Chin. Ann. of Math.
20B: 4(1999),413-424.

MASLOV-TYPE INDEX THEORY FOR SYMPLECTIC PATHS AND SPECTRAL FLOW (I)***

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Abstract

The spectral flow for paths of admissible operators in arbitrary Banach space is defined, and some properties of the spectral flow is studied.

Keywords Relative Morse index, Non-selfadjoint operator, Spectral flow
1991 MR Subject Classification 58E05, 58G99
Chinese Library Classification O176.3, O19

§1. Introduction

The spectral flow for a one parameter family of linear selfadjoint Fredholm operators is introduced by Atiyah-Patodi-Singer[2] in their study of index theory on manifolds with boundary. Since then other significant applications have been found. In [17], J. Robbin and D. Salamon studied in detail the spectral flow for the curves of linear selfadjoint Fredhom operators with invertible operators at the endpoints and proved an index theorem. In [7] and [8] the notion of the spectral flow was generalized to the higher dimensional case by X. Dai and W. Zhang.

In this paper the notion of spectral flow is generalized to the paths of admissible operators (cf. Definition 2.3 below) in arbitrary Banach spaces. This specially includes the case of non-selfadjoint operators. Let X be a Banach space and A_s be a path of admissible operators on X. Following the ideas in [7,8,15] for selfadjoint case, we define the generalized spectral section for A_s, and prove the existence of it provided that A_s is in a sufficiently small neighborhood of A_0. Then we define the spectral flow via Theorem 1.11 of [8] in this case. In the general case, we cut the path into small pieces and define the spectral flow as the sum of all the pieces. Using these results, a relative Morse index is defined for certain not necessarily selfadjoint operators on Banach spaces. For the selfadjoint case, such relative Morse indices have been defined in [3,9,18] and others. We are not aware of results dealing with the non-selfadjoint case. Based on the basic properties of the spectral flow, we give

Manuscript received July 14, 1998. Revised May 20, 1999.
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***Project supported by the National Natural Science Foundation of China, the Mathematical Centre of the Ministry of Education of China, and the Qiu Shi Science and Technology Foundation.

proofs of two index theorems for the Galerkin approximation procedure and the saddle point reduction respectively, and give a method to calculate the spectral flow via the intersection forms defined in [17]. The results of this paper is used in [14] to study the Maslov-type index theory for symplectic paths.

§2. Definition of the Spectral Flow

Let X be a Banach space. We denote the set of bounded linear operators and compact linear operators on X by $\mathcal{B}(X)$ and $\mathcal{CL}(X)$ respectively. Let A be in $\mathcal{B}(X)$. We denote the spectrum and the resolvent set of A by $\sigma(A)$ and $\rho(A)$ respectively. Recall that the resolvent of A is defined by

$$R(\zeta) \equiv R(\zeta, A) = (A - \zeta)^{-1}, \qquad \zeta \in \rho(A). \tag{2.1}$$

Let $P \in \mathcal{B}(X)$ be a projection. We will denote by PAP the operator

$$PAP\colon \mathrm{im}\ P \to \mathrm{im}\ P.$$

Let $A \in \mathcal{B}(X)$. Let Ω be a bounded open subset of $\mathbf{C}$ such that $\partial\Omega \subset \rho(A)$. Then $K \equiv \Omega \cap \sigma(A)$ is compact. By Proposition VII. 4.4 in [4], there is a positively oriented system of curves $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ in $\Omega \setminus K$ such that $K \subset \mathrm{ins}\,\Gamma$ and $\mathbf{C} \setminus \Omega \subset \mathrm{out}\,\Gamma$ (see [4, p.200]). The curves $\gamma_1, \dots, \gamma_m$ can be found such that they are smooth. Define

$$P(A, \Omega) = -\frac{1}{2\pi i} \int_\Gamma R(\zeta, A) d\zeta \tag{2.2}$$

for $\Omega \neq \emptyset$. Define $P(A, \emptyset) = 0 \in \mathcal{B}(X)$ by convention. By Proposition VII.4.6 in [4], (2.2) is well defined.

Lemma 2.1. *Let $A \in \mathcal{B}(X)$. Let Ω_k, $k = 1, 2$ be bounded open subsets of* $\mathbf{C}$. *Suppose $(\partial\Omega_1) \cup (\partial\Omega_2) \subset \rho(A)$. Then we have*

(i) $P(A, \Omega_1) + P(A, \Omega_2) = P(A, \Omega_1 \cup \Omega_2)$ *if* $\Omega_1 \cap \Omega_2 = \emptyset$, *and*

(ii) $P(A, \Omega_1)P(A, \Omega_2) = P(A, \Omega_2)P(A, \Omega_1) = P(A, \Omega_1 \cap \Omega_2)$.

Proof. (i) follows from the definition of $P(A, \Omega)$.

(ii) Set

$$\Omega = (\Omega_1 \cap \Omega_2) \cup (\Omega_1 \setminus \bar{\Omega}_2) \cup (\Omega_2 \setminus \bar{\Omega}_1).$$

Define $f, g\colon \Omega \to \mathbf{C}$ by

$$f(\zeta) = \begin{cases} 1, & \text{if } \zeta \in (\Omega_1 \cap \Omega_2) \cup (\Omega_1 \setminus \bar{\Omega}_2), \\ 0, & \text{if } \zeta \in (\Omega_2 \setminus \bar{\Omega}_1), \end{cases}$$

$$g(\zeta) = \begin{cases} 1, & \text{if } \zeta \in (\Omega_1 \cap \Omega_2) \cup (\Omega_2 \setminus \bar{\Omega}_1), \\ 0 & \text{if } \zeta \in (\Omega_1 \setminus \bar{\Omega}_2). \end{cases}$$

Then f and g are holomorphic in Ω. By Proposition VII.4.6 in [4] and the Riesz functional calculus (see [4, p.201]), we have

$$P(A, \Omega_1)P(A, \Omega_2) = f(A)g(A) = (fg)(A) = P(A, \Omega_1 \cap \Omega_2).$$

Let P and Q be a pair of projections. Suppose that $P - Q$ is compact. Set $T = QP\colon \mathrm{im}\ P \to \mathrm{im}\ Q$.

