Chin. Ann. of Math.
20B: 4(1999),385-392.

UNIQUE CONTINUATION ON A HYPERPLANE FOR WAVE EQUATION****

CHENG Jin* Masahiro YAMAMOTO** ZHOU Qi***

Abstract

One kind of unique continuation property for a wave equation is discussed. The authors show that, if one classical solution of the wave equation vanishes in an open set on a hyperplane, then it must vanish in a larger set on this hyperplane. The result can be viewed as a localized version of Robbiano's result[9]. The approach involves the localized Fourier-Gauss transformation and unique continuation on a line in the Laplace equation.

Keywords Unique continuation, Hyperplane, Wave operator, Localized Fourier-Gauss transform

1991 MR Subject Classification 35B60, 35L05

Chinese Library Classification O175.27

§1. Introduction

The unique continuation is that, if a solution of a partial differential equation vanishes in an open set, then it must vanish on the connected component which contains this open set. It is well understood for elliptic operators of the second order. In 1939, Carleman[2] showed a unique continuation theorem for systems of partial differential equations in the two dimensional case whose coefficients are not analytic. The powerful technique which he proposed is called a Carleman estimate and has played a central role in the development of unique continuation arguments since then. As a general reference book, see Hörmander[7]. Robbiano[9] discussed the unique continuation for second-order hyperbolic partial differential equations by changing the hyperbolic equations to the elliptic ones. The basic idea is an application of the Carleman estimates for an elliptic equation through a localized Fourier-Gauss transformation. See also Robbiano[10]. His result was improved by Hörmander[8] and Tataru[11].

Manuscript received May 17, 1999.

*Department of Mathematics, Fudan University, Shanghai 200433, China
Email: jcheng@fudan.edu.cn

Graduate School of Mathematical Sciences, The University of Tokyo, Komaba 3-8-1, Meguro, Tokyo 153, Japan. **Email myama@ms.u-tokyo.ac.jp

***Graduate School of Mathematical Sciences, The University of Tokyo, Komaba 3-8-1, Meguro, Tokyo 153, Japan. **Email** qzhou@ms.u-tokyo.ac.jp

****The first named author was supported partly by the National Natural Science Foundation of China (No.19501001). The second named author was partly supported by Sanwa Systems Development Co., Ltd (Tokyo, Japan). The third named author was supported by the Monbusho fellowship of Japan Government.

Recently, using the harmonic measure and the complex extension, Cheng, Hon and Yamamoto[3], Cheng and Yamamoto[4] have proved the conditional stability along a line for a harmonic function in an n-dimensional domain. That is, from values of a harmonic function on a smaller part of a straight line which is inside that domain, one can estimate the values of the harmonic function on a larger part of the line. Their result fills some gap between the analytic continuation for a holomorphic function and the unique continuation for a harmonic function.

The purpose of this paper is to apply the unique continuation on a line for the harmonic function to the hyperbolic equation. Combining with the localized Fourier-Gauss transformation and the conditional stability estimate obtained in [3] and [4], we show that the unique continuation on a hyperplane for the wave equation is also true. This means that, if one classical solution of the wave equation vanishes in an open set on a hyperplane, then it must vanish in a larger set on this hyperplane. In our unique continuation, we note that we do not know Cauchy data and our continuation is different from usual ones. We can also construct an example to show that, outside the hyperplane, we can not obtain any information about the solution. Our result can be viewed as a localized Robbiano's result[9] to a hyperplane. In addition, we can extend our result to weaker solutions of the hyperbolic equation (see Remark 2.1).

§2. Notations and Results

Let $n \geq 2$ and let Ω be a simply connected bounded domain in $\mathcal{R}^n$. We consider the D'Alembert operator

$$\mathcal{P} := \partial_t^2 - \sum_{j=1}^{n} \partial_{x_j}^2, \qquad (t,x) \in \mathcal{R}_t^1 \times \mathcal{R}_x^n. \tag{2.1}$$

Henceforth we set

$$B(x,r) := \{y \in \mathcal{R}^n \mid |y-x| < r\}, \qquad x \in \mathcal{R}^n.$$

Denote $x = (x_1, x')$ where $x' \in \mathcal{R}^{n-1}$ and let

$$\Omega' = \Omega \cap \{x' = 0\}, \quad B'(x,r) = B(x,r) \cap \{x' = 0\}.$$

We assume that $0 < r < R$ and $B'(0,R) \subset \Omega$.

