in \mathbf{R};\ \exists x' \in \mathbf{R}^{n-1},\ \text{such that } (x_1, x') \in \Omega\} \equiv \beta > 0. \end{cases} \tag{2.1}$$

Let $T > 0$ and $0 < \sigma < T$ be given. Put

$$\begin{cases} a = \dfrac{T - \sigma}{\beta}, \\ K_\sigma = \{(t, x_1) \in [0, T] \times [0, \beta];\ a x_1 < t < a x_1 + \sigma\} \end{cases} \tag{2.2}$$

and

$$D_\sigma = (K_\sigma \times \mathbf{R}^{n-1}) \cap \Omega_T, \tag{2.3}$$

where $\Omega_T = [0, T] \times \Omega$. Now we make the following assumption on the class $\mathcal{G}$.

Assumption 2.1. *Class $\mathcal{G}$ is a family of set-valued functions $G:[0,\infty)\to 2^{\Omega}$ with the following properties:*

(i) *Any $G(\cdot)\in\mathcal{G}$ is continuous with respect to the Hausdorff metric (defined on 2^{Ω}, the set of all subsets in Ω);*

(ii) *For any $T>0$, there exists a $G(\cdot)\in\mathcal{G}$ and a $\sigma\in(0,T)$, such that*

$$D_\sigma\subset\{(t,x)\in\Omega_T;x\in G(t),t\in[0,T]\}$$

with D_σ being defined by (2.3).

Now we can state our main results concerning rapid exact controllability for System (1.1).

Theorem 2.1. *Let Assumption* 2.1 *hold. Let $f:\mathbf{R}\to\mathbf{R}$ be of class $C^1(\mathbf{R})$ with $f'(\cdot)\in L^\infty(\mathbf{R})$, then System* (1.1) *is rapidly exactly controllable.*

Remark 2.1. If we take $\mathcal{G}=\{\Omega\}$, then it is easy to see that the answer to the rapid exact controllability of System (1.1) is trivially true (In this case we need only to suppose that the mapping H defined by

$$H(\phi(\cdot))\equiv f(\phi(\cdot)),\quad \forall\phi(\cdot)\in H_0^1(\Omega)$$

maps $H_0^1(\Omega)$ into $L^2(\Omega)$). In fact it is well-known that the following system is rapidly exactly controllable,

$$\begin{cases} y''-\Delta y=v(t,x), & \text{in } (0,\infty)\times\Omega,\\ y=0, & \text{on } (0,\infty)\times\partial\Omega,\\ y(0,x)=y_0(x),\quad y'(0,x)=y_1(x), & \text{in } \Omega.\end{cases}$$

That is, for any given $T>0$ and $(y_0,y_1),\ (z_0,z_1)\in H_0^1(\Omega)\times L^2(\Omega)$, there is a control $v(\cdot)\in\mathcal{U}$ such that the solution $y(t,x)$ of the above system satisfies: $y(T,x)=z_0(x),\ y'(T,x)=z_1(x)$, a.e. $x\in\Omega$. Now set $u(t,x)=-f(y(t,x))+v(t,x)$. One can see that under this control the state of System (1.1) is steered to zero at time T. And one may also obtain the same result even if $f(y)$ is replaced by $f(y,y',\nabla y)$.

Remark 2.2. Assumption 2.1 is the same as Assumption 1.1 in [12]. This assumption is physically reasonable. For the existence of $\mathcal{G}$ satisfying Assumption 2.1, we refer the readers to [12]. We note that the control subdomain is allowed to have a relatively small measure, more precisely, there exists a constant $0<\alpha<<1$ such that

$$\operatorname{mes}G(t)<\alpha\operatorname{mes}\Omega,\quad \forall t\in[0,\infty),\quad G(\cdot)\in\mathcal{G}.$$

In such a case, if we fix the controller $G(t)\equiv G(0)$, then, the system might even be not exactly controllable (even for the linear case $f(\cdot)\equiv 0$, see [1])! But by Theorem 2.1, we may make the following conclusion: By allowing the controller to change, one might achieve the rapid exact controllability for the semilinear wave equation with a locally supported distributed controller under some physically reasonable conditions on the controller.

