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ON THE MODULI NUMBER OF PLANE CURVE SINGULARITIES WITH ONE CHARACTERISTIC PAIR**

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Abstract

The author gives another linear-algebraic proof of the famous result of Zariski, Delorme, Briancon-Granger-Maisonobe about the moduli number of plane curve singularities with the same topological type as $X^a + Y^b = 0$ (i.e.,with one characteristic pair). Since the original proof depends very much on the division theorem of Briancon, it cannot be generalized to higher dimensions. It is hopeful that the proof here will be applied to the higher dimensional cases.

Keywords Moduli number, Plane curve, Singularity, Characteristic pair
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§1. Introduction

Since the poineering work of Zariski[10] there have been many papers published on the moduli number of plane curve singularities[5−9]. The problem how to compute the moduli number attracts many attentions[3,6−10]. As far as the authors' knowledge the most generalized algorithm is the work of [3] which depends on the division theorem in [1,2]. Here we give a linear algebraic proof of the Briancon-Granger-Maisonobe's theorem, which avoids the Division theorem. Since the division theorem only holds in dimension 2, it seems quite hopeful that the proof here will be generalized to arbitrary dimension cases[4].

It is well-known that any plane curve singularity with the same topological type as $x^a + y^b = 0$ is defined by

$$F_{t_{ij}}(x,y) = x^a + y^b = \sum_{(i,j)\in I_1} t_{ij}x^iy^j = 0, \tag{1.1}$$

where

$$I = \{(i,j) : 0 \le i \le a-2, 0 \le j \le b-2\},$$
$$I_1 = \{(i,j) : (i,j) \in I, i/a + j/b \ge 1\}.$$

The rational number $i/a+j/b$ is called the weight $\rho(i,j)$ of the lattice point (i,j). Let $\tau_{\min}$ be the minimum Tjurina number in this family of singularities defined by (1.1). Laudal-Pfister theory[7] tells us that the dimension of the generic component of moduli (moduli number)

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is $r(a,b)-\mu+\tau_{\min}$, where $\mu=(a-1)(b-1)$ is the Milnor number of the singularities in this family and $r(a,b)$ is the number of points in the set I. Thus the computation of moduli number is equivalent to the computation of the minimum Tjurina number of this family of singularities.

We recall that the Tjurina number of the singularity defined by $F_{t_{ij}}(x,y)=0$ is the dimension of

$$\tau_{t_{ij}}=\dim(A(t_{ij})),$$

where

$$A(t_{ij})=C\{x,y\}/(F_{t_{ij}},\partial F_{t_{ij}}/\partial x,\partial F_{t_{ij}}/\partial y),\tag{1.2}$$

i.e., the moduli algebra.

Let

$$M(t_{ij})=C\{x,y\}/(\partial F_{t_{ij}}/\partial x,\partial F_{t_{ij}}/\partial y)\tag{1.3}$$

be the moduli algebra. It is well-known that the dimension of the Milnor algebra is $(a-1)(b-1)$ and $\{x^iy^j:(i,j)\in I\}$ is a base for $M(t_{ij})$. Let

$$\begin{aligned}E_{t_{ij}}(x,y)&=F_{t_{ij}}-(x/a)\partial F_{t_{ij}}/\partial x-(y/b)\partial F_{t_{ij}}/\partial y\\&=\sum_{(i,j)\in I_1}(1-i/a-j/b)t_{ij}x^iy^j.\end{aligned}\tag{1.4}$$

We have

$$A(t_{ij})=M(t_{ij})/(E_{t_{ij}}).\tag{1.5}$$

Therefore the computation of (minimum) Tjurina number is equivalent to the computation of the (maximal) number of linear relations on the above-mentioned base imposed by the ideal $(E_{t_{ij}})$.

§2. Briancon-Granger-Maisonobe Construction

In this section we recall the construction in [3].

Let

$$\begin{aligned}&I_1'=\{(i,j):\rho(i,j)>1\},\\&W=\{(i,j):\rho(i,j)<1-2/a-2/b,(i,j)\in I\}.\end{aligned}$$

The Briancon-Granger-Maisonobe construction is as follows.

