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REAL TOMITA-TAKESAKI THEORY**

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Abstract

Tomita-Takesaki theory in the real case is considered. The author introduces the conception of a nondegenerate pair of closed subspaces in a real Hilbert space. Then a satisfactory real Tomita-Takesaki theory is obtained, and it seems to be a special result of the complex case.

Keywords Tomita-Takesaki theory, Non-degenerate pair, Unitary involution, Modular operator
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Let M be a real Von Neumann algebra on a real Hilbert space H, and $\xi_0 \in H$ be cyclic-separating for M. Let $K = \overline{M_h\xi_0}$, and $L = \overline{M_k\xi_0}$. Clearly, $K \cap L = \{0\}$, and $(K\dot{+}L)$ is dense in H. Hence, we must study such a nondegenerate pair (K, L). Many properties of (K, L) are similar to the complex case (§1). Then a satisfactory real Tomita-Takesaki theory is obtained (§2). For σ-finite real W^*-algebra M, we point out that $M_c^{\varphi} = M^{\varphi}\dot{+}iM^{\varphi}$ (§3) and etc.

Moreover, the real Tomita-Takesaki theory seems to be a special result of the complex case.

§1. Nondegenerate Pair of Closed Linear Subspaces

Definition 1.1. *Let H be a real Hilbert space, K, L be two closed (real) linear subspaces of H. (K, L) is called a nondegenerate pair, if*

$$K \cap L = \{0\}, \quad \text{and} \quad (K\dot{+}L) \quad \text{is dense in} \quad H.$$

Remark 1.1. In this case, $(K^{\perp}, L^{\perp})$ must be also nondegnerate, where $K^{\perp}, L^{\perp}$ are the orthogonal parts of K, L in H respectively. Indeed, $K^{\perp} \cap L^{\perp} = (K\dot{+}L)^{\perp} = \{0\}$; and if $\langle \xi, K^{\perp}\dot{+}L^{\perp} \rangle = 0$ for some $\xi \in H$, then $\xi \in K \cap L = \{0\}$, i.e., $(K^{\perp}\dot{+}L^{\perp})$ is also dense in H.

Definition 1.2. *Let (K, L) be nondegenerate in H. Denote the projections from H onto K, L by p, q respectively, $a = p + q$, and let $p - q = jb$ be the polar decomposition.*

Proposition 1.1. *Keep the assumption and notations as in Definition* 1.2.

(i) $0 \le a \le 2$, *and* $\{0, 2\}$ *are not eigenvalues of* a;

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(ii) $b = |p-q| \geq 0$; 0 *is not an eigenvalue of* $b; b = a^{1/2}(2-a)^{1/2}$; *and* b *commutes with* p, q, a, j *respectively;*

(iii) j *is self-adjoint and unitary;* $jp = (1-q)j, jq = (1-p)j$, *and* $ja = (2-a)j$.

Proof. (i) Clearly, $0 \leq a \leq 2$. If $a\xi = 0$ for some $\xi \in H$, then

$$\langle a\xi, \xi\rangle = \|p\xi\|^2 + \|q\xi\|^2 = 0, \quad p\xi = q\xi = 0,$$

and $\xi \in K^\perp \cap L^\perp = \{0\}, \xi = 0$. Hence, 0 is not an eigenvalue of a. Since $(K^\perp, L^\perp)$ is also nondegenerate, it follows that 0 is not an eigenvalue of $(1-p)+(1-q) = 2-a$, i.e., 2 is not an eigenvalue of a.

(ii) Since $b^2 = (p-q)^2 = a(2-a)$, it follows that $b = a^{1/2}(2-a)^{1/2}$, and 0 is not an eigenvalue of b by (i). Moreover, b is a limit of a sequence in real polynomials of $(p-q)^2$. Thus, b commutes with p, q, a respectively. By $(p-q) = jb$ and $(p-q)^* = p-q$, so b commutes with j.

(iii) Clearly, $j^* = j$. Since 0 is not an eigenvalue of b, it follows that $H = \overline{bH} = \overline{(p-q)H}$. Hence, j is also unitary. By

$$bjp = jbp = (p-q)p = (1-q)(p-q) = b(1-q)j,$$

we can see $jp = (1-q)j$. Similarly, $jq = (1-p)j$. Further,

$$ja = (1-q)j + (1-p)j = (2-a)j.$$

The proof is completed.