§3. Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. For this purpose, we consider the following linear wave equation with bounded potential:

$$\begin{cases} w''-\Delta w=q(t,x)w, & \text{in } \Omega_T,\\ w=0, & \text{on } \Sigma\triangleq(0,T)\times\partial\Omega,\\ w(0)=w_0,\ w'(0)=w_1, & \text{in } \Omega.\end{cases}\tag{3.1}$$

We need the following estimate.

Theorem 3.1. *Let $q(\cdot)\in L^\infty(\Omega_T)$. Then for any $T>0$, there is a controller $G(\cdot)\in\mathcal{G}$, such that the weak solution $w(\cdot)\in C([0,T];L^2(\Omega))\cap C^1([0,T];H^{-1}(\Omega))$ of System* (3.1) *satisfies*

$$|w_0|^2_{L^2(\Omega)} + |w_1|^2_{H^{-1}(\Omega)} \le \mathcal{C}(\ell)\int_0^T \int_{G(t)} |w|^2 dxdt, \quad \forall\ (w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega) \tag{3.2}$$

for some constant $\mathcal{C} = \mathcal{C}(\ell)$ *with* $\ell \triangleq |q(\cdot)|_\infty$. *Furthermore the constant* $\mathcal{C}$ *has the following explicit estimate*

$$\mathcal{C} = \mathcal{C}(\ell) = O(\exp(C\ell)) \qquad as\ \ell \to \infty \tag{3.3}$$

for some constant $C > 0$.

Proof. Without loss of generality, we may assume $|a| < 1$ (recall a is defined in (2.2)). We divide the proof into several steps.

Step 1. Let us introduce some notations and transformations. Put

$$q_1(t,x) = \begin{cases} q(t,x), & \text{if } (t,x) \in \Omega_T, \\ 0, & \text{if } (t,x) \in ((-\infty,0) \cup (T,\infty)) \times \Omega. \end{cases}$$

Let $W \in C((-\infty,\infty); L^2(\Omega)) \cap C^1((-\infty,\infty); H^{-1}(\Omega))$ be the weak solution of the following system

$$\begin{cases} W'' - \Delta W = q_1(t,x)W, & \text{in } (-\infty,\infty) \times \Omega, \\ W = 0, & \text{on } (\infty,\infty) \times \Gamma, \\ W(0) = w_0,\ W'(0,x) = w_1, & \text{in } \Omega. \end{cases} \tag{3.4}$$

Let us introduce the following coordinate transformation

$$\begin{cases} t = \bar{t} + a\bar{x}_1, \\ x = \bar{x} \end{cases} \tag{3.5}$$

and denote

$$z(\bar{t},\bar{x}) = W(t,x)(= W(\bar{t} + a\bar{x}_1, \bar{x})), \quad (\bar{t},\bar{x}) \in (-\infty,\infty) \times \Omega. \tag{3.6}$$

Then $z(\cdot)$ solves

$$\begin{cases} (1-a^2) z_{\bar{t}\bar{t}} + 2a z_{\bar{t}\bar{x}_1} - \sum\limits_i z_{\bar{x}_i\bar{x}_i} = q_2 z, & \text{in } (-\infty,\infty) \times \Omega, \\ z = 0, & \text{on } (-\infty,\infty) \times \Gamma, \end{cases} \tag{3.7}$$

where $q_2(\bar{t},\bar{x}) = q_1(\bar{t} + a\bar{x}_1, \bar{x})$. Furthermore, for any fixed $s \in (-\infty,\infty)$, let us denote

$$v(\bar{t},\bar{x}) = \int_s^{\bar{t}} z(s,\bar{x}) ds + \chi(\bar{x}), \tag{3.8}$$

where χ solves

$$\begin{cases} \sum\limits_i \chi_{\bar{x}_i\bar{x}_i} = (1-a^2) z_{\bar{t}}(s) + 2a z_{\bar{x}_1}(s), & \bar{x} \in \Omega, \\ \chi = 0, & \text{on } \Gamma. \end{cases} \tag{3.9}$$

