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THE MODULI SPACE OF COMPLEX LAGRANGIAN SUBMANIFOLDS IN THE HYPER-KAEHLER MANIFOLD

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Abstract

The deformation of a compact complex Lagrangian submanifold in a hyper-Kaehler manifold and the moduli space are studied. It is proved that the moduli space M_{cl} is a special Kaehler manifold, where special means that there is a real flat torsionfree symplectic connection ∇ satisfying $d_\nabla \tilde{I} = 0$ ($\tilde{I}$ is a complex structure of M_{cl}). Thus, following [4], one knows that T^*M_{cl} is a hyper-Kaehler manifold and then that M_{cl} is a complex Lagrangian submanifold in T^*M_{cl}.

Keywords Moduli space, Complex Lagrangian submanifold, Hyper-Kaehler manifold, Special Lagrangian submanifold, Special Kaehler manifold, Deformation
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§1. Introduction

Special Lagrangian submanifolds of a Calabi-Yau manifold are one of the recent attractive subjects in mathematics (see [5-8]). In 1996, R. C. Mclean[7] obtained the deformation theorem of special Lagrangian submanifold, which shows that, given one compact special Lagrangian submanifold L, there is a local moduli space M_{sl} which is a manifold and whose tangent space at L is canonically identified with the space of harmonic 1-forms on L. The L^2 inner product on harmonic forms then gives the moduli space a natural Riemannian metric. Strominger, Yau and Zaslow[10] have studied the moduli space of special Lagrangian tori in the context of mirror symmetry. In 1997, N. J. Hitchin[6] showed that there is a natural embedding of the local moduli space M_{sl} as a Lagrangian submanifold in the product $H^1(L,R) \times H^{n-1}(L,R)$ of two dual vector spaces and that Mclean's metric is the natural induced metric. Moreover, he studied the structure of the moduli space of special Lagrangian submanifold together with flat line bundles and showed that there is a natural complex structure and Kaehler metric on this space. From [8], we can also prove that T^*M_{sl} is a Kaehler manifold. As pointed by Hitchin in [6], examples of special Lagrangian submanifolds are difficult to find, and so far consist of three types. First of them is of complex Lagrangian submanifolds of hyper-Kaehler manifolds.

In this paper, we study the deformation of a compact complex Lagrangian submanifold L in a hyper-Kaehler manifold and the moduli space M_{cl}. We know that complex Lagrangian

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submanifolds are special Lagrangian from [6]. We first show that the moduli space $M_{\rm cl}$ of deformation of a complex Lagrangian submanifold L is identified to the moduli space $M_{\rm sl}$ of deformation of the special Lagrangian submanifold L. But there is a natural complex structure in the space of harmonic 1-forms on L. So, the moduli space $M_{\rm cl}$ has the natural complex structure $\tilde{I}$ and the Riemannian metric $\tilde{g}$. We show that $M_{\rm cl}$ is a *special* Kaehler manifold, where special means that there is a real flat torsionfree symplectic connection ∇ satisfying $d_\nabla \tilde{I} = 0$. Thus, following [4], we know that $T^*M_{\rm cl}$ is a hyper-Kaehler manifold and then that $M_{\rm cl}$ is a complex Lagrangian submanifold in $T^*M_{\rm cl}$, which reminds us that a moduli space of complex submanifolds (when unobstructed) is a complex manifold.

§2. Preliminaries

2.1. Hyper-Kaehler Manifolds and Calabi-Yau Manifolds.

A Calabi-Yau manifold is a Kaehler manifold X of complex dimension n with a covariant constant holomorphic n-form Ω. Equivalently it is a Riemannian manifold with holonomy group contained in $SU(n)$.

A hyper-Kaehler manifold is a Riemannian manifold M endowed with three complex structures I, J and K, such that the following hold:

(i) M is Kaehler with respect to these structures and

(ii) I, J and K, considered as endomorphisms of a real tangent bundle, satisfy the relation $IJ = -JI = K$.

This means that the hyper-Kaehler manifold has the natural action of quaternions H in its real tangent bundle. Therefore its complex dimension is even. In this paper, let the complex dimension of hyper-Kaehler manifold be $2k$. On a hyper-Kaehler manifold M if the complex structure I is fixed, a Kaehler form ω_I and a closed holomorphic (2,0)-form $\sigma_I = \omega_J - i\omega_K$ are determined. Thus $\Omega_I = \sigma_I^k = (\omega_J - i\omega_K)^k$ is a never vanishing covariant constant holomorphic volume form. So, a hyper-Kaehler manifold is a Calabi-Yau manifold with respect to the complex structure I. Similarly, a hyper-Kaehler manifold is a Calabi-Yau manifold with respect to the complex structure J or K.