Definition 1.3. *Let* (K, L) *be nondegenerate in* H. *The operator* j *as above is called the unitary involution with respect to* (K, L). *The operator* $\delta = (2-a)a^{-1} = a^{-1}(2-a)$ *is called the modular operator with respect to* (K, L). *Clearly,* δ *is unbounded, invertible, non-negative and self-adjoint.*

Proposition 1.2. *Let* (K, L) *be nondegenerate in* H, *and* δ *be the modular operator. Then for each almost everywhere finite, real valued, measurable function* f *on* $\mathbb{R}$, *we have*

$$jf(\delta)j = f(\delta^{-1}).$$

Proof. By $jaj = 2-a$, and $j\delta j = \delta^{-1}$, the conclusion is obvious.

Definition 1.4. *Let* (K, L) *be nondegenerate in* H, *and keep above notations. Let* $s(\xi + \eta) = \xi - \eta, \forall \xi \in K, \eta \in L, \mathcal{D}(s) = K \dot{+} L$.

Proposition 1.3. *Let* (K, L) *be nondegenerate in* H.

(i) s *is a linear closed operator on* H *with a dense domain;* $\mathcal{D}(s^*) = K^\perp \dot{+} L^\perp, s^*(\xi+\eta) = -\xi + \eta, \forall \xi \in K^\perp, \eta \in L^\perp$, *and* $jsj = s^*$;

(ii) $s = j\delta^{1/2}, s^* = j\delta^{-1/2}$ *are the polar decompositions of* s, s^* *respectively. In particular,* $\mathcal{D}(\delta^{1/2}) = \mathcal{D}(s) = K \dot{+} L$.

Proof. (i) Let $\{\xi_n\} \subset K$ and $\{\eta_n\} \subset L$ be such that

$$\xi_n + \eta_n \to x \quad \text{and} \quad \xi_n - \eta_n \to y.$$

Then we have $\xi \in K$ and $\eta \in L$ such that

$$\xi_n \to \xi \quad \text{and} \quad \eta_n \to \eta.$$

Hence $x = \xi + \eta$ and $y = \xi - \eta = sx$. This means that the operator s is closed. Clearly,

$$\langle sx, y\rangle = \langle x, s^*y\rangle, \quad \forall x \in \mathcal{D}(s), y \in \mathcal{D}(s^*).$$

Hence $s^* \subset$ the adjoint of s. Now, if $y \in \mathcal{D}$ (the adjoint of s), then there is a z such that

$$\langle \xi - \eta, y\rangle = \langle \xi + \eta, z\rangle, \quad \forall \xi \in K, \eta \in L.$$

Taking $\eta = 0$, we can see $y - z = 2\xi' \in K^\perp$; taking $\xi = 0$, then we have $y + z = 2\eta' \in L^\perp$. Therefore

$$y = \xi' + \eta' \in K^\perp \dot{+} L^\perp, \quad z = -\xi' + \eta' = s^* y,$$

i.e., $s^* =$ the adjoint of s.

By $jp = (1-q)j$ and $jq = (1-p)j$, we have

$$jK = L^\perp, \quad jL = K^\perp,$$

and $jL^\perp = K, jK^\perp = L$. Thus, $jsj = s^*$.

(ii) Let $\xi' \in K^\perp, \eta' \in L^\perp$. Then

$$(p-q)(\xi' + \eta') = p\eta' - q\xi' = (p+q)(-\xi' + \eta) = as^*(\xi' + \eta').$$

Hence, $as^* \subset (p-q) = jb = bj$. Now by $js^*j = s$, we have

$$ajs = as^*j \subset b = a^{1/2}(2-a)^{1/2}.$$

Further, $js \subset \delta^{1/2} = a^{-1/2}(2-a)^{1/2}$. But js and $\delta^{1/2}$ are both self-adjoint on H, so $js = \delta^{1/2}$, and

$$s = j\delta^{1/2}, \quad s^* = \delta^{1/2} j = j\delta^{-1/2}.$$

Since $s^*s = j\delta^{-1/2} j\delta^{1/2} = \delta$, it follows that $s = j\delta^{1/2}$ and $s^* = j\delta^{-1/2}$ are the polar decompositions. The proof is completed.