Theorem 2.1. *Suppose that $u \in C^2((-T,T) \times B(0,R))$ satisfies the equation $\mathcal{P}u = 0$ in $(-T,T) \times B(0,R)$. Let $s_0 \in (0,T)$ be given. Assume*

$$u(t,x) = 0, \qquad (t,x) \in (-T,T) \times B'(0,r). \tag{2.2}$$

Then there exists a constant $K = K(r,R,s_0) > 0$, independent of u, such that

$$\begin{gathered} u(t,x) = 0 \quad \textit{for} \quad (t,x) \in (-T+s_0, T-s_0) \times B'(0,R); \\ |t| + K(R-|x_1|)^{-\frac{1}{2}}(|x_1| - r)^{\frac{1}{2}} < T - s_0. \end{gathered} \tag{2.3}$$

The constant K may tend to ∞ as $s_0 \to 0$.

By this theorem, we can easily obtain

Corollary 2.1. *Suppose that $u \in C^2((-T,T) \times \Omega)$ satisfies the equation $\mathcal{P}u = 0$ in $(-T,T) \times \Omega$. Let $\kappa \in (0,1)$ and $s_0 \in (0,T)$ be given such that $(1-\kappa)T - s_0 > 0$. Assume*

$$u(t,x) = 0, \qquad (t,x) \in (-T,T) \times B'(0,r). \tag{2.4}$$

If we take $T > 0$ so large that

$$\frac{K\epsilon^{-1/2}(R-r-\epsilon)^{1/2}+s_0}{1-\kappa} < T,$$

then

$$u(t, x_1, 0) = 0 \quad \text{for } |t| < \kappa T,\ |x_1| < R-\epsilon. \tag{2.5}$$

In character, our unique continuation is different from usual ones for the hyperbolic equation. In fact, our result is restricted to a hyperplane and independent of characteristic cones of the wave operator.

Remark 2.1. In fact, noticing that by the regularity of the harmonic functions, we can use the weak estimate in the proof of the main result. Therefore, in Theorem 2.1 and Corollary 2.1, we can relax the assumption $u \in C^2((-T,T)\times\Omega)$ to $u \in C((-T,T), H^2(\Omega))$. We omit the proof here.

§3. Proof of the Main Result

3.1. Some Lemmas.

Let us define a transformation by

$$v_{a,\lambda}(s,x) := \sqrt{\frac{\lambda}{2\pi}}\int_{-T}^{T} e^{-\frac{\lambda}{2}(is+a-t)^2}u(t,x)dt, \tag{3.1}$$

where $\lambda > 0$, $a \in \mathcal{R}$ and $i = \sqrt{-1}$. We call it a localized Fourier-Gauss transformation (LFGT for short).

Firstly we show some properties of LFGT which will be used for our proof of the main result (see [9]).

Lemma 3.1. *Let $u \in C^1((-T,T)\times B(0,R))$ and $s_0 \in (-T,T)$ be fixed. Then*

$$v_{a,\lambda}(0,x) \to u(a,x) \quad \text{as } \lambda \to +\infty, \qquad |a| < T, \tag{3.2}$$

$$\begin{aligned} &|v_{a,\lambda}(s,x)|, \quad |(\partial_{x_j} v_{a,\lambda})(s,x)|, \quad |(\partial_s v_{a,\lambda})(s,x)| \\ &\le C\lambda^{1/2}e^{\frac{\lambda}{2}s_0^2} \quad \text{for } 1 \le j \le n, \quad (s,x) \in (-s_0,s_0)\times B(0,R), \end{aligned} \tag{3.3}$$

where $C > 0$ depends on $\|u\|_{C^1([-T,T]\times\overline{B(0,R)})}$.

Proof. From the definition of LFGT, we have

$$\begin{aligned} v_{a,\lambda}(0,x) &= \sqrt{\frac{\lambda}{2\pi}}\int_{-T}^{T} e^{-\frac{\lambda}{2}(a-t)^2}u(t,x)dt \\ &= \sqrt{\frac{\lambda}{2\pi}}\int_{-T-a}^{T-a} e^{-\frac{\lambda}{2}t^2}u(t+a,x)dt \\ &= \sqrt{\frac{1}{2\pi}}\int_{\sqrt{\lambda}(-T-a)}^{\sqrt{\lambda}(T-a)} e^{-\frac{t^2}{2}}u\Big(\frac{t}{\sqrt{\lambda}}+a,x\Big)dt. \end{aligned}$$