Lemma 2.2. *The operator T defined above is Fredholm.*

Proof. Set $T_1 = PQP\colon \mathrm{im}\ P \to \mathrm{im}\ P$ and $T_2 = QPQ\colon \mathrm{im}\ Q \to \mathrm{im}\ Q$. Since $P - Q$ is compact, T_1 and T_2 are Fredholm. Now our assertion follows from $\ker T \subset \ker T_1$ and $\mathrm{im}\ T \supset \mathrm{im}\ T_2$.

Define $[P - Q]$ by

$$[P - Q] = \mathrm{ind}\ (QP\colon \mathrm{im}\ P \to \mathrm{im}\ Q). \tag{2.3}$$

Lemma 2.3. *Let $P, Q, R, P_1, Q_1 \in \mathcal{B}(X)$ be projections such that $P - Q$ and $Q - R$ are compact.*

(i) *We have $[P - Q] + [Q - R] = [P - R]$. In particular we have $[P - Q] = -[Q - P]$.*

(ii) *Suppose P_1 and Q_1 are of finite rank and $P_1P = PP_1 = Q_1Q = QQ_1 = 0$. Then we have*

$$[(P + P_1) - (Q + Q_1)] = [P - Q] + [P_1 - Q_1]. \tag{2.4}$$

(iii) *We have*

$$[P - Q] + [(I - P) - (I - Q)] = 0. \tag{2.5}$$

(iv) *Let $T \in \mathcal{B}(X)$ be invertible. Then we have*

$$[TPT^{-1} - TQT^{-1}] = [P - Q]. \tag{2.6}$$

Proof. (i) Since $P - Q$ and $Q - R$ is compact, we have

$$\begin{aligned}[P - Q] + [Q - R] &= \mathrm{ind}\ (QP\colon \mathrm{im}\ P \to \mathrm{im}\ Q) + \mathrm{ind}\ (RQ\colon \mathrm{im}\ Q \to \mathrm{im}\ R)\\ &= \mathrm{ind}\ (RQP\colon \mathrm{im}\ P \to \mathrm{im}\ R)\\ &= \mathrm{ind}\ (RP + R(Q - P)P\colon \mathrm{im}\ P \to \mathrm{im}\ R)\\ &= \mathrm{ind}\ (RP\colon \mathrm{im}\ P \to \mathrm{im}\ R)\\ &= [P - R].\end{aligned}$$

(ii) By (i) we have

$$\begin{aligned}[(P + P_1) - (Q + Q_1)] &= [(P + P_1) - P] + [P - Q] + [Q - (Q + Q_1)]\\ &= \dim P_1X + [P - Q] - \dim Q_1X\\ &= [P - Q] + [P_1 - Q_1].\end{aligned}$$

(iii) Set $R = QP + (I - Q)(I - P)$. Since $P - Q$ is compact, we have

$$[P - Q] + [(I - P) - (I - Q)] = \mathrm{ind}\ R = \mathrm{ind}\ (I - (P - Q)(I - 2P)) = 0.$$

(iv) follows from the definition.

Let $Q_s \in \mathcal{B}(X)$, $0 \le s \le 1$ be a family of projections (we do not assume that the family is continuous on s here).

Definition 2.1.[15,8] *An s-section for Q_s is a continuous curve of projections P_s such that $P_s - Q_s$ are compact operators.*

Definition 2.2. *Let P_s be an s-section for Q_s. Set*

$$T_s = P_sQ_s\colon \mathrm{im}\ Q_s \to \mathrm{im}\ P_s.$$

Then the s-flow of Q_s is defined by

$$\mathrm{sfl}\{Q_s\} = \mathrm{ind}\ T_1 - \mathrm{ind}\ T_0. \tag{2.7}$$

Lemma 2.4. *The s-flow is well defined if there exists an s-section for Q_s.*

Proof. Let R_s be another s-section for Q_s. We claim that $[R_s - P_s]$ is constant. In fact, fix $s \in [0,1]$. By Lemma I.4.10 in [10], for $t \in [0,1]$ close enough to s, there are invertible operators $U_{s,t}, V_{s,t} \in \mathcal{B}(X)$ such that

$$\begin{aligned} &R_t U_{s,t} = U_{s,t} R_s, \quad P_t V_{s,t} = V_{s,t} P_s, \\ &U_{s,t} \to I, \quad V_{s,t} \to I \quad \text{as } t \to s. \end{aligned}$$

So we have

$$\begin{aligned} \lim_{t\to s}[R_t - P_t] &= \lim_{t\to s} \text{ind}\,(P_t R_t\colon \text{im}\, R_t \to \text{im}\, P_t) \\ &= \lim_{t\to s} \text{ind}\,(V_{s,t}^{-1} P_t R_t U_{s,t}\colon \text{im}\, R_s \to \text{im}\, P_s) \\ &= \lim_{t\to s} \text{ind}\,(P_s V_{s,t}^{-1} U_{s,t} R_s\colon \text{im}\, R_s \to \text{im}\, P_s) \\ &= \text{ind}\,(P_s R_s\colon \text{im}\, R_s \to \text{im}\, P_s) \\ &= [R_s - P_s]. \end{aligned}$$

By the fact that $[R_s - P_s] \in \mathbf{Z}$, we obtain the claim.