Then $v(\cdot)$ satisfies

$$\begin{cases} (1-a^2) v_{\bar{t}\bar{t}} + 2a v_{\bar{t}\bar{x}_1} - \sum\limits_i v_{\bar{x}_i\bar{x}_i} = \int_s^{\bar{t}} q_2(\tau,\bar{x}) v_{\bar{t}}(\tau,\bar{x}) d\tau, & \text{in } (-\infty,\infty) \times \Omega, \\ v = 0, & \text{on } (-\infty,\infty) \times \Gamma, \\ v(s) = \chi, \quad v_{\bar{t}}(s) = z(s), & \text{in } \Omega. \end{cases} \tag{3.10}$$

Step 2. Let us use energy estimate. First, put

$$E(t) \triangleq \frac{1}{2}\Big(|W_t(t,\cdot)|^2_{H^{-1}(\Omega)} + |W(t,\cdot)|^2_{L^2(\Omega)}\Big), \tag{3.11}$$

where $W(\cdot)$ is the weak solution of System (3.4). Using the usual energy estimate and noting the time reversibility of Equation (3.4), we get

$$|w_0|^2_{L^2(\Omega)} + |w_1|^2_{H^{-1}(\Omega)} = 2E(0) \leq Ce^{T\ell} \int_{\varepsilon T}^{(1-\varepsilon)T} E(\tau) d\tau \tag{3.12}$$

for any $\varepsilon \in (0,1)$ (recall $\ell \stackrel{\triangle}{=} |q(\cdot)|_\infty$). Thus we obtain

$$|w_0|^2_{L^2(\Omega)} + |w_1|^2_{H^{-1}(\Omega)} \leq C(1+\ell)e^{T\ell} \int_0^T |W(\tau,\cdot)|^2_{L^2(\Omega)} d\tau. \tag{3.13}$$

Next, for any fixed $s \in [-2T, 2T]$, put

$$\mathcal{E}^s(\bar t\,) \stackrel{\triangle}{=} \frac{1}{2}\Big(|v(\bar t,\cdot)|^2_{H_0^1(\Omega)} + |v_{\bar t}(\bar t,\cdot)|^2_{L^2(\Omega)}\Big), \tag{3.14}$$

where $v(\cdot)$ is the weak solution of System (3.10). Now, multiplying the first equation of (3.10) by $v_{\bar t}$, integrating it on Ω and noting the second equation in (3.10) (which implies that $\int_\Omega v_{\bar t \bar x_1} v_{\bar t} d\bar x = \frac{1}{2}\int_\Omega (v_{\bar t}^2)_{\bar x_1} d\bar x = 0$), we conclude that

$$\begin{aligned}\frac{d}{d\bar t}\mathcal{E}^s(\bar t\,) &\leq C\Big\| \int_\Omega \Big(\int_s^{\bar t} q_2(\tau) v_{\bar t}(\tau) d\tau\Big) v_{\bar t} d\bar x \Big\| \\ &\leq C(1+\ell)\Big[\int_s^{\bar t} \mathcal{E}^s(\tau)d\tau + \mathcal{E}^s(\bar t\,)\Big], \quad \forall\, \bar t \in [s, 2T].\end{aligned} \tag{3.15}$$

Integrating (3.15) with respect to $\bar t$, we arrive at

$$\begin{aligned}\mathcal{E}^s(\bar t\,) &\leq \mathcal{E}^s(s) + C(1+\ell)\int_s^{\bar t} \Big[\int_s^{\tau'} \mathcal{E}^s(\tau)d\tau + \mathcal{E}^s(\tau')\Big] d\tau' \\ &\leq \mathcal{E}^s(s) + C(1+\ell)\int_s^{\bar t} \mathcal{E}^s(\tau) d\tau, \quad \forall\, \bar t \in [s, 2T].\end{aligned} \tag{3.16}$$

Thus by Gronwall's inequality, we get

$$\mathcal{E}^s(\bar t\,) \leq Ce^{C\ell}\mathcal{E}^s(s), \qquad \forall\, \bar t, s \in [-2T, 2T]. \tag{3.17}$$