2.2. Complex Lagrangian Submanifolds and Special Lagrangian Submanifolds.

A submanifold L of a Calabi-Yau manifold (X, ω, Ω) is special Lagrangian if Kaehler form ω and imaginary part of Ω restrict to zero on L and $\dim X = 2 \dim L$. It is easily shown that a special Lagrangian submanifold has least volume in its homology class[5].

A submanifold L of a hyper-Kaehler manifold M is *complex Lagrangian* if L is a complex submanifold for the complex structure I and $\sigma_I = \omega_J - i\omega_K$ vanishes on L. Thus $\dim M = 2\dim L$ and ω_J, ω_K vanish on L. The following observation is due to Hitchin[6].

Proposition 2.1.[6] *A complex Lagrangian submanifold L of a hyper-Kaehler manifold M is a special Lagrangian submanifold.*

Remark 2.1. When $k = 1$, a special Lagrangian submanifold is complex Lagrangian. When $k \geq 2$, a special Lagrangian submanifold need not be complex Lagrangian.

§3. Complex Lagrangian Submanifolds

Now, we consider the complex Lagrangian submanifold L of the hyper-Kaehler manifold

M for the complex structure I.

Lemma 3.1. *The normal bundle $N(L)$ is isomorphic to the tangent bundle $T(L)$.*

Proof. Since $\omega_J(\cdot,\cdot) = g(\cdot, J\cdot)$, $\omega_J|_L = 0$ is equivalent to that J maps tangent vectors of L to normal vectors of L. It follows that J induces an isomorphism of $T(L)$ with $N(L)$. Certainly, K also induces an isomorphism of $T(L)$ with $N(L)$.

Remark 3.1. Using the induced metric isomorphism $\flat : T(L) \to T^*(L)$, we further obtain an identification of normal vector fields to a complex Lagrangian submanifold with differential $1-$forms on this submanifold.

A cross-section V in $N(L)$ will be called holomorphic if, for any vector field U of L,

$$\nabla_{IU} V = I \nabla_U V.$$

Lemma 3.2. *Let W be a vector field of L. Then the following properties are equivalent*

(i) *normal vector field JW is holomorphic;*

(ii) *normal vector field KW is holomorphic;*

(iii) *W is anti-holomorphic, i.e. for any vector field U of L, $\nabla_{IU} W = -I\nabla_U W$.*

Proof. Using the fact that J induces the isomorphism of $T(L)$ with $N(L)$, we have

$$\nabla_{IU} JW = (\bar{\nabla}_{IU} JW)^N = (J\bar{\nabla}_{IU} W)^N = J(\bar{\nabla}_{IU} W)^T = J\nabla_{IU} W,$$
$$I\nabla_U JW = I(\bar{\nabla}_U JW)^N = I(J\bar{\nabla}_U W)^N = IJ\nabla_U W = -JI\nabla_U W.$$

So, JW is holomorphic if and only if W is anti-holomorphic. For the same reason, KW is holomorphic if and only if W is anti-holomorphic.

Lemma 3.3. *Let W be a vector field of L and θ is a dual 1-form by the induced metric isomorphism $\flat : T(L) \to T^*(L)$. Then any two of the following properties imply the third.*

(i) *W is anti-holomorphic;*

(ii) *$d\theta = 0$;*

(iii) *$d(I\theta) = 0$, where $I\theta$ is defined by $(I\theta)(U) = -\theta(IU)$ for any vector field U of L.*

Proof. Choose $e_1, \cdots, e_k, f_1, \cdots, f_k$ a frame in $T_m(L)$ with $f_i = Ie_i$. Extend these to local vector fields $\{E_i, F_j\}$, such that they form a frame at each point, $F_i = IE_i$, and $\nabla_x E_i(m) = \nabla_x F_i(m) = 0$ for all i and all $x \in T_m(M)$. Let $\{\omega^i, \omega^{k+i}\}$ be the dual frame of $\{E_i, F_j\}$ and $W = \theta^i E_i + \theta^{k+i} F_i$. Then $\theta = \theta_i \omega^i + \theta_{k+i}\omega^{k+i}$, where $\theta_i = \theta^i, \theta_{k+i} = \theta^{k+i}$. By a simple calculation at $m \in L$, we have