Now by Lemma 2.3 we have

$$([Q_1 - P_1] - [Q_0 - P_0]) - ([Q_1 - R_1] - [Q_0 - R_0]) = [R_1 - P_1] - [R_0 - P_0] = 0,$$

and our lemma is proved.

Definition 2.3. *Let A be in $\mathcal{B}(X)$. A is said to be admissible if*

(i) $\delta(A) \equiv \inf\limits_{\zeta\in\sigma(A)\setminus i\mathbf{R}} |\Re\zeta| > 0$, *and*

(ii) *$A - \zeta I$ is Fredholm for all $\zeta \in i\mathbf{R}$. Here $\Re\zeta$ is the real part of ζ.*

We denote by $\mathcal{A}(X)$ the set of all admissible operators on X.

Proposition 2.1. *For $A \in \mathcal{A}(X)$, we have*

(i) *$A + B$ is admissible for all $B \in \mathcal{CL}(X)$, and*

(ii) *there is a unique direct sum decomposition*

$$X = X^- \oplus X^0 \oplus X^+, \tag{2.8}$$

where X^-, X^0 and X^+ are invariant closed subspaces of A, and the real part of the spectral points of $A^- \equiv A|X^-$, $A^0 \equiv A|X^0$ and $A^+ \equiv A|X^+$ is negative, zero and positive respectively. In particular there holds $\dim X^0 < +\infty$. *We call X^-, X^0 and X^+ the stable subspace, the centre subspace and the unstable subspace respectively.*

Proof. (i) Let $\Omega := \{\zeta \in \mathbf{C} \mid |\Re\zeta| < \delta(A)\}$. Since A is admissible and B is compact, $A+B-\zeta I$ is Fredholm for all $\zeta \in \Omega$. According to Theorem IV.5.31 in [10], both $\dim\ker(A+B-\zeta I)$ and $\dim\text{coker}\,(A+B-\zeta I)$ are constant for $\zeta \in \Omega$ except for an isolated set of values of ζ. Since $\sigma(A+B)$ is bounded, these intergers must be zero and therefore $\sigma(A+B)\cap\Omega$ is isolated.

We claim that ζ is an isolated eigenvalue of finite algebraic multiplicity for all $\zeta \in \sigma(A+B)\cap\Omega$. In fact, let $\zeta \in \sigma(A+B)\cap\Omega$. By Theorem III.6.17 in [10], there are two closed $(A+B)$-invariant subspaces M and N of X such that $\sigma((A+B)|_M) = \{\zeta\}$ and $\sigma((A+B)|_N) = \sigma(A+B)\setminus\{\zeta\}$. Now Theorem IV.5.33 of [10] shows that $\dim M < +\infty$. From this $A+B$ is admissible.

(ii) follows from Theorems III.6.17 and IV.5.33 in [10].

Definition 2.4. *Let $A \in \mathcal{B}(X)$ be admisible. We call $m^-(A) \equiv \dim X^-$ (may be infinite), $m^+(A) \equiv \dim X^+$ (may be infinite) and $\nu_h(A) \equiv \dim X^0$ the (Morse negative) index, Morse positive index and h-nullity of A respectively. If $\nu_h(A) = 0$, we call A hyperbolic. If $\dim X < \infty$, we call $\operatorname{sign}(A) \equiv m^+(A) - m^-(A)$ the signature of A.*

According to Atiyah-Patodi-Singer[1], we define

Definition 2.5. *Let A be in $\mathcal{A}(X)$. Denote by P_A^-, P_A^0 and P_A^+ the projections defined by (2.8) onto X^-, X^0 and X^+ respectively. The APS projection of A is defined by*

$$Q_A = P_A^+ + P_A^0.$$

Lemma 2.5. *Let A be in $\mathcal{A}(X)$. Then there is an $\epsilon > 0$ such that $B \in \mathcal{A}(X)$ and there is a continuous family of projections $P(B)$ such that $P(B) - Q_B$ is of finite rank for all $B \in \mathcal{B}(A, \epsilon) \equiv \{B \in \mathcal{B}(X) \mid \|B - A\| < \epsilon\}$. We will denote by $\epsilon(A)$ the supremum of all such numbers ϵ.*

Proof. Define

$$\Omega_1 = \Big\{\zeta \in \mathbf{C} \Big| |\Re\zeta| < \frac{1}{2}\delta(A), |\Im\zeta| < \|A\| + 1\Big\},$$
$$\Omega_2 = \Big\{\zeta \in \mathbf{C} \Big| -\frac{1}{2}\delta(A) < \Re\zeta < \|A\| + 1, |\Im\zeta| < \|A\| + 1\Big\}.$$

Set $\gamma = \partial\Omega_1 \cup \partial\Omega_2$. Then $\gamma \subset \rho(A)$. Since γ is compact, there is a $\theta > 0$ such that

$$\|R(\zeta, A)\| \le \theta^{-1}, \qquad \forall \zeta \in \gamma.$$

Set $\epsilon = \min\{\theta, \frac{1}{2}\}$. Then $\gamma \subset \rho(B)$ for all $B \in \mathcal{B}(A, \epsilon)$. For all $B \in \mathcal{B}(A, \epsilon)$, set $P_k(B) = P(B, \Omega_k)$ for $k = 1, 2$. Then P_1 and P_2 are families of continuous projections. Since $P_1(A)$ is of finite rank, $P_1(B)$ is of finite rank by Lemma I.4.10 in [10]. Since every operator in a finite dimentional space is admissible, by Theorem III.6.17 in [10] the spectral points of B near $i\mathbf{R}$ are isolated eigenvalues of finite algebraic multiplicity. Hence $B \in \mathcal{A}(X)$. By Lemma 2.1 we see that $P_2(B)$ is a continuous family of projections and

$$P_1(B)(P_2(B) - Q_B) = (P_2(B) - Q_B)P_1(B) = P_2(B) - Q_B.$$

By taking $P(B) = P_2(B)$, our lemma follows.

