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# THE EXISTENCE OF J-HOLOMORPHIC CURVES AND APPLICATIONS TO THE WEINSTEIN CONJECTURE**

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**Abstract**

The author first proves the existences of $J-$holomorphic curves in the symplectizations of Legendre fibrations and then as an application confirms the Weinstein conjectures on contact manifolds of Legendre fibrations. As a corollary, a new proof on the theorem due to Hofer, Viterbo, Gluck, Ziller, Weinstein and Ljusternik-Fet Theorem is provided, which is quite different from their original proofs.

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## §1. Introductionand and Results

We first recall several definitions in [1]. A contact structure on a manifold is a field of a tangent hyperplanes (contact hyperplanes) that is "nondegenerate" at any point, locally such a field is defined as the field of zeros of a $1-$form $\alpha$, called a contact form. The nondegeneracy condition is that $d\alpha$ is nondegenerate on the hyperplanes on which $\alpha$ vanishes; equivalently, in $(2n+1)$-space:

$$\alpha \wedge (d\alpha)^n \neq 0.$$

The important example of contact manifold is the well-known projective cotangent bundles definded as follows: Let $N = T^*M$ be the cotangent bundle of the smooth connected compact manifold $M$. $N$ carries a canonical symplectic structure $\omega = -d\lambda$, where $\lambda = \sum_{i=1}^{n} y_i dx_i$ is the Liouville form on $N$ (see [1,9]). Let $P = PT^*M$ be the oriented projective cotangent bundle of $M$, i.e. $P = \bigcup_{x\in M} PT^*_x M$. It is well known that $P$ carries a canonical contact structure induced by the Liouville form and the projection $\pi : T^*M \mapsto PT^*M$. Now we fix a Riemannian metric on $M$ and pull back the symplectic structure in $T^*M$ via the "Rieszmap" $TM \simeq T^*M : x \mapsto \langle x, \cdot\rangle$. We get a symplectic structure $\omega = d\lambda$ and the Liouville form $\lambda$ on $TM$. The ristriction of the Liouville form $\lambda$ on the unit sphere bundle $STM$ defines a contact structure $\xi$ on $STM$. It is easy to see that $STM$ is contact diffeomorphic to $PT^*M$.

Let $\Sigma$ be a smooth closed oriented manifold of dimension $2n+1$. A contact form on $\Sigma$ is a 1-form such that $\lambda \wedge (d\lambda)^n$ is a volume form on $\Sigma$. Associated to $\lambda$ there are two important

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structures, first of all the so-called Reed vectorfield $X = X_\lambda$ defined by $i_X\lambda \equiv 1, \ \ i_X d\lambda \equiv 0$ and secondly the contact structure $\xi = \xi_\lambda \mapsto \Sigma$ given by $\xi_\lambda = \ker(\lambda) \subset T\Sigma$. Let $W = R \times \Sigma$ and put $\xi = \ker(\lambda)$. Then $d\lambda$ is a symplectic structure for the vector bundle $\xi \to \Sigma$. We choose a complex structure $J$ for $\xi$ such that $g_J : d\lambda \circ (Id \times J)$ is a metric for $\xi \to \Sigma$. As in [7], we define an almost complex structure $\widetilde{J}$ on $W$ by

$$\widetilde{J}(t,u)(h,k) = (-\lambda(u)(k), J(u)\pi k + hX(u)).$$

$\pi : T\Sigma \mapsto \xi$ is the bundle projection along $RX \mapsto \xi$ and $X$ the Reeb vectorfield associated to $\lambda$.

Let $\Sigma = \{\psi \in C^\infty(R, [\frac{1}{2}, 1]) | 0 \le \psi' \le 1\}$ as in [7]. To $\psi \in \Sigma$ we associate an exact form $\omega_\psi$ on $W$ by $\omega_\psi = d(\psi\lambda)$. Consider a solution of

$$\tilde{u}_s + \widetilde{J}(\widetilde{u})\widetilde{u}_t = 0, \tag{1.1}$$

$$\tilde{u} = (a, u) : C \mapsto W. \tag{1.2}$$

We define the $\Sigma$-energy of a solution of (1.1)–(1.2) as

$$E_\Sigma(\widetilde{u}) = \sup_{\psi \in \Sigma} \int_C \tilde{u}^* \omega_\psi. \tag{1.3}$$

**Theorem 1.1.** *There exists a non-constant solution $\tilde{u} = (a, u) : C \to W$ of the partial differential equations* (1.1)–(1.2) *with $E_\Sigma(\tilde{u}) < \infty$.*

Then as in [3,6,7,10], the above theorem implies

**Theorem 1.2.** *Let $STM$ be a unit sphere bundle of $M$ and $\lambda = f \sum_{i=1}^{n} y_i dx_i$ be a contact form on $STM$. If $M$ is simply connected, then $(STM, \lambda)$ carries a closed orbit of Reeb vectorfield.*