(i) W is anti-holomorphic if and only if $f_i(\theta^j) = e_i(\theta^{k+j}), f_i(\theta^{k+j}) = -e_i(\theta^j)$;

(ii) $d\theta = 0$ if and only if $e_i(\theta_j) = e_j(\theta_i), e_i(\theta_{k+j}) = f_j(\theta_i), f_i(\theta_{k+j}) = f_j(\theta_{k+i})$;

(ii) $d(I\theta) = 0$ if and only if $e_i(\theta_{k+j}) = e_j(\theta_{k+i}), e_i(\theta_j) = -f_j(\theta_{k+i}), f_i(\theta_j) = f_j(\theta_i)$.

Now, Lemma 3.3 can be obtained easily.

Lemma 3.4. *1-form θ of L is harmonic if and only if $d\theta = d(I\theta) = 0$.*

Proof. Since L is a complex submanifold of hyper-Kaehler manifold M, L is a Kaehler manifold. So, when θ is harmonic, $I\theta$ is harmonic too. Thus $d\theta = d(I\theta) = 0$. Conversely, using the calculation of Lemma 3.3, when $d(I\theta) = 0$, we have $e_i(\theta_j) = -f_j(\theta_{k+i})$. So, we have $\Sigma e_i(\theta_i) + f_i(\theta_{k+i}) = 0$, which says that $\delta\theta = 0$.

§4. Deformations

Theorem 4.1. *A normal vector field JW to a compact complex Lagrangian submani-*

fold L is the deformation vector field to a normal deformation through complex Lagrangian submanifold if and only if the corresponding $1-$form $\theta = W^\flat$ is harmonic. There are no obstructions to extending a first order deformation to an actual deformation and the tangent space to such deformations can be identified through the cohomology class of the harmonic forms with $H^1(L, R)$.

Remark 4.1. When JW is a deformation vector field to a normal deformation through complex Lagrangian submanifolds, then is KW also.

Proof. We define a non-linear map

$$F : U \subset \Gamma(N(L)) \to \Omega^2(L) \oplus \Omega^2(L)$$

as followes: For a small normal vector field JW, then

$$F(JW) = ((\exp_{JW})^* \omega_J, (\exp_{JW})^* \omega_K).$$

Here U is an open neighborhood of the zero in $\Gamma(N(L))$ for which $JW \in U$ implies that the exponential map $\exp_{JW}$ is a diffeomorphism of L on to its image L_{JW}. Under the identification of small normal vector fields with nearby submanifolds, it is easy to see that $F^{-1}(0,0)$ is simply the set of normal vector fields JW in U for which ω_J and ω_K restrict to L_{JW} to be zero, i.e. L_{JW} is complex Lagrangian.

We now consider the linearization of F,

$$F'(0) : \Gamma(N(L)) \to \Omega^2(L) \oplus \Omega^2(L),$$

where $F'(0)(JW) = \frac{\partial}{\partial t}\Big|_{t=0} F(tJW)$. Therefore

$$F'(0)(JW) = (d((JW)\rfloor\omega_J)|_L, d((JW)\rfloor\omega_K)|_L).$$

But

$$(JW)\rfloor\omega_J = \omega_J(JW, \cdot) = g(JW, J\cdot) = g(W, \cdot) = \theta,$$
$$(JW)\rfloor\omega_K = \omega_K(JW, \cdot) = g(JW, K\cdot) = g(W, -I\cdot) = I\theta.$$

So, we see that $F'(0)(JW) = (d\theta, d(I\theta))$. Hence, $F'(0)$ as a map $F'(0) : \Omega^1(L) \to \Omega^2(L) \oplus \Omega^2(L)$ is just $d \oplus dI$. By Lemma 3.4, the first order complex Lagrangian deformations (kernel of $F'(0)$) correspond to harmonic $1-$forms. The proof that the deformation theory of complex Lagrangian submanifolds is unobstructed is similar to Mclean's proof of his theroem in [7].

From Theroem 4.1, we know that the moduli space $M_{\rm cl}$ of complex Lagrangian submanifolds near L is a smooth manifold of dimension dim $H^1(L, R)$.