Corollary 2.1. *Let A_s, $0 \le s \le 1$ be a curve in $\mathcal{A}(X)$. Then there is a partition $0 = s_0 < s_1 < \cdots < s_n = 1$ of $[0, 1]$ such that Q_{A_s} possesses an s-section on each subinterval $[s_k, s_{k+1}]$, $k = 0, \cdots, n-1$ of $[0, 1]$.*

Proof. Clearly $\epsilon(B) \ge \epsilon(A) - \|B - A\|$ for all $B \in \mathcal{B}(A, \epsilon(A))$ and $\epsilon(A) > 0$ for all $A \in \mathcal{A}(X)$. Set $\delta = \inf_{s \in [0,1]} \epsilon(A_s)$. Since $[0, 1]$ is compact, $\delta > 0$. Since A_s is uniformly continuous, we can choose $n \in \mathbf{N}$ such that $\|A_s - A_t\| < \delta$ for all $s, t \in [0, 1]$ and $|s - t| \le \frac{1}{n}$. Then A_s on each subinterval $[\frac{k}{n}, \frac{k+1}{n}]$, $k = 0, \cdots, n-1$, possesses an s-section by the definition of $\epsilon(A)$.

Definition 2.6. *Let A_s, $0 \le s \le 1$ be a curve in $\mathcal{A}(X)$. Let $0 = s_0 < s_1 < \cdots < s_n = 1$ be a partition of $[0, 1]$ such that Q_{A_s} possesses an s-section on each subinterval $[s_k, s_{k+1}]$, $k = 0, \cdots, n-1$ of $[0, 1]$. Then the spectral flow of A_s is defined by*

$$\operatorname{sf}\{A_s\} = \sum_{k=0}^{n-1} \operatorname{sfl}\{Q_{A_s}, s_k \le s \le s_{k+1}\}. \tag{2.9}$$

Moreover, we define

$$\mathrm{sf}_-\{A_s\} = -\mathrm{sf}\{-A_s\}. \tag{2.10}$$

Lemma 2.6. *The spectral flow is well-defined.*

Proof. Since any two partitions of $[0, 1]$ possesse a common refinement, the lemma follows from Lemma 2.4.

Definition 2.7.[2] *Let X be a Hilbert space. Let D_s, $0 \le s \le 1$, be a family of selfadjoint Fredholm operators such that 0 is either a discrete spectral point of D_s or not in the spectrum of D_s for all $0 \le s \le 1$ and $D_s(|D_s| + I)^{-1}$ is a continuous curve in $\mathcal{B}(X)$. The spectral flow of D_s is given by*

$$\mathrm{sf}\{D_s\} = \mathrm{sf}\{D_s(|D_s| + I)^{-1}\}, \tag{2.11}$$

$$\mathrm{sf}_-\{D_s\} = \mathrm{sf}_-\{D_s(|D_s| + I)^{-1}\}. \tag{2.12}$$

Remark 2.1. By Theorem 1.11 of [8], the above definition coincides with the spectral flow for a curve of selfadjoint Fredholm operators defined in [2].

Lemma 2.7. *Let A be in $\mathcal{A}(X)$ and $B \in \mathcal{CL}(X)$. Then $Q_{A+B} - Q_A$ is compact.*

Proof. Let $\delta{:}= \min\{\delta(A), \delta(B)\}$ and $M{:}= \|A\| + \|B\| + 1$. Set

$$\Omega{:}= \Big\{\zeta \in \mathbf{C}\Big| -\frac{\delta}{2} < \Re\zeta < M, |\Im\zeta| < M\Big\}.$$

Then by the definition of the APS projection we have

$$\begin{aligned} Q_{A+B} - Q_A &= -\frac{1}{2\pi i}\int_{\partial\Omega}(R(\zeta, A + B) - R(\zeta, A))d\zeta \\ &= \frac{1}{2\pi i}\int_{\partial\Omega} R(\zeta, A + B)BR(\zeta, A)d\zeta. \end{aligned}$$

Since B is compact, $Q_{A+B} - Q_B$ is compact.

Now we can give the definition of relative Morse index.

Definition 2.8. *Let A be in $\mathcal{A}(X)$ and $B \in \mathcal{CL}(X)$. The relative Morse index of the pair A, $A + B$ is defined by*

$$I(A, A + B) = -\mathrm{sf}\{A + B_s\}, \tag{2.13}$$

where B_s, $0 \le s \le 1$ is a curve in $\mathcal{CL}(X)$ such that $B_0 = 0$ and $B_1 = B$.

Remark 2.2. Note that by Lemma 2.7, we can take $P_s = Q_A$ as the s-section of Q_{A+B_s}. So by Lemma 2.4, (2.13) is well defined.

The following proposition gives the basic properties of the spectral flow.