However, by (3.8)–(3.9) and (3.14), we see that

$$\begin{aligned}\mathcal{E}^s(s) &= \frac{1}{2}\Big(|v(s,\cdot)|^2_{H_0^1(\Omega)} + |v_{\bar t}(s,\cdot)|^2_{L^2(\Omega)}\Big) \\ &= \frac{1}{2}\Big(|\chi(\cdot)|^2_{H_0^1(\Omega)} + |z(s,\cdot)|^2_{L^2(\Omega)}\Big) \leq C\Big(|z_{\bar t}(s,\cdot)|^2_{H^{-1}(\Omega)} + |z(s,\cdot)|^2_{L^2(\Omega)}\Big).\end{aligned} \tag{3.18}$$

Thus, combining (3.17) and (3.18), we get

$$\mathcal{E}^s(\bar t\,) \leq Ce^{C\ell}\Big(|z_{\bar t}(s,\cdot)|^2_{H^{-1}(\Omega)} + |z(s,\cdot)|^2_{L^2(\Omega)}\Big), \quad \forall\, \bar t, s \in [-2T, 2T]. \tag{3.19}$$

Finally, define

$$\begin{cases} \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ Az = -\Delta z, \qquad\qquad \forall\, z \in \mathcal{D}(A). \end{cases} \tag{3.20}$$

Now, let us fix S_1 and T_1 satisfying

$$0 < S_1 < T_1 < \sigma, \tag{3.21}$$

where σ is the number appeared in Assumption 2.1 (in the previous section). Denote $\psi(\bar t\,) = \bar t^{\,2}(\sigma - \bar t\,)^2$. Then, by (3.7), using integration by parts, we get

$$\begin{aligned}&\int_0^\sigma \int_\Omega (q_2 z)\psi\Big(A^{-1}z\Big) d\bar t d\bar x \\ &= \int_0^\sigma \int_\Omega \Big[(1-a^2)z_{\bar t\bar t} + 2az_{\bar t\bar x_1} - \sum_i z_{\bar x_i \bar x_i}\Big]\psi\Big(A^{-1}z\Big) d\bar t d\bar x\end{aligned}$$

$$
\begin{aligned}
&= -(1-a^2)\int_0^\sigma\int_\Omega \psi z_{\bar t}\Big(A^{-1}z_{\bar t}\Big)d\bar t d\bar x - (1-a^2)\int_0^\sigma\int_\Omega \psi_{\bar t} z_{\bar t}\Big(A^{-1}z\Big)d\bar t d\bar x \\
&\quad - 2a\int_0^\sigma\int_\Omega \Big[z_{\bar x_1}\psi\Big(A^{-1}z_{\bar t}\Big) + z_{\bar x_1}\psi_{\bar t}\Big(A^{-1}z\Big)\Big]d\bar t d\bar x + \int_0^\sigma\int_\Omega \psi z^2 d\bar t d\bar x \\
&= -(1-a^2)\int_0^\sigma \psi|z_{\bar t}(\bar t,\cdot)|^2_{H^{-1}(\Omega)}d\bar t - \frac{1-a^2}{2}\int_0^\sigma\int_\Omega \psi_{\bar t}\frac{\partial}{\partial \bar t}\Big[\Big(A^{-1/2}z\Big)^2\Big]d\bar t d\bar x \\
&\quad - 2a\int_0^\sigma\int_\Omega \Big[\psi\Big(A^{-1/2}z_{\bar x_1}\Big)\Big(A^{-1/2}z_{\bar t}\Big) + \Big(A^{-1/2}z_{\bar x_1}\Big)\psi_{\bar t}\Big(A^{-1/2}z\Big)\Big]d\bar t d\bar x \\
&\quad + \int_0^\sigma\int_\Omega \psi z^2 d\bar t d\bar x \\
&= -(1-a^2)\int_0^\sigma \psi|z_{\bar t}(\bar t,\cdot)|^2_{H^{-1}(\Omega)}d\bar t + \frac{1-a^2}{2}\int_0^\sigma\int_\Omega \psi_{\bar t\bar t}\Big(A^{-1/2}z\Big)^2 d\bar t d\bar x \\
&\quad - 2a\int_0^\sigma\int_\Omega \Big[\psi\Big(A^{-1/2}z_{\bar x_1}\Big)\Big(A^{-1/2}z_{\bar t}\Big) + \Big(A^{-1/2}z_{\bar x_1}\Big)\psi_{\bar t}\Big(A^{-1/2}z\Big)\Big]d\bar t d\bar x \\
&\quad + \int_0^\sigma\int_\Omega \psi z^2 d\bar t d\bar x.
\end{aligned}
\tag{3.22}
$$