Corollary 4.1. *$M_{\rm cl} = M_{\rm sl}$, where $M_{\rm sl}$ is the moduli space of special Lagrangian submanifolds near L.*

Proof. Corollary 4.1 is from the fact that $M_{\rm cl} \subset M_{\rm sl}$ and $\dim M_{\rm cl} = \dim M_{\rm sl} = \dim H^1(L, R)$.

Remark 4.2. Corollary 4.1 says that special Lagrangian submanifolds L_t obtained from a complex Lagrangian submanifold L by a local deformation of special Lagrangian are again complex Lagrangian.

§5. The Moduli Space

Theorem 5.1. *There are a natural complex structure $\tilde{I}$ and a Riemannian metric $\tilde{g}$*

on the moduli space $M_{\rm cl}$ such that $M_{\rm cl}$ is a special Kaehler manifold, where the special Kaehler manifold means that it is Kaehler and that there is a real flat torsionfree symplectic connection ∇ on it satisfying $d_\nabla \tilde{I} = 0$.

Proof. We follow the method of Hitchin[6]. Suppose now we take local coordinates $t_1, \cdots, t_{2m}$ on the moduli space $M_{\rm cl}$ of deformations of $L = L_0$. Here of course, from Theorem 4.1, we know that $2m = b_1(L) = \dim H^1(L, R)$. Let $F : L \times M_{\rm cl} \to M$, where $F(L,t) = L_t$ is a complex Lagrangian submanifold for any $t \in M_{\rm cl}$. For each tangent vector $\frac{\partial}{\partial t_\alpha}$, $F_*(\frac{\partial}{\partial t_\alpha})$ is a deformation normal vector field on L_t. We define as Theorem 4.1 a corresponding closed 1-form θ_α on L_t for each $t \in M_{\rm cl}$:

$$\Big(F_*\Big(\frac{\partial}{\partial t_\alpha}\Big)\Big)\Big\rfloor \omega_J = \theta_\alpha.$$

Now, we can induce a natural complex structure $\tilde{I}$ on $M_{\rm cl}$. For the tangent vector $\tilde{I}\frac{\partial}{\partial t_\alpha}$, let a corresponding closed 1-form be $-I\theta_\alpha$, i.e.,

$$\Big(F_*\Big(\tilde{I}\frac{\partial}{\partial t_\alpha}\Big)\Big)\Big\rfloor \omega_J := -I\theta_\alpha.$$

By the directly calculation, we have

$$F_*\Big(\tilde{I}\frac{\partial}{\partial t_\alpha}\Big) = IF_*\Big(\frac{\partial}{\partial t_\alpha}\Big). \tag{5.1}$$

Actually, if $F_*(\frac{\partial}{\partial t_\alpha}) = JW$ is a deformation vector field on L_t, then we define $F_*(\tilde{I}\frac{\partial}{\partial t_\alpha}) = KW$, which is also a deformation vector field on L_t by Remark 4.1. So, $\tilde{I}^2 = -id$ and $\tilde{I}$ is an almost complex structure. A natural Riemannian metric $\tilde{g}$ in $M_{\rm cl}$ is defined as follows: Given two tangnet vectors $\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta} \in T_t(M_{\rm cl})$,

$$\tilde{g}\Big(\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta}\Big) := (\theta_\alpha, \theta_\beta) = \int_L \langle \theta_\alpha, \theta_\beta\rangle \mathrm{dvol}(t).$$

From the definitions of $\tilde{I}$ and $\tilde{g}$, we have

$$\tilde{g}\Big(\tilde{I}\frac{\partial}{\partial t_\alpha}, \tilde{I}\frac{\partial}{\partial t_\beta}\Big) = \int_L \langle -I\theta_\alpha, -I\theta_\beta\rangle \mathrm{dvol}(t) = \int_L \langle \theta_\alpha, \theta_\beta\rangle \mathrm{dvol}(t) = \tilde{g}\Big(\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta}\Big),$$

which shows that $M_{\rm cl}$ is an almost Hermitian manifold. Now, we prove that $M_{\rm cl}$ is a special Kaehler manifold.

(i) The complex structure $\tilde{I}$ is integrable.