Proposition 2.2. *Let A_s, $0 \le s \le 1$ be a curve in $\mathcal{A}(X)$.*

(a) (*Catenation*) *Assume $t \in [0, 1]$. Then we have*

$$\mathrm{sf}\{A_s, 0 \le s \le t\} + \mathrm{sf}\{A_s, t \le s \le 1\} = \mathrm{sf}\{A_s, 0 \le s \le 1\}. \tag{2.14}$$

(b) (*Homotopy*) *Let $A(s, t)$, $0 \le s, t \le 1$ be a continuous family in $\mathcal{A}(X)$. Then we have*

$$\mathrm{sf}\{A(s, t), (s, t) \in \partial([0, 1] \times [0, 1])\} = 0. \tag{2.15}$$

(c) *There holds*

$$\mathrm{sf}\{A_s\} - \mathrm{sf}_-\{A_s\} = \nu_h(A_1) - \nu_h(A_0). \tag{2.16}$$

In particular, suppose that $m^-(A_0) < \infty$. Then $m^-(A_1) < \infty$ and we have

$$\mathrm{sf}\{A_s\} = m^-(A_0) - m^-(A_1). \tag{2.17}$$

(d) (*Product*) *Let P_s be a curve of projections on X such that $P_sA_s = A_sP_s$ for all $s \in [0,1]$. Set $Q_s = I - P_s$. Then we have*

$$\mathrm{sf}\{A_s\} = \mathrm{sf}\{P_sA_sP_s\} + \mathrm{sf}\{Q_sA_sQ_s\}. \tag{2.18}$$

(e) *For $A \in \mathcal{A}(X)$, there exists an $\epsilon > 0$ such that for all curves A_s in $\mathcal{B}(A,\epsilon) \equiv \{B \in \mathcal{B}(X) \mid \|B - A\| < \epsilon\}$ with endpoints $A_0 = A$, $A_1 = B$, $I(A,B) \equiv -\mathrm{sf}\{A_s, 0 \le s \le 1\}$ is well defined and satisfies*

$$0 \le I(A,B) \le \nu_h(A) - \nu_h(B). \tag{2.19}$$

(f) (*Zero*) *Suppose that $\nu_h(A_s)$ is constant for every $s \in [0,1]$. Then* $\mathrm{sf}\{A_s\} = 0$.

(g) *Let T_s, $0 \le s \le 1$, be a curve of invertible operators in $\mathcal{B}(X)$. Then we have*

$$\mathrm{sf}\{T_sA_sT_s^{-1}\} = \mathrm{sf}\{A_s\}. \tag{2.20}$$

Proof. By Lemma 2.5 and the definition of the spectral flow, we can assume that Q_{A_s} possesses an s-section R_s and $R_sA_s = A_sR_s$ on $[0,1]$ without loss of generality.

(a) follows from the definition.

(b) Set $\delta = \inf_{(s,t)\in[0,1]\times[0,1]} \epsilon(A(s,t))$, where $\epsilon(A)$ is defined by Lemma 2.5. Since $[0,1]\times[0,1]$ is compact, $\delta > 0$. Since $A(s,t)$ is continuous on $[0,1] \times [0,1]$, we can choose $n \in \mathbf{N}$ such that $\|A(s_1,t_1) - A(s_2,t_2)\| < \delta$ for all $s_1, s_2, t_1, t_2 \in [0,1]$, $|s_1 - s_2| \le \frac{1}{n}$ and $|t_1 - t_2| \le \frac{1}{n}$. Set $\Omega(k,l) = \left\{(s,t)\middle| \frac{k}{n} < s < \frac{k+1}{n}, \frac{l}{n} < t < \frac{l+1}{n}\right\}$ for all $k, l = 0, \cdots, n-1$. By the definition of $\epsilon(A)$, applying Lemma 2.5 to $A(\frac{k}{n},\frac{l}{n}) + (A(s,t) - A(\frac{k}{n},\frac{l}{n}))$ for $(s,t) \in \Omega(k,l)$, we obtain

$$\mathrm{sf}\{A(s,1)\} - \mathrm{sf}\{A(s,0)\} = \sum_{k,l=0}^{n-1} \mathrm{sf}\{A(s,t), (s,t) \in \partial\Omega(k,l)\} = 0.$$

(c) Since $I - R_s$ is an s-section of $P^-_{A_s}$, by Lemma 2.1 and Lemma 2.3 we have

$$\begin{aligned}\mathrm{sf}\{A_s\} &= [Q_{A_1} - R_1] - [Q_{A_0} - R_0] \\ &= -[P^-_{A_1} - (I - R_1)] + [P^-_{A_0} - (I - R_0)] \\ &= -([(Q_{-A_1} - (I - R_1)] - \nu_h(A_1)) + ([Q_{-A_0} - (I - R_0)] - \nu_h(A_0)) \\ &= \mathrm{sf}_-\{A_s\} + \nu_h(A_1) - \nu_h(A_0).\end{aligned} \tag{2.21}$$

So (2.16) is proved. In the special case that $m^-(A_0) < \infty$, $P^-_{A_0}$ is compact. By the Riesz-Schauder theory and Lemma I.4.10 in [10], the projections $P^-_{A_s}$ and $I - R_s$ are of finite rank and $\dim(I - R_s)X$ =constant for all $s \in [0,1]$. So (2.17) holds by (2.21).

(d) Since $P_sA_s = A_sP_s$, and P_s is a projection, $P_sA_sP_s$ and $Q_sA_sQ_s$ are curves of admissible operators. $Q_{P_sA_sP_s} = P_sQ_{A_s}P_s$, and $P_sR_sP_s$ is an s-section of $P_sQ_{A_s}P_s$. Since $Q_s = I - P_s$, by the definition of the spectral flow we have

$$\begin{aligned}\mathrm{sf}\{A_s\} &= [Q_{A_1} - R_1] - [Q_{A_0} - R_0] \\ &= ([P_1Q_{A_1}P_1 - P_1R_1P_1] + [Q_1Q_{A_1}Q_1 - Q_1R_1Q_1]) \\ &\quad - ([P_0Q_{A_0}P_0 - P_0R_0P_0] + [Q_0Q_{A_0}Q_0 - Q_0R_0Q_0]) \\ &= \mathrm{sf}\{P_sA_sP_s\} + \mathrm{sf}\{Q_sA_sQ_s\}.\end{aligned}$$