By the integration equality

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt = \sqrt{2\pi},$$

we have

$$
\begin{aligned}
v_{a,\lambda}(0,x)-u(a,x) &= \sqrt{\frac{1}{2\pi}}\int_{\sqrt{\lambda}(-T-a)}^{\sqrt{\lambda}(T-a)} e^{-\frac{t^2}{2}}u\Big(\frac{t}{\sqrt{\lambda}}+a,x\Big)dt-\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}u(a,x)dt\\
&= \sqrt{\frac{1}{2\pi}}\Big(\int_{\sqrt{\lambda}(-T-a)}^{\sqrt{\lambda}(T-a)} e^{-\frac{t^2}{2}}\Big[u\Big(\frac{t}{\sqrt{\lambda}}+a,x\Big)-u(a,x)\Big]dt\Big)\\
&\quad -\sqrt{\frac{1}{2\pi}}\int_{\sqrt{\lambda}(T-a)}^{\infty} e^{-\frac{t^2}{2}}u(a,x)dt\\
&\quad -\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\sqrt{\lambda}(-T-a)} e^{-\frac{t^2}{2}}u(a,x)dt.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
|v_{a,\lambda}(0,x)-u(a,x)| &\le \frac{1}{\sqrt{2\pi}\sqrt{\lambda}}\|\partial_t u\|_{C([-T,T]\times\overline{B(0,R)})}\\
&\quad +\frac{1}{\sqrt{2}}|u(a,x)|e^{-\frac{\lambda}{4}(T-a)^2}+\frac{1}{\sqrt{2}}|u(a,x)|e^{-\frac{\lambda}{4}(T+a)^2}. \qquad (3.4)
\end{aligned}
$$

Therefore (3.2) is seen.

The estimates (3.3) for $|v_{a,\lambda}(s,x)|$ and $|(\partial_{x_j}v_{a,\lambda})(s,x)|$ are straightforward. For $|(\partial_s v_{a,\lambda})(s,x)|$, it is sufficient to note

$$
\begin{aligned}
\partial_s\int_{-T}^{T} e^{-\frac{\lambda}{2}(is+a-t)^2}u(t,x)dt &= \int_{-T}^{T}\partial_s(e^{-\frac{\lambda}{2}(is+a-t)^2})u(t,x)dt\\
&= \int_{-T}^{T} -i\partial_t(e^{-\frac{\lambda}{2}(is+a-t)^2})u(t,x)dt\\
&= iu(-T,x)e^{-\frac{\lambda}{2}(is+a+T)^2}-iu(T,x)e^{-\frac{\lambda}{2}(is+a-T)^2}\\
&\quad +i\int_{-T}^{T} e^{-\frac{\lambda}{2}(is+a-t)^2}(\partial_t u)(t,x)dt \qquad (3.5)
\end{aligned}
$$

by integration by parts. Thus the proof of the lemma is complete.

In connection with the operator $\frac{\partial^2}{\partial t^2}-\Delta$, we define an elliptic operator by

$$
\Delta_{s,x}:=\partial_s^2+\sum_{j=1}^{n}\partial_{x_j}^2, \qquad (s,x)\in\mathcal{R}_s^1\times\mathcal{R}_x^n,
$$

and we set

$$
\chi_{a,\lambda}:=\Delta_{s,x}v_{a,\lambda}.
$$

Then we have

Lemma 3.2. *Suppose that $u\in C^2([-T,T]\times\overline{B(0,R)})$. Then for any $\lambda>0$, there exists a positive number C such that*

$$
|\chi_{a,\lambda}(s,x)|\le C\lambda^{\frac{3}{2}}e^{-\frac{\lambda}{2}[(T-|a|)^2-s_0^2)]}, \quad (s,x)\in(-s_0,s_0)\times B(0,R),
$$

where $C>0$ depends on s_0, T, a and $\|u\|_{C^2([-T,T]\times\overline{B(0,R)})}$.