The above theorem was proved by means of variational method in [4,8,16]. If $f \equiv 1$ in the above theorem, then the Reeb flow is the well-known geodesic flow and the closed orbit of Reeb flow in this case corresponds to the closed geodesics. Therefore we have

**Theorem 1.3 (Ljusternik-Fet).**[9] *Every simply connected Riemannian manifold has at least one closed geodesics.*

Therefore we get a new proof on the well-known Ljusternik-Fet Theorem without using the classical minimax principle. It is well known that $S^3$ with standard contact structure is a Legendre fibration (see[1]), so we have

**Theorem 1.4.** *There exists a non-constant solution $\tilde{u} = (a, u) : C \to W = S^3 \times R$ of the partial differential equations* $(1.1) - (1.2)$ *with $E_\Sigma(\tilde{u}) < \infty$.*

This result completes the ones in [7] in which the existence of $J-$planes in $S^3 \times R$ with overtwisted contact form $\lambda$ was proved.

**Theorem 1.5.**[12] *Let $S^3$ be a unit sphere with $\lambda = f \sum_{i=1}^{2} y_i dx_i$ being a contact form on $S^3$. Then $(S^3, \lambda)$ carries a closed orbit of Reeb vectorfield.*

Theorem 1.5 was proved by means of variational method in [12] and the other method in [7].

**Sketch of Proofs.** In Section 2, by comparing the topology of the space $\mathcal{D}(\Sigma)$ of disks in manifold $\Sigma$ with the one of the loop space $\Lambda(B)$ of base manifold $B$, we know that the global topologies of these spaces are different. In Section 3, we construct a Fredholm map from $\mathcal{D}(\Sigma)$ to $\Lambda(B)$ as in [6]. In Section 4, we use a theorem due to K. K. Mukherjea[11] to conclude that the Fredholm map constructed in Section 5 is not proper. In the final section, we use the Sacks-Uhlenbeck-Gromov's trick as in [3,6,7,10,13] to conclude that the non-properness of the Fredholm map implies the existence of $J-$holomorphic planes, and then we observed that the existence of $J$-holomorphic plane implies the existence of periodic orbit (see [3,6,7,10]).

## §2. The Topology of Disk Space and Loop Space

Let $\Sigma \to B$ be a contact manifold of Legendre fibration with the contact form $\lambda$. Let $W = \Sigma \times R \to B$ be the Lagrangian fibration induced by the Legendre fibration and its symplectization $(\Sigma \times R, d(e^t\lambda))$. Let $D \subset C$ be the closed unit disk in the complex plane, i.e. $D = \{Z \in C||z| \leq 1\}$, and $\mathcal{K}(W) = C^\infty(D, W)$ be the space of all smooth maps from $D$ to $W$ with compact open topology. Thus a sub-basis for the open sets in $\mathcal{K}(W)$ is given by taking $K \subset D$ a compact subset, $U \subset W$ an open set, and letting $\langle K, U\rangle$ be all the maps $f : D \to W$ with $f(K) \subset U$. Let $\mathcal{K}_p(W) = \{f \in \mathcal{K}(W)|f(1) = p\}$. We call $\mathcal{K}$ a disk space and $\mathcal{K}_p$ the disk space with base point $p$. Similarly let $\Omega(B)$ be the space of all smooth maps from $S^1$ to $B$ with compact open topology. Let $\Omega(B, b)$ be the subspace of $\Omega(B)$ with base point $p$, i.e. $\Omega(B, b) = \{z \in \Omega(B)|z(0) = b\}$.

**Proposition 2.1.** *The disk space $\mathcal{K}_p(W)$ is contractible.*

**Proof.** Define $\mathcal{K}_p(W) \times I \to \mathcal{K}_p(W)$ by $h(u, t)(z) = u(1 - t + tz)$. Clearly, $h(u, 0)$ is the constant map at $p$ for any $u \in \mathcal{K}_p(W)$ and $h(u, 1) = u$.

In the following, we shall study the algebraic topology of loop space $\Omega(B, b)$. Define the path space $\mathcal{L}(B, b)$ based at $b \in B$ to be the set of all paths given by $w$: $I \to B, w(0) = q$, and with the compact open topology as above. Define $\pi : \mathcal{L}(B) \to B$ by $\pi(w) = w(1)$.

**Lemma 2.1.**[5,p.13] *$\pi : \mathcal{L}(B) \to B$ is a fibration.*

**Lemma 2.2.**[5] *$\mathcal{L}(B, b)$ is contractible (i.e., homotopically equivalent to a point).*

**Proposition 2.2.**[9] *If $B$ is a simply connected compact manifold, then the loop space $\Omega(B, b)$ is not contractible.*

Now we consider the Hilbert topology on the disk space and the loop space. Assume that $H^{k,2}(D, W)$ is the Sobolev space and $\mathcal{D}(W) = H^{k,2}(D, W)$.