(e) Let ϵ and $P_i = P_i(B)$, $i = 1, 2$ be defined in the proof of Lemma 2.5. Then $P_1(B)$ and $P_2(B)$ are countinuous family of projections parametrized for $B \in \mathcal{B}(A,\epsilon)$. Hence $P_2(A_s)$ is an s-section of Q_{A_s} for all curves A_s, $s \in [0,1]$ in $\mathcal{B}(0,\epsilon)$. By the definition of the spectral flow, Lemma 2.1, Lemma 2.3 and Theorem I.4.10 in [10], we have

$$\begin{aligned} I(A,B) &= -\mathrm{sfl}\{Q_{A_s}, 0 \le s \le 1\} = -([Q_B - P_2(B)] - [Q_A - P_2(A)]) \\ &= -([Q_B - (Q_B + P_1(B)P_B^-)] - [Q_A - Q_A]) = \dim\mathrm{im}\, P_1(B)P_B^- \\ &\le \dim\mathrm{im}\, P_1(B) - \dim\mathrm{im}\, P_1(B)P_B^0 = \dim\mathrm{im}\, P_1(A) - \dim\mathrm{im}\, P_B^0 \\ &= \dim\mathrm{im}\, P_A^0 - \dim\mathrm{im}\, P_B^0 = \nu_h(A) - \nu_h(B), \end{aligned}$$

and $I(A,B) = \dim\mathrm{im}\, P_1(B)P_B^0 \ge 0$.

(f) follows from (e) and the compactness of $[0,1]$. Note that in this case we have

$$\inf_{s\in[0,1]} \delta(A_s) > 0,$$

where $\delta(A)$ is defined in Definition 2.3. In fact it also follows from the compactness of $[0,1]$ and the proof of (e).

(g) Clearly $T_sA_sT_s^{-1}$ is a curve in $\mathcal{A}(X)$ and the APS projection of $T_0A_sT_0^{-1}$ is $T_0Q_{A_s}T_0^{-1}$. Since $T_0R_sT_0^{-1}$ is an s-section of $T_0Q_{A_s}T_0^{-1}$, by (a), (b), (e), Lemma 2.3 and the definition of the spectral flow we have

$$\begin{aligned} \mathrm{sf}\{T_sA_sT_s^{-1}\} &= \mathrm{sf}\{T_0A_sT_0^{-1}\} + \mathrm{sf}\{T_sA_1T_s^{-1}\} = \mathrm{sf}\{T_0A_sT_0^{-1}\} \\ &= [T_0Q_{A_1}T_0^{-1} - T_0R_1T_0^{-1}] - [T_0Q_{A_0}T_0^{-1} - T_0R_0T_0^{-1}] \\ &= [Q_{A_1} - R_1] - [Q_{A_0} - R_0] = \mathrm{sf}\{A_s\}. \end{aligned}$$

Let A be in $\mathcal{A}(X)$. Define $d(A)$ by the maximum positive number of ϵ such that (e) of Proposition 2.2 holds.

Remark 2.3. Suppose X is a Hilbert space and $A \in \mathcal{A}(X)$ is a bounded selfadjoint Fredholm operator. Then we have

$$d(A) = \begin{cases} \gamma(A), & \text{if } A \text{ is invertible,} \\ \dfrac{1}{2}\gamma(A), & \text{if } A \text{ is not invertible,} \end{cases} \tag{2.22}$$

where $\gamma(A) = \min\{|\lambda| \mid \lambda \in \sigma(A) \text{ and } \lambda \ne 0\}$.

Corollary 2.2. *Let X be a Hilbert space. Let A be a closed selfadjoint Fredholm operator on X with compact resolvent, and B be a bounded selfadjoint operator on X. Set $K = (|A| + I)^{-1}$. Then we have*

$$I(KA, K(A+B)) = I(A, A+B), \tag{2.23}$$

where KA, $K(A+B)$ are linear operators defined on the Hilbert space $V = D(|A|^{\frac{1}{2}})$ with graph norm $\|x\|_V = (\||A|^{\frac{1}{2}}x\|_X^2 + \|x\|_X^2)^{\frac{1}{2}}$.

Proof. Note that KA and KB are selfadjoint on V. Set $L_t = (|A+tB| + I)^{-\frac{1}{2}}$ for all $0 \le t \le 1$, then L_t are compact operators on X. By the definition of the spectral flow and Proposition 2.2, we have

$$\begin{aligned} I(KA, K(A+B)) &= -\mathrm{sf}\{K(A+sB), 0 \le s \le 1\} = -\mathrm{sf}\{L_0(A+sB)L_0, 0 \le s \le 1\} \\ &= -\mathrm{sf}\{L_s(A+sB)L_s, 0 \le s \le 1\} + \mathrm{sf}\{L_t(A+B)L_t, 0 \le t \le 1\} \\ &= -\mathrm{sf}\{L_s^2(A+sB), 0 \le s \le 1\} = -\mathrm{sf}\{(A+sB, 0 \le s \le 1\} \\ &= I(A, A+B). \end{aligned}$$

From (e) of Proposition 2.2 we have

Corollary 2.3. *The set* $\{A \in \mathcal{A}(X) \mid \nu_h(A) = 0\}$ *is open and dense in* $\mathcal{A}(X)$.

§3. Two Index Theorems

In this subsection we will give elementary proofs of the index Theorems 3.1 and 3.2 for the Galerkin approximation procedure and the saddle point reduction respectively. These theorems have been studied by many authors (for the first one, see [3] and [9], and for the second, see [5, 11, 12, 13]).

Firstly we give two lemmata to show that these theorems are applicable.