Proposition 1.4. *Let (K, L) be a pair of closed linear subspaces in a real Hilbert space H. Then (K, L) is nondegenerate, if and only if the closed real linear subspace $(K\dot{+}iL)$ of the complex Hilbert space $H_c = (H\dot{+}iH)$ is nondegenerate* (*see* [1, *Definition* 8.1.1]).

Proof. Let (K, L) be nondegenerate. Then

$$(K\dot{+}iL) \cap i(K\dot{+}iL) = (K\dot{+}iL) \cap (L\dot{+}iK) = (K \cap L)\dot{+}i(K \cap L) = \{0\},$$
$$(K\dot{+}iL)\dot{+}i(K\dot{+}iL) = (K\dot{+}iL)\dot{+}(L\dot{+}iK) = (K\dot{+}L)\dot{+}i(K\dot{+}L)$$

will be dense in H_c, i.e., $(K\dot{+}iL)$ is nondegenerate in H_c.

Similarly, if $(K\dot{+}iL)$ is nondegenerate in H_c, then (K, L) is also degenerate in H. The proof is completed.

Definition 1.5. *Let (K, L) be nongenerate in H. Denote the projections from $(H_c)_r$ onto $(K\dot{+}iL), i(K\dot{+}iL) = L\dot{+}iK$ by P, Q respectively, where $(H_c)_r = (H_c, Re\langle,\rangle)$ is a real Hilbert space.*

Clearly, $(H_c)_r \cong H \oplus H$, and under this unitary equivalence, $P \cong p \oplus q$ and $Q \cong q \oplus p$.

Further, let $A = P + q$, and $P - Q = JB$ be the polar decomposition in $(H_c)_r$.

Proposition 1.5. *Let (K, L) be nondegenerate in H, and keep the notations of Definition* 1.5.

(i) *$Pi = iQ, Qi = iP$ in H_c;*

(ii) *$0 \le A \le 2$ in H_c, and $\{0, 2\}$ are not eigenvalues of A;*

(iii) *$B \ge 0$ in H_c, $B = A^{1/2}(2-A)^{-1/2}$, 0 is not an eigenvalue of B, and B commutes with P, Q, A, J respectively;*

(iv) *J is self-adjoint and unitary in $(H_c)_r, Ji = -iJ$ in H_c, $\langle Jx, y\rangle = \langle Jy, x\rangle, \forall x, y \in H_c$, and*

$$JP = (1-Q)J, \quad JQ = (1-P)J, \quad JA = (2-A)J.$$

Proof. From Proposition 1.4 and [1, Lemma 8.1.2], it is obvious. Moreover, we can also get the proof from Proposition 1.1.

Definition 1.6. *Let (K, L) be nondegenerate in H, and keep the notations of Definition* 1.5. *Let*

$$\begin{gathered}\triangle = A^{-1}(2-A) = (2-A)A^{-1},\\ S((\xi+\eta)+i(\xi'+\eta')) = (\xi-\eta)+i(-\xi'+\eta'),\ \forall \xi,\xi' \in K,\ \eta,\eta' \in L,\\ \mathcal{D}(S) = (K\dot{+}L)\dot{+}i(K\dot{+}L),\end{gathered}$$

and

$$\begin{gathered}S^{+}((\xi+\eta)+i(\xi'+\eta')) = (-\xi+\eta)+i(\xi'-\eta'),\ \forall \xi,\ \xi' \in K,\ \eta,\eta' \in L,\\ \mathcal{D}(S^{+}) = (K^{\perp}\dot{+}L^{\perp})\dot{+}i(K^{\perp}\dot{+}L^{\perp}).\end{gathered}$$

Proposition 1.6. *Let (K, L) be nondegenerate in H, and keep all notations as above.*

(i) *$\triangle$ is a unbounded, invertible, (complex) linear, non-negative, self-adjoint operator on H_c, and*

$$Jf(\triangle)J = f(\triangle^{-1}),$$

$\forall$ almost everywhere finite, real valued, measurable function f on $\mathbb{R}$;

(ii) *S and S^{+} are two conjugate linear closed operators on H_c with dense domain; S^{+} is the adjoint operator of S on $(H_c)_r, JSJ = S^{+}; S = J\triangle^{1/2}, S^{+} = J\triangle^{-1/2}$ are the polar decompositions on $(H_c)_r$; and*

$$\mathcal{D}(\triangle^{1/2}) = \mathcal{D}(S) = (K\dot{+}L)\dot{+}i(K\dot{+}L).$$

Proof. From Proposition 1.4 and [1, Lemma 8.1.3, Lemma 8.1.4], it is obvious. Moreover, we can also get the proof from Propositions 1.2 and 1.3.