**Lemma 2.3.**[6,9,13] *For $k \geq 2$, $\mathcal{D}(W)$ is a Hilbert manifold.*

**Lemma 2.4.** *For $k \geq 2$, $\mathcal{D}(W)$ and $\mathcal{D}(W, p) = \{u \in \mathcal{D}(W)|u(1) = p\}$ are weakly homotopically equivalent to $\mathcal{K}(W)$ and $\mathcal{K}(W, p)$ respectively.*

**Proof.** Apply the Palais-Svarc Lemma, and see [9, chapter 1] for a similar proof.

Similarly we have

**Lemma 2.5.** *For $k \geq 1$, let $\Lambda(B) = H^{k,2}(S^1, B)$ and $\Lambda(B, b) = \{z \in \Lambda(B)|z(0) = b\}$. Then $\Lambda(B)$ and $\Lambda(B, b)$ are Hilbert manifolds. Moreover $\Lambda(B)$ and $\Lambda(B, b)$ are weakly homotopically equivalent to $\Omega(B)$ and $\Omega(B, b)$ respectively.*

Finally, by Lemmas 2.3– 2.5 and Propositions 2.1, 2.2, one has the following crucial result.

**Proposition 2.3.** *The Hilbert manifold $\mathcal{D}(W, p)$ is contractible, but the Hilbert manifold $\Lambda(B, b)$ is non-contractible if $B$ is a simply connected closed manifold.*

## §3. Fredholm Theory

### 3.1. Linear Fredholm Operator.

For $3 < k < \infty$, consider the Hilbert space $V_k$ consisting of all maps $u \in H^{k,2}(D, C^n)$, such that $u(z) \in R^n \subset C^n$ for almost all $z \in \partial D$. $L_{k-1}$ denotes the usual Hilbert $L_{k-1}$-space

$H_{k-1}(D, C^n)$. We define an operator $\bar{\partial} : V_p \mapsto L_p$ by

$$\bar{\partial} u = u_s + i u_t, \tag{3.1}$$

where the coordinates on $D$ are $(s,t) = s + it$, $D = \{z||z| \leq 1\}$. The following result is well known.

**Proposition 3.1.**[6] *$\bar{\partial} : V_p \mapsto L_p$ is a surjective real linear Fredholm operator of index $n$. The kernel consists of the constant real valued maps.*

Let $(C^n, \sigma = -\text{Im}(\cdot,\cdot))$ be the standard symplectic space. We consider a real $n$-dimensional plane $R^n \subset R^{2n}$. It is called Lagrangian if the skew-scalar product of any two vectors of $R^n$ equals zero. For example, the plane $p = 0$ and $q = 0$ are Lagrangian subspaces. The manifold of all (nonoriented) Lagrangian subspaces of $R^{2n}$ is called the Lagrangian-Grassmanian $\Lambda(n)$. One can prove that the fundamental group of $\Lambda(n)$ is free cyclic, i.e. $\pi_1(\Lambda(n)) = Z$. Next assume that $(\Gamma(z))_{z \in \partial D}$ is a smooth map associating to a point $z \in \partial D$ a Lagrangian subspace $\Gamma(z)$ of $C^n$, i.e. $(\Gamma(z))_{z\in\partial D}$ defines a smooth curve $\alpha$ in the Lagrangian-Grassmanian manifold $\Lambda(n)$. Since $\pi_1(\Lambda(n)) = Z$, we have $[\alpha] = ke$. We call integer $k$ the Maslov index of curve $\alpha$, and denote it by $m(\Gamma)$ (see [1,6]).

Now let $z : S^1 \mapsto W$ be a smooth curve, $\pi \circ z : S^1 \to W \to B$ be a smooth curve in $B$, $L_t = \pi^{-1}(\pi(z))$, and $TW$ be the tangent bundle of the symplectic manifold $W$. Let

$$TW|_z \equiv S^1 \times R^{2n}$$

be the trivialization of $TW|_z$. Then $TL_t$ defines a family of Lagrangian subspace of $R^{2n}$, i.e. defines a loop $\alpha$ in Lagrangian-Grassmanian manifold $\Lambda(n)$. This loop defines the Maslov index $m(\alpha)$ of the map $z$.

**Lemma 3.1.** *Let $u : D^2 \to W$ be a $C^k$−map$(k \geq 1)$. Then $m(u|_{\partial D}) = 0$.*

**Proof.** Let

$$TW|_D \equiv D \times R^{2n}, \quad h(r,t) = u(re^{2\pi i t}).$$

Then $h$ defines a homotopy from $h(1,t) = u|_{\partial D}$ to $h(0,t) = p$ which induces a homotopy $\bar{h}$ in Lagrangian-Grassmanian manifold. Note that $m(\bar{h}(0,\cdot)) = 0$. By the homotopy invariance of Maslov index, we know that $m(u|_{\partial D}) = 0$.