However it is easy to check that

$$
\int_\Omega \|A^{-1/2}z_{\bar x_1}(\bar t,\bar x)\|^2 d\bar x \le C|z_{\bar x_1}(\bar t,\cdot)|^2_{H^{-1}(\Omega)} \le C|z(\bar t,\cdot)|^2_{L^2(\Omega)}. \tag{3.23}
$$

Thus, by (3.22) and noting (3.23), we end up with

$$
\begin{aligned}
&\int_0^\sigma \psi|z_{\bar t}(\bar t,\cdot)|^2_{H^{-1}(\Omega)}d\bar t \\
&= \frac{1}{1-a^2}\Big\{-\int_0^\sigma\int_\Omega (q_2 z)\psi\Big(A^{-1}z\Big)d\bar t d\bar x + \frac{1-a^2}{2}\int_0^\sigma\int_\Omega \psi_{\bar t\bar t}\Big(A^{-1/2}z\Big)^2 d\bar t d\bar x \\
&\quad - 2a\int_0^\sigma\int_\Omega \Big[\psi^{1/2}\Big(A^{-1/2}z_{\bar x_1}\Big)\Big(\psi^{1/2}A^{-1/2}z_{\bar t}\Big) + \Big(A^{-1/2}z_{\bar x_1}\Big)\psi_{\bar t}\Big(A^{-1/2}z\Big)\Big]d\bar t d\bar x \\
&\quad + \int_0^\sigma\int_\Omega \psi z^2 d\bar t d\bar x\Big\} \\
&\le C(1+\ell)\int_0^\sigma |z(\bar t,\cdot)|^2_{L^2(\Omega)}d\bar t + \frac{1}{2}\int_0^\sigma \psi|z_{\bar t}(\bar t,\cdot)|^2_{H^{-1}(\Omega)}d\bar t,
\end{aligned}
\tag{3.24}
$$

which gives

$$
\int_0^\sigma \psi|z_{\bar t}(\bar t,\cdot)|^2_{H^{-1}(\Omega)}d\bar t \le C(1+\ell)\int_0^\sigma |z(\bar t,\cdot)|^2_{L^2(\Omega)}d\bar t. \tag{3.25}
$$

Thus

$$
\int_{S_1}^{T_1} |z_{\bar t}(\bar t,\cdot)|^2_{H^{-1}(\Omega)}d\bar t \le C\int_0^\sigma \psi|z_{\bar t}(\bar t,\cdot)|^2_{H^{-1}(\Omega)}d\bar t \le C(1+\ell)\int_0^\sigma |z(\bar t,\cdot)|^2_{L^2(\Omega)}d\bar t. \tag{3.26}
$$

Step 3. Let us complete the proof of Theorem 3.1. By (3.6), (3.8) and (3.13)–(3.14), we get

$$
\begin{aligned}
&|w_0|^2_{L^2(\Omega)} + |w_1|^2_{H^{-1}(\Omega)} \\
&\le C(1+\ell)e^{T\ell}\int_{-2T}^{2T} |z(\bar t,\cdot)|^2_{L^2(\Omega)}d\bar t = C(1+\ell)e^{T\ell}\int_{-2T}^{2T} |v_{\bar t}(\bar t,\cdot)|^2_{L^2(\Omega)}d\bar t \\
&\le C(1+\ell)e^{T\ell}\int_{-2T}^{2T} \mathcal{E}^s(\bar t)d\bar t, \quad \forall\ s\in[-2T,2T].
\end{aligned}
\tag{3.27}
$$

Combining (3.27) and (3.19), we obtain

$$|w_0|^2_{L^2(\Omega)}+|w_1|^2_{H^{-1}(\Omega)}\le C(1+\ell)e^{C\ell}\Big(|z_{\bar t}(s,\cdot)|^2_{H^{-1}(\Omega)}+|z(s,\cdot)|^2_{L^2(\Omega)}\Big),\quad \forall\ s\in[-2T,2T]. \tag{3.28}$$

Now, integrating (3.28) with respect to s from S_1 to T_1 (where S_1 and T_1 are given in (3.21)), and using (3.26), we have

$$|w_0|^2_{L^2(\Omega)}+|w_1|^2_{H^{-1}(\Omega)}\le C(1+\ell^2)e^{C\ell}\int_0^\sigma |z(\bar t,\cdot)|^2_{L^2(\Omega)}d\bar t. \tag{3.29}$$

Thus, by (3.6) and using Assumption 2.1, we conclude the desired estimate (3.2) immediately.