Denote by $\tilde{N}$ and N the Nijenhuis tensors of $M_{\rm cl}$ and M respectively. If $\tilde{N}(\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta}) \neq 0$ at $t_0 \in M_{\rm cl}$, then there is a point $p \in L$ such that

$$F_{*(p,t_0)}\Big(\tilde{N}\Big(\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta}\Big)\Big) \neq 0. \tag{5.2}$$

Now, define the map

$$f(\cdot) := F(p, \cdot) : M_{\rm cl} \to M.$$

By (5.1), we have

$$f_*\Big(\tilde{I}\frac{\partial}{\partial t_\alpha}\Big) = If_*\Big(\frac{\partial}{\partial t_\alpha}\Big),$$

i.e., f is a pseudo-holomorphic map. Therefore we have (see [1,p.70])

$$N\Big(f_*\Big(\frac{\partial}{\partial t_\alpha}\Big), f_*\Big(\frac{\partial}{\partial t_\beta}\Big)\Big) = f_*\Big(\tilde{N}\Big(\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta}\Big)\Big).$$

But since M is hyper-Kaehler, $N=0$. So, $f_*\Big(\tilde N\Big(\frac{\partial}{\partial t_\alpha},\frac{\partial}{\partial t_\beta}\Big)\Big)=0$, i.e.,

$$F_{*(p,t)}\Big(\tilde N\Big(\frac{\partial}{\partial t_\alpha},\frac{\partial}{\partial t_\beta}\Big)\Big)=0.$$

It is in contradiction with (5.2).

(ii) The symplectic structure $\tilde\omega(\cdot,\cdot)=\tilde g(\cdot,\tilde I\cdot)$ is closed.

Let $A_1,\cdots,A_{2m}$ be a basis for $H_1(L,Z)$ (modulo torsion). Then we can evalute the closed form $F_t^*\theta_\beta$ on the homology class A_α to obtain a period matrix $\lambda_{\alpha\beta}$ which is a function on the moduli space :

$$\lambda_{\alpha\beta}=\int_{A_\alpha}(F_t^*\theta_\beta), \tag{5.3}$$

where $F_t(\cdot)=F(\cdot,t):L\to L_t\subset M$. Since by Theorem 4.1 the harmonic forms θ_β are linearly independent, it follows that $\lambda_{\alpha\beta}$ is invertible. We can now be explicit about the identification of the tangent space to $M_{\rm cl}$ with the cohomology group $H^1(L,R)$. Let $a_1,\dots,a_{2m}\in H^1(L,R)$ be the basis dual to $A_1,\dots,A_{2m}$. It follows that

$$\frac{\partial}{\partial t_\beta}\mapsto\Big[F_*\Big(\frac{\partial}{\partial t_\beta}\Big)\Big\rfloor\omega_J\Big]=[\theta_\beta]=\sum\lambda_{\alpha\beta}a_\alpha$$

identifies $T_tM_{\rm cl}$ with $H^1(L,R)$. We need the following

Lemma 5.1.[6] *The 1-forms $\xi_\alpha=\sum\lambda_{\alpha\beta}dt_\beta$ on $M_{\rm cl}$ are closed.*

From Lemma 5.1, we have

$$\frac{\partial\lambda_{\alpha\beta}}{\partial t_\gamma}=\frac{\partial\lambda_{\alpha\gamma}}{\partial t_\beta}. \tag{5.4}$$

Because L_t is a complex submanifold in the hyper-Kaehler manifold M with complex structure I, L_t is Kaehler. It can be easily obtained that

$$*\theta=C^{-1}I\theta\wedge\omega_I^{k-1}(t) \tag{5.5}$$

for any 1-form θ on L_t, where $\omega_I(t)$ is the Kaehler form of L_t and C is constant and independent of t. Now, we define the matrix $l_{\alpha\beta}$ as

$$l_{\alpha\beta}:=C^{-1}\int_L a_\alpha\wedge a_\beta\wedge F_t^*\omega_I^{k-1}. \tag{5.6}$$

The matrix $l_{\alpha\beta}$ has the following propoties:

(1) $l_{\alpha\beta}$ is a constant matrix;

(2) $l_{\alpha\beta}$ is anti-symmetric.