Consider the partial differential equation

$$\bar{\partial} u + A(z) u = 0 \text{ on } D, \tag{3.2}$$

$$u(z) \in \Gamma(z) \text{ for } z \in \partial D, \tag{3.3}$$

$$m(\Gamma) = 0. \tag{3.4}$$

For $3 < k < \infty$, consider the Banach space $\bar{V}_k$ consisting of all maps $u \in H^{k,2}(D, C^n)$ such that $u(z) \in \Gamma(z)$ for almost all $z \in \partial D$. Let $L_{k-1}$ be the usual $L_{k-1}$−space $H_{k-1}(D, C^n)$ and $L_{k-1}(S^1) = \{u \in H^{k-1}(S^1)|u(z) \in \Gamma(z) \text{ for } z \in \partial D\}$. We define an operator $P$: $\bar{V}_k \to L_{k-1} \times L_{k-1}(S^1)$ by

$$P(u) = (\bar{\partial} u + Au, u|_{\partial D}), \tag{3.5}$$

where $D$ as in (3.1).

**Proposition 3.2.**[6] *$\bar{\partial} : \bar{V}_p \to L_p$ is a real linear Fredholm operator of index $n$.*

**3.2. Nonlinear Fredholm Operator.**

Now let $\pi : \Sigma \to B$ be a contact manifold of Legendre fibration with contact form $\lambda$. Let $L = R \times \Sigma \to B$ be the Lagrangian fibration induced by the Legendre fibration $\Sigma \to B$ and its symplectization $(R \times \Sigma, de^t\lambda)$. Recall that

$$\mathcal{D}^k(L,p) = \{u \in H^k(D, R^N) | u(x) \in L \text{ a.e. for } x \in D \text{ and } u(1) = p\},$$
$$\Lambda^k(B,b) = \{z \in H^k(S^1, R^N) | u(z) \in B \text{ a.e. for } z \in S^1 \text{ and } z(1) = b\}.$$

**Lemma 3.2.** *Let $\pi : L \to B$ be the Lagrangian fibration as above and $\pi(p) = b$. Let*

$$\mathcal{D}^k_z(L,p) = \{u \in \mathcal{D}_k(L,p) | u(\tau) \in \pi^{-1}(z(\tau)) \ for \ \tau \in \partial D\}.$$

*Then, $\mathcal{D}^k_z(L,p)$ is a smooth Hilbert manifold and*

$$\mathcal{D}^k(L,p) = \bigcup_{z \in \Lambda^{k-1}(B,b)} \mathcal{D}^k_z(L,p) \tag{3.6}$$

*is a Hilbert fibration defined by*

$$P_r \circ u = \pi \circ (u|_{\partial D}). \tag{3.7}$$

**Proof.** First we prove $\mathcal{D}^k_z(L,p) = \{u \in \mathcal{D}^k(L,p) | \pi \circ (u|_{\partial D}) = z\}$ is a Hilbert manifold, i.e. the space of disks with boundary in the family of Lagrangian submanifolds $L_{z(t)} = R \times \pi^{-1}(z(t))$ is a Hilbert manifold. In principle, we can use the Gauss normal charts on $L$ to construct the Hilbert chart on $\mathcal{D}^k_z(L,p)$ (see [9]) in the loop space of a Riemannian manifold. However, because of the boundary conditions, we have to introduce the $t-$dependent metric. Let therefore $g_t$ be a smooth family of metrics so that $L_t$ is totally geodesic with respect to $g_t$. Then we define

$$\exp : S^1 \times TP \mapsto P, \quad \exp : (t,p,\xi) = \exp_{g_t}(p,\xi).$$

Then one can use the above exp to construct Hilbert chart on $\mathcal{D}^k(L,p)$ (for the detail see [3]). Since $\pi : L \mapsto B$ is a fibration it is obvious that it induces the fibration