1st step.

Find (e_1,e_2) in I_1' such that

$$\rho_1=\rho(e_1,e_2)=\min\{\rho(i,j):(i,j)\in I\}.$$

Let

$$\begin{aligned}&E_1=(e_1,e_2)+N^2,\quad \Delta_1=I_1'-E_1,\\&D_1=\{(i,j):0\le i\le a-2-e_1,0\le j\le b-2-e_2\}.\end{aligned}$$

Define

$$\begin{aligned}&\rho_1'=\rho(q_1,q_2)=\min\{\rho(i,j):(i,j)\in W-D_1\},\\&\rho_2=\rho_1+\rho_1'=\min\{\rho(e_1,b-1),\rho(a-1,e_2)\}.\end{aligned}$$

If

$$I_2 = \Delta_1 \cap \{(i,j) : \rho(i,j) \geq \rho_2\} = \emptyset,$$

the process stops here. If it is not empty, we continue the 2nd step.

nth step.

If the $(n-1)$th step is completed, (q_{2n-3}, q_{2n-2}) and I_n are defined. Find (e_{2n-1}, e_{2n}) such that

$$\rho_{2n-1} = \rho(e_{2n-1}, e_{2n}) = \min\{\rho(i,j) : (i,j) \in I_n\}.$$

Let

$$E_n = (e_{2n-1}, e_{2n}) + N^2, \quad \Delta_n = \Delta_{n-1} - E_n,$$
$$D_n = \Big\{(i,j) : (i,j) + (e_{2n-1}, e_{2n}) \in E_n - \bigcup_{i=1}^{n-1} E_i\Big\}.$$

Let

$$\begin{aligned}\rho'_{2n-1} &= \rho(q_{2n-1}, q_{2n}) \\ &= \min\{\rho(i,j) : (i,j) \in W - D_1 - \cdots - D_n\}, \\ \rho_{2n} &= \rho_{2n-1} + \rho'_{2n-1}.\end{aligned}$$

Define

$$I_{n+1} = \{(i,j) : (i,j) \in \Delta_n, \rho(i,j) \geq \rho_{2n}\}.$$

We continue this process until it stops where the algorithm is completed.

Lemma 2.1. *The above algorithm is completed after finite steps.*

Proof. It is clear that $\Delta_k \supseteq \Delta_{k-1}$ and $\Delta_k \ddagger \Delta_{k-1}$ if the $(k-1)$th step is not the final step. On the other hand, there are only finitely many points in I_1, the algorithm has to stop after finite steps.

The last step's Δ_m is denoted by Δ.

Lemma 2.2. *For nth step,*

$$(e_{2n-1} - q_{2n-3}, e_{2n} - q_{2n-2}) \in \{(i,j) : 0 \leq i \leq a-2, 0 \leq j \leq b-2, \rho(i,j) > 1\}.$$

Proof. By the defination

$$\rho(e_{2n-1}, e_{2n}) = \min\{\rho(i,j) : (i,j) \in I_n\},$$

we have

$$\rho(e_{2n-1}, e_{2n}) \geq \rho(q_{2n-3}, q_{2n-2}) + \rho(e_1, e_2).$$

Since the weight function is linear, our conclusion is proved.

Briancon-Granger-Maisonobe Theorem. *For generic parameters t_{ij}'s, $\{x^i y^j : (i,j) \in \Delta\}$ is a C-linear base of the moduli algebra $A(t_{ij})$.*

§3. A Linear-Algebraic Proof of Briancon-Granger-Maisonobe Theorem

In this section we give a linear-algebraic proof of the above main theorem in [3].

As in [7, Section 4] we give a negative weight $1-\rho(i,j)$ to the variable t_{ij}, thus $F_{t_{ij}}(x,y)$ is a quasi-homogeneous polynomial of x, y, t_{ij}'s of degree 1. Since the set

$$\{x^i y^j : 0 \le i \le a-2, 0 \le j \le b-2\}$$

is the base of the Milnor algebra $M(t_{ij})$, we can write

$$E_{t_{ij}}(x,y)x^{\alpha_1}y^{\alpha_2} = \sum_{(\beta_1,\beta_2)\in I} k_{\alpha\beta}(t_{ij})x^{\beta_1}y^{\beta_2} \tag{3.1}$$

in the $M(t_{ij})$. It is clear that $k_{\alpha\beta}$ is a quasi-homogeneous polynomial of the variables t_{ij}'s.