Proof. We have

$$
\begin{aligned}
\partial_s v_{a,\lambda}(s,x) &= i\sqrt{\frac{\lambda}{2\pi}}\int_{-T}^{T} e^{-\frac{\lambda}{2}(is+a-t)^2}\partial_t u(t,x)dt - i\sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(is+a-T)^2}u(T,x)\\
&\quad + i\sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(is+a+T)^2}u(-T,x)
\end{aligned}
$$

by (3.5). Therefore we can similarly obtain

$$\begin{aligned}
\partial_s^2 v_{a,\lambda}(s,x) = &-\sqrt{\frac{\lambda}{2\pi}}\int_{-T}^{T} e^{-\frac{\lambda}{2}(is+a-t)^2}\partial_t^2 u(t,x)dt + \sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(is+a-T)^2}\partial_t u(T,x)\\
&-\sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(is+a+T)^2}\partial_t u(-T,x)\\
&-\sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(is+a-T)^2}\lambda(is+a-T)u(T,x)\\
&+\sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(is+a+T)^2}\lambda(is+a+T)u(-T,x).
\end{aligned}$$

Therefore since $\mathcal{P}u = 0$, we have

$$\begin{aligned}
\chi_{a,\lambda} = \Delta_{s,x} v_{a,\lambda} = &\sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(is+a-T)^2}\partial_t u(T,x) - \sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(is+a+T)^2}\partial_t u(-T,x)\\
&-\sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(is+a-T)^2}\lambda(is+a-T)u(T,x)\\
&+\sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(is+a+T)^2}\lambda(is+a+T)u(-T,x).
\end{aligned}$$

Thus the proof is complete.

3.2. Conditional Stability in Unique Continuation on a Line for the Laplace Equation

Our proof is heavily dependent on the conditional stability (see [3, 4]) in line unique continuation for the Laplace equation. Such stability is stated as follows:

Theorem 3.1.[3,4] *Let $\varphi \in C^2((-s_0,s_0)\times B(0,R))$ satisfy $\Delta_{s,x}\varphi = 0$ in $(-s_0,s_0)\times B(0,R)$ and*

$$\|\varphi\|_{C([-s_0,s_0]\times\overline{B(0,R)})} \le M.$$

Then, for $\rho \in (r,R)$, there exist positive constants $C_1 = C_1(r,R,s_0)$ and $\alpha = \alpha(\rho,r,R,s_0) \in (0,1)$ such that

$$\|\varphi(0,\cdot,0)\|_{L^\infty(-\rho,\rho)} \le C_1 M^{1-\alpha}\|\varphi(0,\cdot,0)\|^{\alpha}_{L^\infty(-r,r)}.$$

Moreover

$$\lim_{\rho\uparrow R}\alpha(\rho,r,R,s_0) = 0, \qquad \lim_{\rho\downarrow r}\alpha(\rho,r,R,s_0) = 1, \tag{3.6}$$

and for $\rho \in (r,R)$,

$$\alpha(\rho,r,R,s_0) \ge C_2(R-\rho), \qquad 1-\alpha(\rho,r,R,s_0) \le C_3(\rho-r)^{1/2}, \tag{3.7}$$

where $C_2 = C_2(r,R,s_0) > 0$ and $C_3 = C_3(r,R,s_0) > 0$ are constants.

In [3] and [4], the coefficient $C_1M^{1-\alpha}$ at the right side is not specified, but we can derive such dependency on M easily from the proof in [3, 4]. Moreover from the proof in [3, 4] (in particular, see Lemma 4.2 in [4]), we can verify (3.6) and the first inequality in (3.7). For the second inequality in (3.7), we refer to Lemma 2.3 in [1] or [5].

Proof of Theorem 2.1. First we have

$$\Delta_{s,x} v_{a,\lambda} \equiv \partial_s^2 v_{a,\lambda} + \sum_{j=1}^{n}\partial_{x_j}^2 v_{a,\lambda} = \chi_{a,\lambda}, \quad (s,x)\in(-s_0,s_0)\times B(0,R), \tag{3.8}$$

and we can directly verify that

$$v_{a,\lambda}(s,x) = 0$$

for $(s,x) \in (-s_0, s_0) \times B'(0,r)$.

From (3.8) we can represent $v_{a,\lambda}$ as

$$v_{a,\lambda} = \varphi_{a,\lambda} + N\chi_{a,\lambda}. \tag{3.9}$$

Here φ satisfies

$$\Delta_{s,x}\varphi_{a,\lambda} = 0, \qquad (s,x) \in (-s_0, s_0) \times B(0,R),$$

and $N\chi_{a,\lambda}$ is the Newtonian potential of $\chi_{a,\lambda}$ in the domain $(-s_0, s_0) \times B(0,R)$:

$$(N\chi_{a,\lambda})(\xi) := \int_{(-s_0,s_0)\times B(0,R)} \Gamma(\xi - \eta)\chi_{a,\lambda}(\eta)d\eta, \quad \xi = (s,x) \in \mathcal{R}^{n+1},$$

where Γ is the fundamental solution of the Laplaces equation given by

$$\Gamma(\xi - \eta) = \frac{1}{(n+1)(1-n)\omega_{n+1}}|\xi - \eta|^{1-n}, \quad n+1 \geq 3,$$
$$\Gamma(\xi - \eta) = \frac{1}{2\pi}\log|\xi - \eta|, \qquad n+1 = 2,$$

and ω_{n+1} is the volume of the unit ball in $\mathcal{R}^{n+1}$ (see e.g. [6]).

