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INTEGRO-DIFFERENTIAL EQUATIONS ON UNBOUNDED DOMAINS IN BANACH SPACES**

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Abstract

This paper investigates the maximal and minimal solutions of initial value problem for n-th order nonlinear integro-differential equations of Volterra type on an infinite interval in a Banach space by establishing a comparison result and using the monotone iterative technique.

Keywords Integro-differential equation, Initial value problem, Ordered Banach space, Comparison result, Monotone iterative technique

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§1. Introduction

In [1, Section 3.3], we have discussed the initial value problem (IVP) for first order integro-differential equations of Volterra type on infinite interval $J = [0, \infty)$ in a real Banach space E by means of fixed point theory. Now, in this paper, we shall investigate the IVP for n-th order such equations by means of completely different method, that is, by establishing a comparison result and using the monotone iterative technique. Consider the IVP for n-th order nonlinear integro-differential equation of Volterra type in E:

$$\begin{cases} u^{(n)}(t) = f(t, u(t), u'(t), \cdots, u^{(n-1)}(t), (Tu)(t)), \quad \forall\, t \in J, \\ u(0) = u_0, \quad u'(0) = u_1, \cdots, \quad u^{(n-1)}(0) = u_{n-1}, \end{cases} \tag{1.1}$$

where $J = [0, \infty)$, $u_i \in E$ $(i = 0, 1, \cdots, n-1)$, $f \in C[J \times E \times E \times \cdots \times E, E]$ and

$$(Tu)(t) = \int_0^t k(t, s) u(s) ds, \quad \forall\, t \in J, \tag{1.2}$$

$k \in C[D, R_+]$, $D = \{(t, s) \in J \times J : \ t \geq s\}$ and R_+ denotes the set of all nonnegative numbers.

Let P be a cone in E which defines a partial ordering in E by $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E, and P is said to be regular if $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ implies $\|x_n - x\| \to 0$ as $n \to \infty$ for some $x \in E$. It is well known that the regularity of P implies the normality of P. For details on cone theory see [2].

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§2. Several Lemmas

Lemma 2.1(Comparison Result). *Assume that* $p \in C^n[J,E]$ *satisfies*

$$\begin{cases} p^{(n)}(t) \le -\sum\limits_{i=0}^{n-1} a_i(t)p^{(i)}(t) - b(t)(Tp)(t), & \forall\ t \in J, \\ p^{(n-1)}(0) \le p^{(n-2)}(0) \le \cdots \le p'(0) \le p(0) \le \theta, \end{cases} \tag{2.1}$$

where $a_i,\ b \in C[J,R_+]$ $(i = 0,1,\cdots,n-1)$ *and* $p^{(0)}(t) = p(t)$ $(t \in J)$. *Then* $p^{(i)}(t) \le \theta$ *for* $t \in J$ $(i = 0,1,\cdots,n-1)$ *provided*

$$\int_0^\infty \Big[\sum_{i=0}^{n-1}\Big(\sum_{m=0}^{n-i-1}\frac{t^m}{m!}\Big)a_i(t)\Big]dt + \int_0^\infty b(t)dt\int_0^t \Big[\frac{t^{n-1}-(t-s)^{n-1}}{(n-1)!} + \sum_{m=0}^{n-2}\frac{s^m}{m!}\Big]k(t,s)ds \le 1. \tag{2.2}$$

Proof. Let $p_1(t) = p^{(n-1)}(t)$ $(t \in J)$. Then $p_1 \in C^1[J,E]$ and

$$p^{(n-2)}(t) = p^{(n-2)}(0) + \int_0^t p_1(s)ds,$$

$$p^{(n-3)}(t) = p^{(n-3)}(0) + tp^{(n-2)}(0) + \int_0^t ds_1\int_0^{s_1} p_1(s_2)ds_2,$$

$$\cdots\cdots\cdots\cdots,$$

$$p'(t) = p'(0) + tp''(0) + \cdots + \frac{t^{n-3}}{(n-3)!}p^{(n-2)}(0) + \int_0^t ds_1\int_0^{s_1}ds_2\cdots\int_0^{s_{n-3}} p_1(s_{n-2})ds_{n-2},$$

$$p(t) = p(0) + tp'(0) + \cdots + \frac{t^{n-2}}{(n-2)!}p^{(n-2)}(0) + \int_0^t ds_1\int_0^{s_1}ds_2\cdots\int_0^{s_{n-2}} p_1(s_{n-1})ds_{n-1}.$$