Lemma 3.1.[7] *$k_{\alpha\beta}(t_{ij})$ satisfies*

(1) *$k_{\alpha\beta}(t_{ij})$ is not zero polynomial only if $\rho(\alpha_1,\alpha_2) < 1-2/a-2/b$ and*

$$\rho(\beta_1,\beta_2) > 1+\rho(\alpha_1,\alpha_2);$$

(2) *$k_{\alpha\beta}(t_{ij}) = t_{\beta-\alpha} \mod (t^2 ideal)$ if $\beta-\alpha \in I_1$ and $k_{\alpha\beta}(t_{ij}) = 0 \mod (t^2 ideal)$ if $\beta-\alpha \notin I_1$.*

Let V be the linear subspace of $M(t_{ij})$ which is spanned by the set

$$\{E_{t_{ij}}(x,y)x^iy^j : (i,j) \in N^2\}.$$

We know from Lemma 3.1 that V is a C-linear span of the set

$$\{E_{t_{ij}}(x,y)x^iy^j : (i,j) \in W\}.$$

We need to prove the following two lemmas.

Lemma 3.2. *For the nth step of Briancon-Granger-Maisonobe construction V is contained in the span of*

$$\{E_{t_{ij}}(x,y)x^iy^j : (i,j) \in D_1 \cup \cdots \cup D_n\} \cup \{x^iy^j : \rho(i,j) \ge \rho_{2n}\}.$$

Lemma 3.3. *For the nth step in the above construction vectors*

$$\{E_{t_{ij}}(x,y)x^iy^j : (i,j) \in D_1 \cup \cdots \cup D_n\}$$

are linear independent in $M(t_{ij})$ for generic parameter t_{ij}'s.

It is clear that Briancon-Granger-Maisonobe theorem follows from the above Lemmas directly.

Proof of Lemma 3.2. By Lemma 3.1 the ideal in $M(t_{ij})$ generated by $E_{t_{ij}}(x,y)$ is the C-linear space V. Since

$$W \subset D_1 \cup \cdots \cup D_n \cup \{(i,j) : \rho(i,j) \ge \rho'_{2n-1}\},$$

we know that V is contained in the span of

$$\{E_{t_{ij}}(x,y)x^iy^j : (i,j) \in D_1 \cup \cdots \cup D_n\} \cup \{E_{t_{ij}}(x,y)x^iy^j : \rho(i,j) \ge \rho'_{2n-1}\}.$$

On the other hand, the second set is contained in the span of $\{x^iy^j : \rho(i,j) \ge \rho_{2n}\}$. The conclusion is proved.

Proof of Lemma 3.3. We only prove the case of $n=2$. The similar argument can be applied to general n case.

For $n=2$, let

$$c_1 = \max\{\rho(i,j) : (i,j) \in D_2\},$$
$$c_2 = \rho_1 + c_1.$$

Divide D_1 and E_1 to two parts as follows:

$$\begin{aligned} D_1' &= \{(i,j) : (i,j) \in D_1 \rho(i,j) > c_1\}, \\ D_1'' &= D_1 - D_1', \\ E_1' &= \{(i,j) : \rho(i,j) > c_2\}, \\ E_1'' &= E_1 - E_1'. \end{aligned} \tag{3.2}$$