$$P_r : \mathcal{D}^k(L,p) \to B^{k-1}(B,b) \tag{3.8}$$

by

$$P_r \circ u = \pi \circ (u|_{\partial D}). \tag{3.9}$$

Now we consider the tangent bundle of Hilbert manifold $\mathcal{D}^k(L,p)$. Let

$$W^k(u) = H^{k,2}(u^*TL), \tag{3.10}$$

$$W^k(L) = \bigcup_{u \in \mathcal{D}^{k+1}(L,p)} W^k(u), \tag{3.11}$$

$$W^k(L,o) = \{v \in W^k(L) | v(1) = o\}. \tag{3.12}$$

$$P_r : W^k(L,o) \mapsto \mathcal{D}^{k+1}(L,p), \tag{3.13}$$

$$P_r \circ v = u, \ v \in W^k(u) \tag{3.14}$$

is the tangent bundle of $\mathcal{D}^{k+1}(L,p)$ (see [3,6]). Now we construct a nonlinear Fredholm operator from $\mathcal{D}^k(L,p)$ to $T\mathcal{D}^k(L,p) \times \Lambda(B,b)$ follows in [3,6]. Let $\bar\partial : \mathcal{D}^k(L,p) \to T\mathcal{D}^k(L,p)$ be the Cauchy-Riemmann Section induced by the Cauchy-Riemann operator, locally,

$$\bar\partial u = \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} \tag{3.15}$$

for $u \in \mathcal{D}^k(W,p)$. Since the space $\mathcal{D}^k(W,p)$ is contractible, the tangent space $T\mathcal{D}^k(W,p)$ is trivial, i.e. there exists a bundle isomorphism

$$\Phi : T\mathcal{D}^k(W,p) \to \mathcal{D}^k(W,p) \times E,$$

where $E$ is a Hilbert Space. Then the Cauchy-Riemann section $\bar{\partial}$ on $T\mathcal{D}^k(W,p)$ induces a nonlinear map $\Phi \circ \bar{\partial} : \mathcal{D}^k(W,p) \mapsto E$. In the following, we still denote $\Phi \circ \bar{\partial}$ by $\bar{\partial}$ for convenience. Now we define

$$F : \mathcal{D}^k(W,p) \to E \times \Lambda(B,b), \tag{3.16}$$

$$F(u) = (\bar{\partial}u, P_r \circ u), \tag{3.17}$$

where $P_r : \mathcal{D}^k(W,p) \mapsto \Lambda(B,b)$ is the projection defined by $P_r \circ u = \pi \circ (u|_{\partial D})$.

**Theorem 3.1.** *The nonlinear operator $F$ defined in* (3.16)–(3.17) *is a nonlinear Fredholm operator of index zero.*

**Proof.** According to the definition of the nonlinear Fredholm operator, we need to prove that $u \in \mathcal{D}^k(L,p)$, the linearization $DF(u)$ of $F$ at $u$ is a linear Fredholm operator. Note that

$$DF(u) = (D\bar{\partial}_{[u]}, (DP_r)_{[u]}), \tag{3.18}$$

where

$$(D\bar{\partial}_{[u]})v = \frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} + A(u)v \tag{3.19}$$

and $A(u)$ is a $2n \times 2n$ matrix induced by the torsion of almost complex structure (see [3,6] for the computation). Here the second term $(DP_r)[u]$ can be computed as follows:

$$(DP_r)_{[u]} = (D\pi)(D\tau)_{[u]}, \tag{3.20}$$

where $D\pi : TL \to TB$ and $D\tau$ are tangent maps of projection $\pi$ and ordinary trace operator $\tau$, i.e., $\tau \circ u = u|_{\partial D}$.

Observe that the linearization $DF(u)$ of $F$ at $u$ is equivalent to the following Lagrangian boundary value problem

$$\frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} + A(u)v = f, \quad v \in W^k(u^*TW), \tag{3.21}$$

$$v(t) \in T_{\pi \circ u(t)} = (T\pi)^{-1}(h), \quad t \in \partial D, \tag{3.22}$$

where $(f,h) \in E \times T_{(\pi\circ\tau)(u)}\Lambda(B,b)$. One can check that (3.18)–(3.19) or (3.21)–(3.22) defines a linear Fredholm operator. In fact, by Proposition 3.2 and Lemma 3.2, since the operator $A(u)$ is compact, we know that the operator $F$ is a nonlinear Fredholm operator of index zero.

**Definition 3.1.** *A nonlinear Fredholm $F : X \to Y$ operator is proper if any $y \in Y$, $F^{-1}(y)$ is finite or for any compact set $K \subset Y$, $F^{-1}(K)$ is compact in $X$.*

**Definition 3.2.** $\deg(F,y) = \sharp\{F^{-1}(y)\} \bmod 2$ *is called the Fredholm degree of a nonlinear proper Fredholm operator* (*see* [6, 14]).

**Theorem 3.2.** *Assume that the nonlinear Fredholm operator $F : \mathcal{D}^k(W,p) \to E \times \Lambda^{k-1}(B,b)$ constructed in* (3.16)–(3.17) *is proper and $\pi(p) = b$. Then* $\deg(F,(0,b)) = 1$.