It is easy to see by induction that

$$\int_0^t ds_1\int_0^{s_1}ds_2\cdots\int_0^{s_{m-1}} p_1(s_m)ds_m = \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}p_1(s)ds, \quad m = 1,2,\cdots.$$

So, we have

$$\begin{cases} p^{(n-1)}(t) = p_1(t), \\ p^{(n-2)}(t) = p^{(n-2)}(0) + \int_0^t p_1(s)ds, \\ p^{(n-3)}(t) = p^{(n-3)}(0) + tp^{(n-2)}(0) + \int_0^t (t-s)p_1(s)ds, \\ \qquad\cdots\cdots\cdots\cdots, \\ p'(t) = p'(0) + tp''(0) + \cdots + \frac{t^{n-3}}{(n-3)!}p^{(n-2)}(0) + \frac{1}{(n-3)!}\int_0^t (t-s)^{n-3}p_1(s)ds, \\ p(t) = p(0) + tp'(0) + \cdots + \frac{t^{n-2}}{(n-2)!}p^{(n-1)}(0) + \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}p_1(s)ds. \end{cases} \tag{2.3}$$

Substituting (2.3) into (2.1), we get

$$p_1'(t) \le -c_0(t)p(0) - c_1(t)p'(0) - \cdots - c_{n-2}(t)p^{(n-2)}(0) - a_{n-1}(t)p_1(t) - \int_0^t k_1(t,s)p_1(s)ds,\ \forall\ t \in J, \tag{2.4}$$

where

$$c_0(t) = a_0(t) + b(t)\int_0^t k(t,s)ds,$$

$$c_1(t) = ta_0(t) + a_1(t) + b(t)\int_0^t sk(t,s)ds,$$
$$\cdots\cdots\cdots\cdots,$$
$$c_{n-2}(t) = \frac{t^{n-2}}{(n-2)!}a_0(t) + \frac{t^{n-3}}{(n-3)!}a_1(t) + \cdots + a_{n-2}(t) + \frac{b(t)}{(n-2)!}\int_0^t s^{n-2}k(t,s)ds,$$
$$k_1(t,s) = \frac{(t-s)^{n-2}}{(n-2)!}a_0(t) + \frac{(t-s)^{n-3}}{(n-3)!}a_1(t) + \cdots + a_{n-2}(t) + \frac{(t-s)^{n-2}}{(n-2)!}b(t)\int_s^t k(t,r)dr.$$

For any $g \in P^*$ (P^* denotes the dual cone of P, see [2]), let $v(t) = g(p_1(t))$. Then $v \in C^1[J, R^1]$. By (2.4) and (2.1), we have

$$v'(t) \le -c_0(t)g(p(0)) - c_1(t)g(p'(0)) - \cdots - c_{n-2}(t)g(p^{(n-2)}(0)) - a_{n-1}(t)v(t) - \int_0^t k_1(t,s)v(s)ds, \quad \forall\, t \in J \tag{2.5}$$

$$v(0) \le g(p^{(n-2)}(0)) \le \cdots \le g(p'(0)) \le g(p(0)) \le 0. \tag{2.6}$$