Finally, let's give the proof of Theorem 2.1.

Proof of Theorem 2.1. We suppose that the initial state $(y_0,y_1)\in\mathcal{H}\stackrel{\triangle}{=}H_0^1(\Omega)\times L^2(\Omega)$ and the terminal state $(z_0,z_1)\in\mathcal{H}$ are given. Set

$$F(z)=\int_0^1 f'(sz)ds,\quad z\in L^2(\Omega). \tag{3.30}$$

Fixing any given $z(\cdot)\in C([0,T];L^2(\Omega))$, we consider the following linearized controlled system:

$$\begin{cases} y''(t)-\Delta y(t)=F(z(t))y+f(0)+Bu(t), & \text{in } Q_T,\\ y=0, & \text{on } \Sigma,\\ y(0)=y_0,\ y'(0)=y_1, & \text{in } \Omega.\end{cases} \tag{3.31}$$

First by means of Theorem 3.1 and HUM, we can show that System (3.31) is exactly controllable in $\mathcal{H}$ with controls in $L^2(\Omega_T)$ at time T.

Then we can define a map $Y:\ C([0,T];L^2(\Omega))\to C([0,T];L^2(\Omega))$ by

$$Y(z(\cdot))(\cdot)=y(z(\cdot))(\cdot), \tag{3.32}$$

where $y(z(\cdot))(\cdot)\in C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ is the mild solution of System (3.31) with control u given by HUM which drives the system from the initial state (y_0,y_1) to the terminal state (z_0,z_1). Now, in a way similar to [18], we can show that this map admits a fixed point by Schauder's fixed point theorem, which proves Theorem 2.1.

§4. Final Remarks and Some Other Results

Remark 4.1. By means of the inverse function theorem, one can prove a rapid controllability of local nature:

Theorem 4.1. *Let Assumption* 2.1 *hold. Assume that the nonlinear function* $f(\cdot):\mathbf{R}\to\mathbf{R}$ *is of class* $C^1(\mathbf{R})$ *with* $f(0)=0$ *and satisfies*

$$|f(x)-f(y)|\le C(1+|x|^{p-1}+|y|^{p-1})|x-y|,\quad \forall x,y\in\mathbf{R}, \tag{4.1}$$

$$1<p\le\frac{n}{n-2}\ \textit{if}\ n\ge 3;\ 1<p<\infty\ \textit{if}\ n=1,2. \tag{4.2}$$

Then the System (1.1) *is rapidly locally exactly controllable.*

Remark 4.2. In a way similar to [16], one can easily prove the following result by means of Leray-Schauder's fixed point theorem:

Theorem 4.2. *Let Assumption* 2.1 *hold. Assume that the nonlinear function* $f(\cdot):\mathbf{R}\to\mathbf{R}$ *is continuous and satisfies*

$$|f(x)|\le C(1+|x|^p),\qquad \forall x\in\mathbf{R} \tag{4.3}$$

with some constants $C>0$ *and* $p\in[0,1)$. *Then the System* (1.1) *is rapidly exactly controllable.*

Remark 4.3. All the results of this paper extend easily to semilinear wave equations with variable coefficients like

$$y'' - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial y}{\partial x_i}) = f(y) + \chi_{G(t)}(x)u(t,x), \tag{4.4}$$

$$a_{ij} \in W^{1,\infty}(\Omega), \quad \big(a_{ij}(x)\big) \geq a_0 I, \quad \forall x \in \Omega \tag{4.5}$$

for some constant $a_0 > 0$.

Remark 4.4. The same results (Theorem 2.1, Theorem 4.1 and Theorem 4.2) hold for System (1.1) with Neumann boundary conditions or mixed Dirichlet-Neumann boundary conditions.

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