Theorem 1.1. *Let (K, L) be nondegenerate in H. Then $\{\triangle^{it}|t \in \mathbb{R}\}$ is the unique one-parameter strongly continuous group of unitary operators on H_c, such that $\{\triangle^{it}|t \in \mathbb{R}\}$ satisfies the KMS condition relative to $(K\dot{+}iL)$ (see* [1, *Definition* 8.1.7]) *and*

$$\triangle^{it}(K\dot{+}iL) = (K\dot{+}iL),\ \forall t \in \mathbb{R}.$$

Moreover, $J \triangle^{it} J = \triangle^{it}, \forall t \in \mathbb{R}$.

Proof. It is obvious from Proposition 1.4, [1, Theorem 8.1.3 and Lemma 8.1.3.].

§2. Real Tomita-Takesaki Theory

Let M be a real Von Neumann (VN) algebra on a real Hilbert space H, and $\xi_0 \in H$ be a cyclic-separating vector for M. We have following facts.

(1) ξ_0 is also a cyclic-separating vector for the (complex) VN algebra $M_c = M\dot{+}iM$ on the (complex) Hilbert space $H_c = H\dot{+}iH$.

(2) $(M_c)_h = M_h\dot{+}iM_k$, where $(M_c)_h, M_h$ are self-adjoint parts of M_c, M respectively, and M_k is the skew self-adjoint part of M.

Indeed, if $(a + ib) \in (M_c)_h$, where $a, b \in M$, then

$$a^* - ib^* = (a + ib)^* = a + ib.$$

Hence, $a^* = a \in M_h$ and $b^* = -b \in M_k$.

(3) Let $K = \overline{M_h}\xi_0,\ L = \overline{M_k}\xi_0$. Then (K, L) is nondegenerate in H.

Indeed, let $h' \in M'_h,\ k \in M_k$. Then

$$\langle h'\xi_0, k\xi_0\rangle = -\langle k\xi_0, h'\xi_0\rangle = -\langle h'\xi_0, k\xi_0\rangle = 0,$$

i.e., $M'_h\xi_0 \perp M_k\xi_0$ in H. Similarly, $M'_k\xi_0 \perp M_h\xi_0$ in H. Thus

$$M'\xi_0 \subset (M_h\xi_0)^\perp + (M_k\xi_0)^\perp \subset (\overline{M_h\xi_0} \cap \overline{M_k\xi_0})^\perp = (K \cap L)^\perp.$$

Since ξ_0 is also cyclic for M', it follows that $K \cap L = \{0\}$. On the other hand, $M\xi_0 \subset (K \dot{+} L)$ and ξ_0 is cyclic for M. Hence, $(K \dot{+} L)$ is dense in H. Therefore, (K, L) is nondegenerate in H.

Proposition 2.1. *Keep all assumptions and notations in the above and* §1. *Then*

$$\begin{aligned} q\xi_0 &= Q\xi_0 = 0; \\ P\xi_0 &= p\xi_0 = A\xi_0 = a\xi_0 = J\xi_0 \\ &= j\xi_0 = B\xi_0 = b\xi_0 = \xi_0; \\ \triangle^{it}\xi_0 &= \xi_0,\ \forall t \in \mathbb{R}; \\ M\xi_0 &\subset \mathcal{D}(\delta^{1/2}); \end{aligned}$$

and the operator s on H is the closure of operator $x\xi_0 \to x^*\xi_0 (\forall x \in M)$.

Proof. We can get the conclusions on $P, Q, A, B, \triangle^{it} (t \in \mathbb{R})$ by [1, Proposition 8.2.2]. From the relations between P, Q, A, B, J and p, q, a, b, j, then we get other conclusions. Moreover, we can give a direct proof of the conclusions on p, q, a, b, j and s.

Theorem 2.1. *Keep all assumptions and notations in the above and* §1. *Then*

$$JM_cJ = M'_c, \quad jMj = M', \quad \triangle^{it}M_c\triangle^{-it} = M_c, \quad \forall t \in \mathbb{R}.$$

Proof. By [1, Theorem 8.2.7], we just need to prove $jMj = M'$.