We now show that

$$v(t) \le 0, \quad \forall\, t \in J. \tag{2.7}$$

Assume that (2.7) is not true, i.e. there exists a $0 < t_0 < \infty$ such that $v(t_0) > 0$. Let $\min\{v(t):\ 0 \le t \le t_0\} = -\lambda$. Then $\lambda \ge 0$ and $v(t_1) = -\lambda$ for some $0 \le t_1 < t_0$. From (2.6) we have

$$g(p(0)) \ge g(p'(0)) \ge \cdots \ge g(p^{(n-2)}(0)) \ge -\lambda,$$

so (2.5) implies that

$$v'(t) \le \lambda\Big[c_0(t) + c_1(t) + \cdots + c_{n-2}(t) + a_{n-1}(t) + \int_0^t k_1(t,s)ds\Big], \quad \forall\, 0 \le t \le t_0.$$

Consequently

$$0 < v(t_0) = v(t_1) + \int_{t_1}^{t_0} v'(s)ds \le -\lambda + \lambda\int_0^\infty [c_0(t) + c_1(t) + \cdots + c_{n-2}(t) + a_{n-1}(t)]dt + \lambda\int_0^\infty dt\int_0^t k_1(t,s)ds,$$

which implies that $\lambda > 0$ and

$$\int_0^\infty [c_0(t) + c_1(t) + \cdots + c_{n-2}(t) + a_{n-1}(t)]dt + \int_0^\infty dt\int_0^t k_1(t,s)ds > 1. \tag{2.8}$$

It is easy to see by simple calculation that

$$\begin{aligned}
&\int_0^\infty [c_0(t) + c_1(t) + \cdots + c_{n-2}(t) + a_{n-1}(t)]dt + \int_0^\infty dt\int_0^t k_1(t,s)ds \\
&= \int_0^\infty \Big[\Big(1 + t + \cdots + \frac{t^{n-1}}{(n-1)!}\Big)a_0(t) + \Big(1 + t + \cdots + \frac{t^{n-2}}{(n-2)!}\Big)a_1(t) \\
&\quad + \cdots + (1+t)a_{n-2}(t) + a_{n-1}(t)\Big]dt \\
&\quad + \int_0^\infty b(t)dt\int_0^t \Big(1 + s + \cdots + \frac{s^{n-2}}{(n-2)!} + \frac{t^{n-1} - (t-s)^{n-1}}{(n-1)!}\Big)k(t,s)ds.
\end{aligned} \tag{2.9}$$

Evidently, (2.8) and (2.9) contradict (2.2). Hence, (2.7) holds. Since $g \in P^*$ is arbitrary, we get from (2.7) that $p_1(t) \le \theta$ for $t \in J$, i.e. $p^{(n-1)}(t) \le \theta$ for $t \in J$. Finally, we have by

(2.1),

$$p^{(n-2)}(t)=p^{(n-2)}(0)+\int_0^t p^{(n-1)}(s)ds\le\theta,\quad \forall\ t\in J,$$
$$p^{(n-3)}(t)=p^{(n-3)}(0)+\int_0^t p^{(n-2)}(s)ds\le\theta,\quad \forall\ t\in J,$$
$$\cdots\cdots\cdots\cdots,$$
$$p'(t)=p'(0)+\int_0^t p''(s)ds\le\theta,\quad \forall\ t\in J,$$
$$p(t)=p(0)+\int_0^t p'(s)ds\le\theta,\quad \forall\ t\in J,$$

and the lemma is proved.

Consider the IVP of n-th order linear integro-differential equation in E:

$$\begin{cases} u^{(n)}(t)=-\sum\limits_{i=0}^{n-1}a_i(t)u^{(i)}(t)-b(t)(Tu)(t)+y(t),\quad \forall\ t\in J,\\ u(0)=u_0,\ u'(0)=u_1,\ \cdots,u^{(n-1)}(0)=u_{n-1}\end{cases}\tag{2.10}$$

and the linear integral equation in E:

$$u(t)=u_0+tu_1+\cdots+\frac{t^{n-1}}{(n-1)!}u_{n-1}+\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}\Big[y(s)-\sum_{i=0}^{n-1}a_i(s)u^{(i)}(s)-b(s)(Tu)(s)\Big]ds,\ \forall\ t\in J.\tag{2.11}$$