**Proof.** Since $b$ is a constant loop in $\Lambda(B,b)$, $\pi^{-1}(b) = L_b$ is a Lagrangian submanifold by noting that

$$e^t\lambda|_{L_b} = e^t\lambda|_{\Sigma_b \times R} = 0. \tag{3.23}$$

Now we assume that $u : D \mapsto W$ is a $J$-holomorphic disk with boundary $u(\partial D) \subset L_b$. Since almost complex structure $\widetilde{J}$ tamed by the symplectic form $de^t\lambda$, by Stokes formula and (4.17), we conclude that $u : D^2 \to w$ is a constant map. Because $u(1) = p$, we know that $F^{-1}(0,b) = p$. Next we show that the linearization $DF(p)$ of $F$ at $p$ is an isomorphism from $T^p\mathcal{D}(W,p)$ to $E\times T^p\Lambda(B,b)$. This is equivalent to solving the equations

$$\frac{\partial v}{\partial s}+J\frac{\partial v}{\partial t}=f, \tag{3.24}$$

$$v|_{\partial D}\subset T_pL_b. \tag{3.25}$$

By Lemma 3.1, we know that $DF(p)$ is an isomorphism. Therefore $\deg(F,(b,0))=1$.

**Corollary 3.1.** $\deg(F,w)=1$ *for any* $w\in E\times\Lambda$.

**Proof.** Use the connectedness of $E\times\Lambda$ and the homotopy invariance of deg.

## §4. The Non-Properness of the Fredholm Operator

We shall prove in this section that the operator $F:\mathcal{D}\to E\times\Lambda$ constructed in the above section is non proper. We recall some basic definitions on Fredholm structures which were discussed widely in 1960's by many mathematicians (see [2]).

**Definition 4.1.**[2,11] *A Fredholm structure on* $M$ *is an integrable reduction of its principal bundle* $\pi:PM\to M$ *to* $GC(E)$. *A Fredholm manifold is a Banach manifold together with a Fredholm structure.*

For the Fredholm manifold, one can define the infinite dimentional Cohomology theory on it (see [11]). Especially, he proved the following celebrated theorem.

**Proposition 4.1.** *Let* $M$, $N$ *be Fredholm manifolds and* $f:M\to N$ *be a proper,* $C^\infty$*-Fredholm map of index zero. Suppose* $M$ *is contractible and* $\deg(f)=1$. *Then,* $N$ *is contractible.*

Note that Proposition 4.1 is slightly different from Theorem 4.4 in [11]. K. K. Mukherjea used the integer coeffient for cohomology of $M$, $N$ and assume that $f:M\to N$ is orientation preserving. However, his proof can be carried to the case of proposition 4.1 (see [2,11]).

**Theorem 4.1.** *The Fredholm operator* $F:\mathcal{D}^k(W,p)\to E\times\Lambda^{k-1}(B,b)$ *is not proper.*

**Proof.** By Theorems 3.1 and 3.2, we know that the index of $F$ is zero and $\deg(F)=1$. By Theorem 2.1, $\mathcal{D}^(W,p)$ is contractible and $\Lambda^{k-1}(B,b)$ is nontractible. By Mukherjea's theorem, $F$ is not proper.

## §5. The Existences of Holomorphic Planes and Periodic Solutions

In this section, we use the Sacks-Uhlenbeck-Gromov's trick to prove the existence of $J$-holomorphic plane as in [3,6,7,10] for symplectization of contact manifolds. Then this implies the existence of periodic orbit of Reeb vector field by the method as in [3,6,7,10]. Now, we fix a point $q\in\Lambda(B,b)$ and $F\in T\mathcal{D}(W,p)$. Consider the inverse image $F^{-1}(f,q)$ as

$$u:D^2\to W, \tag{5.1}$$

$$\bar\partial_J u=f, \tag{5.2}$$

$$u(e^{2\pi it})\in L_{q(t)}. \tag{5.3}$$

In order to get the estimate of energy on solution $u$, we consider $(\Sigma,\lambda)=\Big(S(M),f\sum\limits_{i=1}^n p_i dq_i\Big)$, where $S(M)$ is the unit sphere bundle of the Riemann manifold $M$ and $f$ is a positive function on $S(M)$.

Now we recall $\Sigma=S(M)$ and $W=R\times\Sigma$ and put $\xi=\ker(\lambda)$. Then $d\lambda$ is a symplectic structure for the vectorbundle $\xi\to\Sigma$. We choose a complex structure $J$ for $\xi$ such that

$g_J := d\lambda \circ (Id \times J)$ is a metric for $\xi \to \Sigma$. As before we define an almost complex structure $\widetilde{J}$ on $W$ by

$$\widetilde{J}(t,u)(h,k) = (-\lambda(u)(k), J(u)\pi k + hX(u)), \tag{5.4}$$

where $\pi : T\Sigma \to \xi$ is the bundle projection along $RX \to \Sigma$ and $X$ the Reeb vector field associated to $\lambda$. We define a complete metric $\tilde{g}$ on $W = R \times \Sigma$ by

$$\langle (h_1,k_1),(h_2,k_2)\rangle = h_1h_2 + \lambda(k_1)\lambda(k_2) + g_J(\pi k_1, \pi k_2). \tag{5.5}$$