We arrange the set of vectors

$$\Big\{E_{t_{ij}}(x,y)x^{\alpha_1}y^{\alpha_2} = \sum_{(\beta_1,\beta_2)\in I} k_{\alpha\beta}(t_{ij})x^{\beta_1}y^{\beta_2} : (\alpha_1,\alpha_2) \in D_1' \cup D_1''\Big\} \tag{3.3}$$

as follows:

$$\begin{pmatrix} & \begin{matrix} x^{\beta_1}y^{\beta_2} \\ (\beta_1,\beta_2)\in E_1' \end{matrix} & \begin{matrix} x^{\beta_1}y^{\beta_2} \\ (\beta_1,\beta_2)\in E_1'' \end{matrix} & \begin{matrix} x^{\beta_1}y^{\beta_2} \\ (\beta_1,\beta_2)\in E_2 \end{matrix} & \begin{matrix} x^{\beta_1}y^{\beta_2} \\ (\beta_1,\beta_2)\notin E_1\cup E_2 \end{matrix} \\ \begin{matrix} E_{t_{ij}}(x,y)x^{\alpha_1}y^{\alpha_2} \\ (\alpha_1,\alpha_2)\in D_1' \end{matrix} & A & 0 & 0 & 0 \\ \begin{matrix} E_{t_{ij}}(x,y)x^{\alpha_1}y^{\alpha_2} \\ (\alpha_1,\alpha_2)\in D_1'' \end{matrix} & A' & B & B' & F \\ \begin{matrix} E_{t_{ij}}(x,y)x^{\alpha_1}y^{\alpha_2} \\ (\alpha_1,\alpha_2)\in D_2 \end{matrix} & A'' & C' & C & F' \end{pmatrix}$$

$$P = \begin{pmatrix} B & B' \\ C' & C \end{pmatrix}.$$

Claim 1. A, B, C are nonsingular matrixes for generic parameters;

Claim 2. The matrix P as above is a nonsingular matrix for generic parameters.

For Claim 1 we can arrange $x^{\beta_1}y^{\beta_2}$ and $E_{t_{ij}}(x,y)x^{\alpha_1}y^{\alpha_2}$ by its increasing order of weights in the matrix A. When a and b are coprime, there are no two lattice points with the same weights; thus we have an upper triangular matrix with its diagonal entries of the form t_{e_1,e_2}+order 2 functions of t_{ij}'s from Lemma 3.1. Hence it is a nonsingular matrix for the generic parameters. When a and b are not coprime, we have a blocked upper trangular matrix with the nonsingular diagonal blocks; thus we have a nonsingular matrix similarly. The proofs for B and C are similar.

For Claim 2 we note that the diagonal entries are t_{e_1,e_2}+ order 2 functions of t_{ij}'s and $t_{e_3-q_1,e_4-q_2}$+order 2 functions of t_{ij}'s. It is clear that the undiagonal entries in B and C cannot be of the above form. For a position in B' its entry is $k_{\alpha\beta}(t_{ij}) = t_{\beta-\alpha}$+order 2 functions of t_{ij}'s from Lemma 3.1. We want to argue that these $t_{\beta-\alpha}$ cannot be t_{e_1,e_2} and $t_{e_3-q_1,e_4-q_2}$. It is clear that they cannot be t_{e_1,e_2} from the definition of (e_1,e_2). If for some $(\alpha_1,\alpha_2) = \alpha$ and $\beta = (\beta_1,\beta_2)$, we have

$$t_{\beta-\alpha} = t_{e_3-q_1,e_4-q_2}, \quad (\beta_1,\beta_2) = (e_3 - q_1 + \alpha_1, e_4 - q_2 + \alpha_2).$$

From the definition of (q_1,q_2) we know $\alpha_1 > q_1$ or $\alpha_2 > q_2$. Thus $\beta_1 < e_3$ or $\beta_2 < e_4$, this is a contradiction to the fact that $\beta \in E_2$. Similarly we can argue that no positions in C' with the entries as the diagonal entries of B and C. Thus for a suitable choice of parameters the matrix P is nonsingular.

From Claim 1 and Claim 2 we get the conclusion of Lemma 3.3. Thus the conclusion of Briancon-Granger-Maisonobe theorem is valid.

From our above approach we note that the computation of minimum Tjurina number in higher dimensions depends on the generalized Briancon-Granger-Maisonobe construction and a careful analysis of its effect. This would lead us to an explaination of "jumping" of Tjurina numbers found in [7]. This part of our work will appear elsewhere[4].

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