Lemma 2.2. *Let* $y\in C[J,E]$ *and* $b\in C[J,R^1]$, $a_i\in C^i[J,R^1]$ $(i=0,1,\cdots,n-1)$. *Then*

(a) $u\in C^n[J,E]$ *is a solution of* IVP (2.10) *if and only if* $u\in C^{n-1}[J,E]$ *is a solution of the integral equation* (2.11);

(b) *integral equation* (2.11) *has a unique solution in* $C^{n-1}[J,E]$ *given by*

$$u(t)=z(t)+\sum_{i=1}^{\infty}(-1)^i\int_0^t h_i(t,s)z(s)ds,\ \forall\ t\in J,\tag{2.12}$$

where

$$z(t)=\sum_{m=1}^{n-1}\Big[\frac{t^{m-1}}{(m-1)!}+\frac{1}{(n-1)!}\sum_{i=m}^{n-1}\sum_{j=0}^{i-m}(-1)^j(n-1)(n-2)\cdots(n-i+m+j)c_j^{i-m}\cdot t^{n-i+m+j-1}a_i^{(j)}(0)\Big]u_{m-1}+\frac{t^{n-1}}{(n-1)!}u_{n-1}+\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}y(s)ds,\ \forall\ t\in J,\tag{2.13}$$

$$h_1(t,s)=\frac{1}{(n-1)!}\Big[\sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^j(n-1)(n-2)\cdots(n-i+j)c_j^i(t-s)^{n-i+j-1}a_i^{(j)}(s)+\int_s^t(t-r)^{n-1}b(r)k(r,s)dr],\ \forall\ (t,s)\in D,\tag{2.14}$$

$$h_i(t,s)=\int_s^t h_1(t,r)h_{i-1}(r,s)dr,\ \forall\ (t,s)\in D,\ i=2,3,4,\cdots.\tag{2.15}$$

The series in the right-hand side of (2.12) *converges uniformly on* $J_r=[0,r]$ *for any* $r>0$.

Proof. (a) From (2.3) we have a formula

$$u(t)=u(0)+tu'(0)+\cdots+\frac{t^{n-1}}{(n-1)!}u_{n-1}(0)+\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}u^{(n)}(s)ds,\forall\ u\in C^n[J,E]. \tag{2.16}$$

So, if $u\in C^n[J,E]$ is a solution of IVP (2.10), then by substituting (2.10) into (2.16), we see that $u(t)$ satisfies (2.11). Conversely, if $u\in C^{n-1}[J,E]$ is a solution of (2.11), then direct differentiation of (2.11) gives

$$u'(t)=u_1+\cdots+\frac{t^{n-2}}{(n-2)!}u_{n-1}+\frac{1}{(n-2)!}\int_0^t(t-s)^{n-2}\Big[y(s)-\sum_{i=0}^{n-1}a_i(s)u^{(i)}(s)$$
$$-b(s)(Tu)(s)\Big]ds,\ \forall\ t\in J,$$
$$\cdots\cdots\cdots\cdots,$$
$$u^{(n-1)}(t)=u_{n-1}+\int_0^t\Big[y(s)-\sum_{i=0}^{n-1}a_i(s)u^{(i)}(s)-b(s)(Tu)(s)\Big]ds,\ \forall\ t\in J,$$
$$u^{(n)}(t)=y(t)-\sum_{i=0}^{n-1}a_i(t)u^{(i)}(t)-b(t)(Tu)(t),\ \forall\ t\in J.$$

Hence $u\in C^n[J,E]$ and $u(t)$ satisfies (2.10).