Now we introduce a family of pseudo-Riemannian metrics on $W$, i.e.,

$$g_\varphi(\cdot,\cdot) = \omega_\varphi(\cdot, \widetilde{J}\cdot) = (d\varphi\lambda)(\cdot, \widetilde{J}\cdot). \tag{5.6}$$

Here

$$\Sigma = \Big\{\varphi \in C^\infty\Big(R, \Big[\frac{1}{2}, 1\Big]\Big)\Big| 0 \le \varphi' \le 1\Big\}. \tag{5.7}$$

Now for $\tilde{u} \in F^{-1}(f,q)$, define

$$E_\Sigma(\tilde{u}) = \sup_{\varphi\in\Sigma}\Big\{\int_D \Big(g_\varphi\Big(\frac{\partial\tilde{u}}{\partial x}, \tilde{J}\frac{\partial\tilde{u}}{\partial x}\Big) + g_\varphi\Big(\frac{\partial\tilde{u}}{\partial y}, \widetilde{J}\frac{\partial u}{\partial y}\Big)\Big)d\sigma\Big\}. \tag{5.8}$$

**Lemma 5.1.** *Let $\Big(S(M), g\sum\limits_{i=1}^n p_i dq_i\Big)$ be the contact manifold induced by Riemanian manifold with the contact form $\lambda = g\sum\limits_{i=1}^n p_i dq_i$. Let $q \in \Lambda(M)$ and $\tilde{u} \in F^{-1}(f,q)$. Then, one has the following estimates*

$$E_\Sigma(\tilde{u}) = \sup_{\varphi\in\Sigma}\Big\{\int_D \Big(g_\varphi\Big(\frac{\partial\tilde{u}}{\partial x}, \tilde{J}\frac{\partial\tilde{u}}{\partial x}\Big) + g_\varphi\Big(\frac{\partial\tilde{u}}{\partial y}, \widetilde{J}\frac{\partial u}{\partial y}\Big)\Big)d\sigma\Big\} \le c(q), \tag{5.9}$$

*where $\Sigma = \{\varphi \in C^\infty(R, [\frac{1}{2}, 1]) | 0 \le \varphi' \le 1\}$ and $\omega_\varphi = d(\phi\lambda)$.*

**Proof.** Since $u(z) = (a, q(z), p(z)) \in R \times S(M)$ and $\lambda = g\sum\limits_{i=1}^n p_i dq_i$, by Stokes formula,

$$\int_D \tilde{u}^*\omega_\varphi = \int_D \tilde{u}^* d\varphi\lambda = \int_{\partial D} (\tilde{u}|\partial D)^*(\varphi\lambda) = \int_{\partial D} g\varphi p(z) dq(z). \tag{5.10}$$

So

$$\Big|\int_D u^* d\varphi\lambda\Big| \le M_1 M_2 M_3, \tag{5.11}$$

where

$$M_1 = \max_{a\in R}\varphi(a) = 1, \quad M_2 = \max_{x\in S(M)} f(x), \quad M_3 = \max_{t\in S^1}|q(t)|.$$

Since $f \in T\mathcal{D}(W,p)$, we have

$$\|f\|_{L^2(D)} \le c_1. \tag{5.12}$$

Note that the Pseudo-Riemann norm $|\cdot|_{g_\varphi} \le |\cdot|_{\tilde{g}}$ since

$$\langle (k_1,h_1),(k_2,h_2)\rangle_{g_\varphi} = \varphi'(a)(h_1h_2 + \lambda(k_1)\lambda(k_2)) + \varphi(a)(g_J(\pi k_1, \pi k_2)). \tag{5.13}$$

So, we get

$$\|f\|_{L^2(D),\varphi} \le \|f\|_{L^2(D)} \le c_1 \tag{5.14}$$

with $\varphi \in \Sigma$. By the following formula,

$$\tilde{u}^*\omega_\varphi = \frac{1}{4}(|\partial\tilde{u}|^2_\varphi - |\bar{\partial}\tilde{u}|^2), \tag{5.15}$$

$$|\nabla\tilde{u}|_\varphi = \frac{1}{2}(|\partial\tilde{u}|^2_\varphi + |\bar{\partial}\tilde{u}|^2_\varphi) \tag{5.16}$$

and the (5.2), (5.11)–(5.12) and (5.15)–(5.16), we finish the proof of Lemma 5.1.