It is obviously that $l_{\alpha\beta}$ is anti-symmetric. What $l_{\alpha\beta}$ is constant follows from the fact that ω_I^{k-1} is closed on L_t and L_t is homotopic to L. From (5.3), (5.5) and (5.6), we have

$$\begin{aligned}\tilde\omega\Big(\frac{\partial}{\partial t_\alpha},\frac{\partial}{\partial t_\beta}\Big)&=\tilde g\Big(\frac{\partial}{\partial t_\alpha},\tilde I\frac{\partial}{\partial t_\beta}\Big)=-\int_L\langle\theta_\alpha,I\theta_\beta\rangle{\rm dvol}(t)\\&=-\int_{L_t}\theta_\alpha\wedge*(I\theta_\beta)=-\int_{L_t}C^{-1}\theta_\alpha\wedge I(I\theta_\beta)\wedge\omega_I^{k-1}\\&=C^{-1}\int_L F_t^*\theta_\alpha\wedge F_t^*\theta_\beta\wedge F_t^*\omega_I^{k-1}=C^{-1}\int_L\lambda_{\gamma\alpha}\lambda_{\delta\beta}a_\gamma\wedge a_\delta\wedge F_t^*\omega_I^{k-1}\\&=\lambda_{\gamma\alpha}\lambda_{\delta\beta}l_{\gamma\delta}.\end{aligned}$$

Thus

$$\tilde\omega=\lambda_{\gamma\alpha}\lambda_{\delta\beta}l_{\gamma\delta}dt_\alpha\wedge dt_\beta. \tag{5.7}$$

Because $\xi_\alpha = \lambda_{\alpha\beta} dt_\beta$ is closed, let

$$du_\alpha = \xi_\alpha = \lambda_{\alpha\beta} dt_\beta. \tag{5.8}$$

Since $\lambda_{\alpha\beta}$ is invertible, $u_1, \dots, u_{2m}$ are local coordinates on M_{cl}. So, from (5.7), we have

$$\tilde{\omega} = l_{\gamma\delta} du_\gamma \wedge du_\delta.$$

Now, it is evidently that $\tilde{\omega}$ is closed.

(iii) There is a special Kaehler structure on M_{cl}.

Define a connection ∇ by $\nabla_{\frac{\partial}{\partial u_\alpha}} \frac{\partial}{\partial u_\beta} = 0$. Then obviously ∇ is a real flat torsionfree symplectic connection, where symplectic connection means $\nabla\tilde{\omega} = 0$. We need only to prove $d_\nabla \tilde{I} = 0$.

Let

$$\tilde{I}\frac{\partial}{\partial t_\beta} \mapsto -[I\theta_\beta] := \Sigma \nu_{\alpha\beta} a_\alpha.$$

Then

$$\tilde{I}\frac{\partial}{\partial t_\beta} = \nu_{\alpha\beta}\lambda^{-1}_{\gamma\alpha}\frac{\partial}{\partial t_\gamma} \quad \text{and} \quad \tilde{I} = \nu_{\alpha\beta}\lambda^{-1}_{\gamma\alpha} dt_\beta \otimes \frac{\partial}{\partial t_\gamma}.$$

Using the method of Hitchin in [6], we can prove that $\zeta_\alpha = \Sigma \nu_{\alpha\beta} dt_\beta$ is closed. We only notice that $\zeta_\alpha = -p_* F^* \omega_K = -\int_{A_\alpha} F^* \omega_K$, where p_* takes closed forms to closed forms (see [2]). So, we have

$$\frac{\partial \nu_{\alpha\beta}}{\partial t_\gamma} = \frac{\partial \nu_{\alpha\gamma}}{\partial t_\beta}. \tag{5.9}$$

From (5.8) and (5.9), we can obtain

$$\begin{aligned} d_\nabla \tilde{I} &= d(\nu_{\alpha\beta}\lambda^{-1}_{\gamma\alpha}) \wedge dt_\beta \otimes \frac{\partial}{\partial t_\gamma} - \nu_{\alpha\beta}\lambda^{-1}_{\gamma\alpha} dt_\beta \wedge \nabla \frac{\partial}{\partial t_\gamma} \\ &= -\frac{\partial(\nu_{\alpha\beta}\lambda^{-1}_{\gamma\alpha})}{\partial t_\delta} dt_\beta \wedge dt_\delta \otimes \frac{\partial}{\partial t_\gamma} - \nu_{\alpha\beta}\lambda^{-1}_{\gamma\alpha} dt_\beta \wedge \nabla \frac{\partial}{\partial t_\gamma} \\ &= -\nu_{\alpha\beta}\frac{\partial \lambda^{-1}_{\gamma\alpha}}{\partial t_\delta} dt_\beta \wedge dt_\delta \otimes \frac{\partial}{\partial t_\gamma} - \nu_{\alpha\beta}\lambda^{-1}_{\gamma\alpha} dt_\beta \wedge \nabla \frac{\partial}{\partial t_\gamma}. \end{aligned}$$