Lemma 3.1. *Let X be a reflexive Banach space. Let T be a compact linear operator defined on X, and $\{P_n\}$ be a sequence of projections on X such that $P_n \to I$ strongly as $n \to \infty$. Suppose that $P_mP_n = P_{\min\{m,n\}}$ for all $m, n \in \mathbf{N}$. Then for all $\epsilon > 0$, there exists an integer n_0 such that $\|T - P_nTP_n\| < \epsilon$ for all $n \geq n_0$.*

Proof. Since $P_n \to I$ strongly, by Example 21.3 in [6], $P_n^* \to I$ strongly. By Banach's uniform boundedness principle, P_n is uniformly bounded. By Schauder theorem, T^* is also compact. Now standard trick shows that $\|T - P_nT\| \to 0$ and $\|T^* - P_n^*T^*\| \to 0$ as $n \to \infty$. Since

$$\|T - P_nTP_n\| \leq \|T - P_nT\| + \|P_n\| \cdot \|T^* - P_n^*T^*\|,$$

we have $\|T - P_nTP_n\| \to 0$.

The following lemma can be found in standard functional analysis text books.

Lemma 3.2. *Let X be a Hilbert space. Let T be a compact linear operator defined on X, and $\{P_n\}$ be a sequence of orthogonal projections such that $P_n \to I$ strongly as $n \to \infty$. Then for all $\epsilon > 0$, there exists an integer n_0 such that $\|T - P_nTP_n\| < \epsilon$ for all $n \geq n_0$.*

Theorem 3.1 (Galerkin Approximation). *Let A be in $\mathcal{A}(X)$ and $B \in \mathcal{CL}(X)$ be such that $\nu_h(A) = \nu_h(A + B) = 0$. Let P be a finite dimensional projection such that*

$$\|PA + AP - 2PAP\| < d(A), \tag{3.1}$$

$$\|PA + AP - 2PAP + B - PBP\| < d(A + B). \tag{3.2}$$

Then we have

$$I(A, A + B) = I(PAP, P(A + B)P). \tag{3.3}$$

Proof. Set $Q = I - P$. Since $PAP + QAQ - A = 2PAP - AP - PA$, by (d) of Proposition 2.2 we have

$$\begin{aligned} I(A, A + B) &= I(A, PAP + QAQ) + I(PAP + QAQ, PAP + QAQ + PBP) \\ &\quad + I(PAP + QAQ + PBP, A + B) \\ &= I(PAP + QAQ, PAP + QAQ + PBP) \\ &= I(PAP, P(A + B)P) + I(QAQ, QAQ) \\ &= I(PAP, P(A + B)P), \end{aligned}$$

where the hyperbolicity of A, $A + B$ is used.

In the above proof we have used ideas of [3] and [9].

Remark 3.1. When $PA = AP$, the condition that A is hyperbolic and P is finite dimensional can be dropped in the above theorem.

Theorem 3.2 (Saddle Point Reduction). *Let X be a Hilbert space. Let A be a bounded selfadjoint Fredholm operator on X and B be a compact selfadjoint operator on X. Let P be an orthogonal projection such that A and B are in the block form*

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{3.4}$$

with respect to the orthogonal decomposition $\mathcal{H} = \mathrm{im}\, P + \mathrm{im}\,(I-P)$. If A_2 is invertible and $\|B_{22}\| < C(A_2)$, we have

$$I(A, A+B) = I(A_1, A_1 + B_{11} + B_{12}(A_2 + B_{22})^{-1} B_{21}). \tag{3.5}$$

Proof. Let $D_s = A + sB$ and

$$T_s = \begin{pmatrix} I & 0 \\ -s(A_2 + sB_{22})^{-1} B_{21} & I \end{pmatrix}$$

for all $s \in [0,1]$. Let $H(s,t) = T_{st}^* D_s T_{st}$ for all $0 \le s \le 1$. By Proposition 2.2 we have

$$\begin{aligned} I(A, A+B) &= -\mathrm{sf}\{D_s; 0 \le s \le 1\} \\ &= -\mathrm{sf}\{H(s,0); 0 \le s \le 1\} - \mathrm{sf}\{H(1,t); 0 \le t \le 1\} \\ &= I(H(0,0), H(1,0)) + I(H(1,0), H(1,1)) \\ &= I(H(0,0), H(1,1)) = I(A, T_1^*(A+B)T_1) \\ &= I(A_1, A_1 + B_{11} + B_{12}(A_2 + B_{22})^{-1} B_{21}) + I(A_2, A_2 + B_{22}) \\ &= I(A_1, A_1 + B_{11} + B_{12}(A_2 + B_{22})^{-1} B_{21}). \end{aligned}$$

§4. Calculation of the Spectral Flow

The object of this section is to prove Theorem 4.1 below. This theorem can be used to calculate the spectral flow in some special cases.

Definition 4.1. *Let A_s, $0 \le s \le 1$, be a curve in $\mathcal{A}(X)$.*

(i) *A crossing for A_s is a number $t \in [0,1]$ such that $\nu_h(A_t) \neq 0$.*

(ii) *Set $P_s = P^0_{A_s}$. A crossing t is called regular if A_s is differentiable at $s = t$ and $P_t \dot{A}_t P_t$ is hyperbolic, where " $\cdot$ " denotes $\frac{d}{ds}$.*