**Theorem 5.1.** *There exists a non-constant solution* $\tilde{u} = (a, u) : C \to W$ *of the partial differential equations*

$$\tilde{u}_s + \tilde{J}(\tilde{u})\tilde{u}_t = 0, \tag{5.17}$$

*where*

$$E_\Sigma(\tilde{u}) < \infty, \tag{5.18}$$

$$E_\Sigma(\tilde{u}) = \sup_{\varphi \in \Sigma} \int_C \tilde{u}^* \omega_\varphi \tag{5.19}$$

*and*

$$\Sigma = \Big\{\varphi \in C^\infty\Big(R, \Big[\frac{1}{2}, 1\Big]\Big)\Big| 0 \le \varphi' \le 1\Big\}, \tag{5.20}$$

$$\omega_\varphi = d(\varphi\lambda). \tag{5.21}$$

**Proof.** (1) Gradient bounds imply $C^0-$bounds. In fact, for a given point $q \in \Lambda(B, b)$ and $f \in T\mathcal{D}(W, p)$, the point $\tilde{u} \in F^{-1}(f, q)$ considered as a map $u : D^2 \mapsto W$ has a bounded image. Since

$$|\nabla u| \le C \quad \text{for } u \in F^{-1}(f, q), \tag{5.22}$$

$$u(1) = p \in W. \tag{5.23}$$

By the mean value formula we know that the image of $u$ for $u \in F^{-1}(f, q)$ is contained in a bounded set $K \subset W$.

(2) Gradient bounds imply the $C^\infty-$bounds as in [6].

(3) By Theorem 4.1 we know that there exists a point $(f, q) \in E \times \Lambda^{k-1}(B, b)$ such that $F^{-1}(f, q)$ is not compact. By the above (1) and (2), there exists a sequence $\{\tilde{u}_n\}$ of solutions of equation

$$\bar{\partial}_J \tilde{u}_k = f, \tag{5.24}$$

$$\tilde{u}_k(t) \in L_{q(t)} \tag{5.25}$$

such that

$$E_\Sigma(u_k) = \sup_{\varphi \in \Sigma} \int_D u_k^* \omega_\varphi \le c_1(q), \tag{5.26}$$

$$\varepsilon_k |\nabla \tilde{u}_k(z_k)| \to +\infty, \tag{5.27}$$

$$\varepsilon_k \to 0, \ z_k \to z_0 \in D, \tag{5.28}$$

$$|\nabla \tilde{u}(z)| \le 2|\nabla \tilde{u}(z_k)| \text{ if } |z - z_k| \le \varepsilon_k. \tag{5.29}$$

Let $r_k = \text{dist}(z_k, \partial D)/\varepsilon_k$. If $r_k \to +\infty$, we define

$$\tilde{v}_k = \Big(a\Big(z_k + \frac{z}{R_k}\Big) - a(z_k), u(z_k + \frac{z}{R_k}\Big)\Big) \tag{5.30}$$

on $B_{\varepsilon_k R_k} \subset C$ with $R_k = |\nabla \tilde{u}(z_k)|$. Then

$$\sup_{\varphi \in \Sigma} \int_{B_{\varepsilon_k R_k}} \tilde{v}_k^* \omega_\varphi \le E_\Sigma(\tilde{u}_k) \le c_1(q), \tag{5.31}$$

$$\frac{\partial}{\partial s}\tilde{v}_k + \widetilde{J}(\tilde{v}_k)\frac{\partial}{\partial t}\tilde{v}_k = \frac{1}{R_k} f, \tag{5.32}$$

$$|\nabla \tilde{v}_k(z)| \le 2 \quad \text{on } B_{\varepsilon_k R_k}, \tag{5.33}$$

$$|\nabla \tilde{v}_k(0)| = 1. \tag{5.34}$$

As in [6], we may assume after perhaps taking a subsequence that $\tilde{v}_k \to \tilde{v}$ in $C^\infty_{\text{loc}}$. Hence we find

$$\tilde{v}_s + \widetilde{J}(\tilde{V})\tilde{v}_t = 0, \tag{5.35}$$

$$\int_C \tilde{v}^* d\varphi\lambda \le c_1(q), \tag{5.36}$$

$$|\nabla\tilde{v}(0)| = 1, \tag{5.37}$$

$$|\nabla\tilde{v}(z)| \le 2 \quad \text{on } C. \tag{5.38}$$

So, the same arguments as in [6] finish the proof of Theorem 5.1.

**Proof of Theorem 1.1**. By the assumption of Theorem 1.1, we know that the space of disks with base point $p$ in $W$ is contractible and the loop space of base manifold $B$ is non-contractible (see Section 2). By Section 3, we have a Fredholm map from the disk space of total space to the loop space of base space. By the Mukherjea's theorem in Section 4, we know that the operator is non-proper. The non-properness of the operator implies the existence of $J-$holomorphic plane. By the arguments as in [3,6,7,10], the bubble concludes the existence of closed orbit of Reeb vectorfield.

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