(b) Let $u\in C^{n-1}[J,E]$ be a solution of the integral equation (2.11). Let $t\in J$ be fixed and $b_i(s)=(t-s)^{(n-1)}a_i(s)$ $(0\le s\le t,\ i=0,1,\cdots,n-1)$. By integrating by parts, it is easy to find

$$\int_0^t\Big[\sum_{i=0}^{n-1}b_i(s)u^{(i)}(s)\Big]ds=\Big\{\sum_{m=1}^{n-1}\Big[\sum_{i=m}^{n-1}(-1)^{i-m}b_i^{(i-m)}(s)\Big]u^{(m-1)}(s)\Big\}\Big|_{s=0}^{s=t}$$
$$+\int_0^t\Big[\sum_{i=0}^{n-1}(-1)^i b_i^{(i)}(s)\Big]u(s)ds. \tag{2.17}$$

Using the formula of n-th derivative for a product, we get

$$b_i^{(m)}(s)=\sum_{j=0}^{m}(-1)^{m-j}(n-1)(n-2)\cdots(n-m+j)c_j^m(t-s)^{n-m+j-1}a_i^{(j)}(s),$$
$$\forall\ 0\le s\le t,\ 0\le m\le i\le n-1, \tag{2.18}$$

where $c_j^m=\frac{m(m-1)\cdots(m-j+1)}{j!}$, so

$$b_i^{(m)}(t)=0,\ \forall\ m<i. \tag{2.19}$$

It follows from (2.17)–(2.19) that

$$\int_0^t(t-s)^{n-1}\Big[\sum_{i=0}^{n-1}a_i(s)u^{(i)}(s)\Big]ds$$
$$=\sum_{m=1}^{n-1}\Big[\sum_{i=m}^{n-1}\sum_{j=0}^{i-m}(-1)^{j+1}(n-1)(n-2)\cdots(n-i+m+j)c_j^{i-m}t^{n-i+m+j-1}a_i^{(j)}(0)\Big]u_{m-1}$$
$$+\int_0^t\Big[\sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^j(n-1)(n-2)\cdots(n-i+j)c_j^i(t-s)^{n-i+j-1}a_i^{(j)}(s)\Big]u(s)ds,$$
$$\forall\ t\in J. \tag{2.20}$$

On the other hand, it is easy to find

$$\int_0^t (t-s)^{n-1} b(s)(Tu)(s)ds = \int_0^t u(s) \int_s^t (t-r)^{n-1} b(r)k(r,s)dr, \ \forall \ (t,s) \in D. \tag{2.21}$$

Now, (2.11), (2.20) and (2.21) imply that

$$u(t) = z(t) - \int_0^t h_1(t,s)u(s)ds, \ \forall \ t \in J, \tag{2.22}$$

where $z(t)$ and $h_1(t,s)$ are defined by (2.13) and (2.14) respectively. From (2.13) and (2.14) we see that $z \in C^n[J,E]$ and $\frac{\partial^i h_1}{\partial t^i} \in C[D,R^1]$ $(i = 0,1,\cdots,n)$. So, if $u \in C[J,E]$ satisfies (2.22), then $u \in C^n[J,E]$, and consequently, (2.17)-(2.21) hold and $u(t)$ satisfies (2.11). Thus, we have proved that $u \in C^{n-1}[J,E]$ is a solution of (2.11) if and only if $u \in C[J,E]$ is a solution of (2.22). Obviously, (2.22) is a linear integral equation of Volterra type in E and, by a known result (see [1, Theorem 1.4.2]), it has a unique solution in $C[J,E]$ given by (2.12), where $h_i(t,s)$ are defined by (2.15) and the series in the right-hand side of (2.12) converges uniformly on $J_r = [0,r]$ for any $r > 0$.

§3. Main Theorems

Let us list some conditions for convenience.