Next, we compute $\nabla \frac{\partial}{\partial t_\gamma}$ as follows:

$$\begin{aligned} \nabla \frac{\partial}{\partial t_\gamma} &= \Big(\nabla_{\frac{\partial}{\partial t_\delta}} \frac{\partial}{\partial t_\gamma}\Big) \otimes dt_\delta = \nabla_{\frac{\partial}{\partial t_\delta}} \Big(\frac{\partial u_\sigma}{\partial t_\gamma}\frac{\partial}{\partial u_\sigma}\Big) \otimes dt_\delta \\ &= \nabla_{\frac{\partial}{\partial t_\delta}} \Big(\lambda_{\sigma\gamma}\frac{\partial}{\partial u_\delta}\Big) \otimes dt_\delta = \frac{\partial \lambda_{\sigma\gamma}}{\partial t_\delta}\frac{\partial}{\partial u_\sigma} \otimes dt_\delta \\ &= \frac{\partial \lambda_{\sigma\gamma}}{\partial t_\delta}\lambda^{-1}_{\tau\sigma}\frac{\partial}{\partial t_\tau} \otimes dt_\delta. \end{aligned}$$

So

$$\begin{aligned} \nu_{\alpha\beta}\lambda^{-1}_{\gamma\alpha} dt_\beta \wedge \nabla \frac{\partial}{\partial t_\gamma} &= \nu_{\alpha\beta}\lambda^{-1}_{\gamma\alpha}\frac{\partial \lambda_{\sigma\gamma}}{\partial t_\delta}\lambda^{-1}_{\tau\sigma} dt_\beta \wedge dt_\delta \otimes \frac{\partial}{\partial t_\tau} \\ &= -\nu_{\alpha\beta}\lambda^{-1}_{\gamma\alpha}\lambda_{\sigma\gamma}\frac{\partial \lambda^{-1}_{\tau\sigma}}{\partial t_\delta} dt_\beta \wedge dt_\delta \otimes \frac{\partial}{\partial t_\tau} \\ &= -\nu_{\alpha\beta}\frac{\partial \lambda_{\tau\alpha}}{\partial t_\delta} dt_\beta \wedge dt_\delta \otimes \frac{\partial}{\partial t_\tau}. \end{aligned}$$

Thus, $d_\nabla \tilde{I} = 0$.

Corollary 5.1. *$M_{\rm cl}$ is not compact.*

Proof. For any $p \in L$, we have defined the pseudo-holomorphic map $f : M_{\rm cl} \to M$. Now, from Theorem 5.1, $M_{\rm cl}$ is Kaehler, and f is holomorphic. So, if $M_{\rm cl}$ is compact, then f is constant. It is impossible.

Corollary 5.2. *The cotangent bundle $T^*M_{\rm cl}$ carries a canonical hyper-Kaehler structure. So, $M_{\rm cl}$ is a complex Lagrangian submanifold in $T^*M_{\rm cl}$.*

Proof. Corollary 5.2 is directly obtained from Theorem 2.1 in [4].

§6. Further Remark

Special Kaehler manifolds arise in global supersymmetry and have received more attention recently due to their prominent role in the seminal work of Seiberg and Witten on $N = 2$ supersymmetric Yang-Mills theories. See [4] for extensive references. D. S. Freed has proven that under a suitable integrality hypothesis a special Kaehler manifold parametrizes an algebraic completely integrable system [4]. We have proven that the moduli space of a compact complex Lagrangian submanifold in the hyper-Kaehler manifold is a special Kaehler manifold. Conversely, suppose (X, ω, ∇) is a special Kaehler manifold. Suppose further that there is a lattice $\wedge^* \subset TX$, flat with respect to ∇, whose dual $\wedge \subset T^*X$ is a complex lagrangian submanifold. Then $M = T^*X/\wedge$ is a hyper-Kaehler manifold and the fibers of $M \to X$ are complex Lagrangian submanifolds of M. So we have the following fact:

• Under a suitable integrality hypothesis a special Kaehler manifold can be realized as the moduli space of a complex Lagrangian submanifold.

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