We have

$$|N\chi_{a,\lambda}(s,x)| \leq C||\chi_{a,\lambda}||_{L^\infty((-s_0,s_0)\times B(0,R))} \text{ for } (s,x) \in (-s_0, s_0) \times B(0,R). \tag{3.10}$$

Therefore, since $v_{a,\lambda}(0,x) = 0$ for $x \in B'(0,r)$, we obtain

$$|\varphi_{a,\lambda}(0,x)| = |N\chi_{a,\lambda}(0,x)|.$$

By Lemma 3.2 and (3.10), we see

$$|\varphi_{a,\lambda}(0,x)| \leq C\lambda^{\frac{3}{2}}e^{-\frac{\lambda}{2}[(T-|a|)^2 - s_0^2]}, \quad x \in B'(0,r). \tag{3.11}$$

On the other hand, for $(s,x) \in (-s_0, s_0) \times B(0,R)$, by (3.10) and Lemmas 3.1, 3.2, we have

$$\begin{aligned}|\varphi_{a,\lambda}(s,x)| &\leq |v_{a,\lambda}(s,x)| + |N\chi_{a,\lambda}(s,x)| \\ &\leq C\lambda^{\frac{1}{2}}e^{\frac{\lambda}{2}s_0^2} + C\lambda^{\frac{3}{2}}e^{-\frac{\lambda}{2}[(T-|a|)^2 - s_0^2]}.\end{aligned} \tag{3.12}$$

In terms of (3.11) and (3.12), application of Theorem 3.1 in Subsection 3.2 to $\varphi_{a,\lambda}$ yields

$$|\varphi_{a,\lambda}(0,x)| \leq C_4\lambda^{\frac{3}{2}}e^{-\frac{\lambda}{2}[(T-|a|)^2\alpha - s_0^2]}, \qquad x \in B'(0,\rho),$$

where $C_4 > 0$ is dependent on r, T, a, s_0 but independent of $\lambda > 0$. We recall that $\alpha = \alpha(\rho, r, R, s_0)$ is given in Theorem 3.1. Henceforth we simply write $\alpha = \alpha(\rho)$, omitting the dependency on r, R and s_0.

Consequently by (3.9), (3.10) and Lemma 3.2, we obtain

$$\begin{aligned}|v_{a,\lambda}(0,x)| &\leq |\varphi_{a,\lambda}(0,x)| + |N\chi_{a,\lambda}(0,x)| \\ &\leq C_4\lambda^{\frac{3}{2}}[e^{-\frac{\lambda}{2}[(T-|a|)^2\alpha - s_0^2]} + e^{-\frac{\lambda}{2}[(T-|a|)^2 - s_0^2]}] \\ &\leq C_4\lambda^{\frac{3}{2}}e^{-\frac{\lambda}{2}[(T-|a|)^2\alpha(|x_1|) - s_0^2]}, \quad x \in B'(0,\rho).\end{aligned} \tag{3.13}$$

Let us set $\beta(\rho)=\frac{s_0}{\sqrt{\alpha(\rho)}}$ for $0<\rho<R$. Then for $r\le\rho<R$, we see $s_0\le\beta(\rho)<\infty$ and

$$\begin{aligned}0\le\beta(\rho)-s_0&=s_0\left|\frac{1}{\sqrt{\alpha(\rho)}}-\frac{1}{\sqrt{\alpha(r)}}\right|\\&\le s_0|\alpha(\rho)-\alpha(r)|\alpha(\rho)^{-1/2}\alpha(r)^{-1/2}(\sqrt{\alpha(\rho)}+\sqrt{\alpha(r)})^{-1}\\&\le s_0|\alpha(\rho)-1|\alpha(\rho)^{-1/2}\\&\le\frac{s_0C_3(\rho-r)^{1/2}}{\sqrt{C_2}(R-\rho)^{1/2}}\end{aligned}$$

by (3.6) and (3.7). Hence

$$0\le\beta(|x_1|)-s_0\le K(R-|x_1|)^{-1/2}(|x_1|-r)^{1/2},\quad r\le|x_1|<R. \tag{3.14}$$