If $a, b \in M, \xi, \eta \in H$, then

$$\begin{aligned} J(a + ib)J(\xi + i\eta) &= J(a + ib)(j\xi - ij\eta) \\ &= J[(aj\xi + bj\eta) + i(bj\xi - aj\eta)] \\ &= (jaj\xi + jbj\eta) + i(jaj\eta - jbj\xi) \\ &= (jaj - ijbj)(\xi + i\eta), \end{aligned}$$

i.e., $J(a + ib)J = jaj - ijbj$. Thus

$$M' \dot{+} iM' = M'_c = JM_cJ = JMJ + iJMJ = jMj \dot{+} ijMj,$$

i.e., $jMj = M'$. The proof is completed.

Remark 2.1. From $\triangle^{it}M_c\triangle^{-it} = M_c, \forall t \in \mathbb{R}$, we can define a one-parameter $*$ automorphism group $\{\sigma_t(\cdot) = \triangle^{it} \cdot \triangle^{-it} | t \in \mathbb{R}\}$ of M_c. But we do not have $\sigma_t(M) \subset M, \forall t \in \mathbb{R}$. On the other hand, it is obvious that

$$\overline{\sigma_t(x)} = \sigma_{-t}(x), \quad \forall x \in M, \quad t \in \mathbb{R},$$

where "$-$" is defined by the decomposition $B(H_c) = B(H) \dot{+} iB(H)$.

Remark 2.2. Of course, for M_c we have the result of [1, Theorem 8.2.10].

§3. The Case of a σ-Finite Real W^*-Algebra

Let M be a σ-finite real W^*-algebra. Clearly, M is σ-finite, if and only if $M_c = M\dot{+}iM$ is σ-finite.

If φ is a faithful normal real state on M, then so is φ_c on M_c. Indeed, let $x, y \in M$ be such that

$$0 = \varphi_c((x+iy)^*(x+iy)) = \varphi(x^*x) + \varphi(y^*y) + i[\varphi(x^*y - y^*x)].$$

Then $\varphi(x^*x) = \varphi(y^*y) = 0$, and $x = y = 0$.

Using above φ, we get a cyclic faithful $*$ representation $\{\pi_\varphi, H_\varphi, \xi_\varphi\}$. Clearly, ξ_φ is also separating for $\pi_\varphi(M)$. Then by §2, we have one-parameter $*$ automorphism group $\{\sigma_t^\varphi | t \in \mathbb{R}\}$ of M_c, which satisfies the KMS condition with respect to φ_c, and φ_c is invariant about $\{\sigma_t^\varphi | t \in \mathbb{R}\}$ (see [1, §8.3]).

Proposition 3.1. *Let M be a σ-finite real W^*-algebra, $M_c = M\dot{+}iM$, and φ be a faithful normal real state on M. Then*

$$M_c^\varphi = M^\varphi \dot{+} iM^\varphi,$$

where $M_c^\varphi = \{x \in M_c | \sigma_t^\varphi(x) = x, \forall t \in \mathbb{R}\}$, *and*

$$M^\varphi = \{a \in M | \sigma_t^\varphi(a) = a, \forall t \in \mathbb{R}\}.$$

Proof. By [1, Proposition 8.3.2],

$$\begin{aligned} &(a+ib) \in M_c^\varphi \quad (a, b \in M) \\ \Longleftrightarrow\ &\varphi_c((a+ib)(c+id) - (c+id)(a+ib)) = 0, \quad \forall c, d \in M \\ \Longleftrightarrow\ &\varphi_c((a+ib)c - c(a+ib)) = 0, \quad \forall c \in M \\ \Longleftrightarrow\ &\varphi(ac - ca) = \varphi(bc - cb) = 0, \quad \forall c \in M \\ \Longleftrightarrow\ &a, b \in M^\varphi, \end{aligned}$$

i.e., $M_c^\varphi = M^\varphi \dot{+} iM^\varphi$. The proof is completed.

Remark 3.1. For M_c, we also have other results of [1, §8.3].

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References

[1] Li Bingren, Introduction to operator algebras, World Sci., Singapore, 1992.

[2] Rieffel, M. A. & Van Dacle, A., A bounded operator approach to Tomita-Takesaki theory, *Pacific J. Math.*, **69**(1977), 187–221.