(iii) *A crossing t is called simple if it is regular and $\nu_h(A_t) = 1$.*

Theorem 4.1. *Let A_s, $-\epsilon \le s \le \epsilon$ ($\epsilon > 0$), be a curve in $\mathcal{A}(X)$. Suppose that 0 is a regular crossing of A_s. Set $P = P^0(A_0)$, $A = A_0$ and $B = \frac{d}{ds}\big|_{s=0} A_s$. Assume that*

$$P(AB - BA)P = 0. \tag{4.1}$$

Then there is a $\delta \in (0, \epsilon)$ such that $\nu_h(A_s) = 0$ for all $s \in [-\delta, 0) \cup (0, \delta]$ and

$$\mathrm{sf}\{A_s, 0 \le s \le \delta\} = -m^-(PBP), \tag{4.2}$$

$$\mathrm{sf}\{A_s, -\delta \le s \le 0\} = m^+(PBP). \tag{4.3}$$

Proof. Let $I(A, A_s)$ be defined in the part (e) of Proposition 2.2 for s small. Let $P_1(A_s)$ be the projections defined in the proof of Lemma 2.5 for s small. Then $P_1(A) = P$. Define

$$U(s) = (P_1(A_s)P + (I - P_1(A_s))(I-P))(I - (P_1(A_s) - P)^2)^{-\frac{1}{2}}$$

for small s. Then $I - (P_1(A_s) - P)^2$ commutes with $P_1(A_s)$ and P. By §I.4.6 in [10], $U(s) \in \mathcal{B}(X)$ is invertible and $U(s)P = P_1(A_s)U(s)$. By the proof of Proposition 2.2 (e), we have

$$\begin{aligned} I(A, A_s) &= \dim \operatorname{im} P_1(A_s)P^-(A_s) = m^-(P_1(A_s)A_sP_1(A_s)) \\ &= m^-(U(s)PU(s)^{-1}A_sU(s)PU(s)^{-1}) = m^-(PU(s)^{-1}A_sU(s)P) \\ &= I(PAP, PU(s)^{-1}A_sU(s)P). \end{aligned} \tag{4.4}$$

Similarly we have

$$\nu_h(A_s) = \nu_h(PU(s)^{-1}A_sU(s)P). \tag{4.5}$$

Claim. 0 is a regular crossing of $PU(s)^{-1}A_sU(s)P$ and there holds

$$P\left\{\frac{d}{ds}\Big|_{s=0}(U(s)^{-1}A_sU(s))\right\}P = PBP.$$

In fact, by the definition of $P_1(s)$, we have

$$P_1(A_s) - P = -\frac{1}{2\pi i}\int_{\partial\Omega_1} R(\zeta, A_s)(A_s - A)R(\zeta, A).$$

Since 0 is a regular crossing of A_s, there holds

$$\frac{d}{ds}\Big|_{s=0}P_1(A_s) = -\int_{\partial\Omega_1} R(\zeta, A)BR(\zeta, A).$$

Now our claim follows from direct computation and (4.1).

Now we can work in the finite dimensional vector space im P. Since PBP commutes with PAP, we can assume that they are both in Jordan normal forms. Since PBP is hyperbolic, our theorem is valid for the curve $PU(s)^{-1}A_sU(s)P$. Since im P is finite dimensional, by the proof of Lemma 2.5 and (e) of Proposition 2.2, there is a constant $\delta > 0$ such that $d(P(A + sB)P) > \delta s$ for s small. By (4.4) and (4.5) our theorem is proved.

Next we derive two well-known results for selfadjoint operators from Theorem 4.1.

Corollary 4.1.[17] *Let X be a Hilbert space and A be a selfadjoint operator on X with compact resolvent. Let $B_s \in \mathcal{B}(X)$ be a C^1 curve of selfadjoint operators. Set $P = P^0(A)$ and $B = \frac{d}{ds}|_{s=0} B_s$. Assume that PBP is nondegenerate. Then there is a $\delta \in (0, \epsilon)$ such that* $\dim\ker(A + sB) = 0$ *and*

$$I(A, A + B_s) = m^-(PBP) \tag{4.6}$$

for all $s \in (0, \delta]$.

Proof. Set $L = (|A| + I)^{-\frac{1}{2}}$. Then the curve $L(A + B_s)L$ satisfies the assumption of Theorem 4.1. By Corollary 2.2 and Theorem 4.1 we have

$$\begin{aligned} &\dim\ker(A + B_s) = \dim\ker L(A + B_s)L = 0, \\ &\quad I(A, A + B_s) = I(LAL, L(A + B_s)L) = m^-(PLBLP) = m^-(PBP). \end{aligned}$$

Motivated by the signature identity (see Lemma 5.2 of [16]), we have

Corollary 4.2. *Let X be a Hilbert space. Let A be a closed selfadjoint operator with compact resolvent and $B \in \mathcal{B}(X)$ be selfadjoint. Suppose that A is invertible and B is definite. Then*

$$I(A, A + B) = I(-B^{-1}, -B^{-1} - A^{-1}). \tag{4.7}$$

Proof. Let P_s and Q_s be the orthogonal projection onto $\ker(A + sB)$ and $\ker(-B^{-1} -$

sA^{-1}) respectively. By Theorem 4.1 and Corollary 4.1 we have

$$I(A, A+B) = -\sum_s \operatorname{sign}(P_s B P_s) - m^+(P_1 B P_1),$$

$$\begin{aligned} I(-B^{-1}, -B^{-1}-A^{-1}) &= -\sum_s \operatorname{sign}(-Q_s A^{-1} Q_s) - m^+(-Q_1 A^{-1} Q_1) \\ &= -\sum_s \operatorname{sign}(s^{-1} Q_s B^{-1} Q_s) - m^+(Q_1 B^{-1} Q_1). \end{aligned}$$

Since $\ker(-B^{-1} - sA^{-1}) = B\ker(A + sB)$, Equation (4.7) is proved.

Acknowledgment. We would like to thank Prof. Zhang Weiping for his suggestion of using the spectral flow to study the Maslov-type index theory.

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