Here we set

$$K=K(r,R,s_0)\equiv\frac{s_0C_3(r,R,s_0)}{\sqrt{C_2(r,R,s_0)}}.$$

Let $K(R-|x_1|)^{-1/2}(|x_1|-r)^{1/2}+|a|<T-s_0$. Then

$$(K(R-|x_1|)^{-1/2}(|x_1|-r)^{1/2}+s_0)^2<(T-|a|)^2.$$

Hence (3.14) yields

$$\begin{aligned}\frac{\alpha(|x_1|)}{s_0^2}=\frac{1}{\beta^2(|x_1|)}&\ge\frac{1}{[K(R-|x_1|)^{-\frac12}(|x_1|-r)^{1/2}+s_0]^2}\\&>\frac{1}{(T-|a|)^2},\end{aligned}$$

namely,

$$(T-|a|)^2\alpha(|x_1|)-s_0^2>0.$$

Therefore

$$\lim_{\lambda\to\infty}|v_{a,\lambda}(0,x)|=0$$

in view of (3.13). Lemma 3.1 yields $u(a,x)=0$. Thus this completes the proof of our main result.

§4. Conclusions and Remarks

In this paper, we prove unique continuation which is restricted to a hyperplane $\{(t,x_1,...,x_n);x_2=...=x_n=0\}$ for the d'Alembertian. To authors' knowledge, such unique continuation is not known.

Remark 4.1. Our result shows that the unique continuation is true in the hyperplane $\{x'=0\}$. The next example shows that our unique continuation is impossible outside the hyperplane.

Example. We consider a function $u(t,x)=t\sum_{j=2}^{n}x_j$ which satisfies $\mathcal{P}u=0$ in $(-T,T)\times\mathcal{R}^n$. Then $u(t,x)=0$ for $x'=0$, and $u(t,x)\ne0$ for $\sum_{j=2}^{n}x_j\ne0$ and $t\ne0$. This means that, from the information on the hyperplane $\{x'=0\}$, we can not obtain the information about the value of u outside the hyperplane.

Remark 4.2. In this paper, we only treat the d'Alembertian. Actually the result is also true for a hyperbolic operator whose coefficients are analytic in x and independent of t.

Remark 4.3. From the viewpoint of Holmgren's theorem, we conjecture that, if $|x_1| + |t| < T$, then $u(t, x_1, 0) = 0$, like in the standard continuation for the d'Alembertian. However we do not know such sharp uniqueness in our continuation on a hyperplane.

Acknowledgement. This paper was finished during the second named author was visiting the Sino-France Institute of Applied Mathematics, Fudan University at Shanghai, China. He would like to thank Prof. Li Tatsien for his kind hospitality.

References

[1] Alessandrini, G., Stable determination of a crack from boundary measurements, *Proceedings of the Royal Soc. Edinburgh.,* **123A**(1993), 497–516.

[2] Carleman, T., Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes, *Ark. Mat. Astr. Fys.,* **26B**(1939), 1–9.

[3] Cheng, J., Hon, Y. C. & Yamamoto, M., Stability in line unique continuation of harmonic functions: general dimensions, *Journal of Inverse and* Ill-*Posed Problems,* **6**(1998), 319–326.

[4] Cheng, J. & Yamamoto, M., Unique continuation on a line for harmonic functions, *Inverse Problems,* **14**(1998), 869–882.

[5] Friedman, A. & Vogelius, M., Determining cracks by boundary measurements, *Indiana Univ. Math. Journal,* **38**(1989), 527–556.

[6] Gilbarg, D. & Trudinger, N. S., Elliptic partial differential equations of second order, Springer-Verlag, Berlin, 1977.

[7] Hörmander, L., The analysis of linear partial differential operators, **I-IV**, Springer-Verlag, Berlin, 1983.

[8] Hörmander, L., A uniqueness theorem for second order hyperbolic differential equations, *Communications in Partial Differential Equations,* **17**(1992), 699–714.

[9] Robbiano, L., Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques, *Communications in Partial Differential Equations,* **16**(1991), 789–800.

[10] Robbiano, L., Fonction de coût et contrôle des solutions des équations hyperboliques, *Asymptotic Analysis,* **10**(1995), 95–115.

[11] Tataru, D., Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem, *Communications in Partial Differential Equations,* **20**(1995), 855–884.