(H_1) there exist $v_0,\ w_0 \in C^n[J,E]$ such that $v_0^{(i)}(t) \le w_0^{(i)}(t)$ for $t \in J$ $(i = 0,1,\cdots,n-1)$,

$$\begin{cases} v_0^{(n)}(t) \le f(t,v_0(t),v_0'(t),\cdots,v_0^{(n-1)}(t),(Tv_0)(t)), & \forall \ t \in J, \\ v_0(0) \le u_0, \quad v_0^{(i)}(0) - v_0^{(i-1)}(0) \le u_i - u_{i-1}, & i = 1,2,\cdots,n-1, \end{cases}$$

$$\begin{cases} w_0^{(n)}(t) \ge f(t,w_0(t),w_0'(t),\cdots,w_0^{(n-1)}(t),(Tw_0)(t)), & \forall \ t \in J, \\ w_0(0) \ge u_0, \quad w_0^{(i)}(0) - w_0^{(i-1)}(0) \ge u_i - u_{i-1}, & i = 1,2,\cdots,n-1. \end{cases}$$

(H_2) there exist $a_i \in C^i[J,R_+]$ $(i = 0,1,\cdots,n-1)$ and $b \in C[J,R_+]$ such that

$$f(t,x_0,x_1,\cdots,x_{n-1},x) - f(t,\overline{x}_0,\overline{x}_1,\cdots,\overline{x}_{n-1},\overline{x}) \ge -\sum_{i=0}^{n-1} a_i(t)(x_i - \overline{x}_i) - b(t)(x - \overline{x}),$$

whenever $t \in J$, $v_0^{(i)}(t) \le \overline{x}_i \le x_i \le w_0^{(i)}(t)$ $(i = 0,1,\cdots,n-1)$ and $(Tv_0)(t) \le \overline{x} \le x \le (Tw_0)(t)$.

(H_3) for any $r > 0$, there exist nonnegative constants c_{ir} $(i = 0,1,\cdots,n)$ such that

$$\alpha(f(J_r,U_0,U_1,\cdots,U_n)) \le \sum_{i=0}^{n} c_{ir}\alpha(U_i), \ \forall \ U_i \subset B_r \ (i = 0,1,\cdots,n),$$

where $J_r = [0,r]$, $B_r = \{x \in E : \ \|x\| \le r\}$ and α denotes the Kuratowski measure of noncompactness in E.

We write $[v_0,w_0] = \{u \in C^n[J,E] : v_0^{(i)}(t) \le u^{(i)}(t) \le w_0^{(i)}(t), \ \forall t \in J, \ i = 0,1,\cdots,n-1\}$.

Theorem 3.1. *Let cone P be normal and conditions (H_1), (H_2) and (H_3) be satisfied. Assume that inequality (2.2) holds. Then IVP (1.1) has minimal and maximal solutions $\overline{u}$ and u^* in $[v_0,w_0]$ respectively. Define the iterative sequences $\{v_k(t)\}$ and $\{w_k(t)\}$ by*

$$v_k(t) = z_{k-1}(t) + \sum_{i=1}^{\infty}(-1)^i \int_0^t h_i(t,s)z_{k-1}(s)ds, \ \forall \ t \in J, \ k = 1,2,3,\cdots, \tag{3.1}$$

$$w_k(t) = \overline{z}_{k-1}(t) + \sum_{i=1}^{\infty}(-1)^i \int_0^t h_i(t,s)\overline{z}_{k-1}(s)ds, \ \forall \ t \in J, \ k = 1,2,3,\cdots, \tag{3.2}$$

where

$$z_{k-1}(t) = \sum_{m=1}^{n-1}\Big\{\frac{t^{m-1}}{(m-1)!} + \frac{1}{(n-1)!}\sum_{i=m}^{n-1}\sum_{j=0}^{i-m}(-1)^j(n-1)(n-2)$$

$$\cdots(n-i+m+j)c_j^{i-m}t^{n-i+m+j-1}a_i^{(j)}(0)\Big\}u_{m-1} + \frac{t^{n-1}}{(n-1)!}u_{n-1}$$

$$+\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}[f(s,v_{k-1}(s),v'_{k-1}(s),\cdots,v_{k-1}^{(n-1)}(s),(Tv_{k-1})(s))$$

$$+\sum_{i=0}^{n-1}a_i(s)v_{k-1}^{(i)}(s) + b(s)(Tv_{k-1})(s)]ds, \tag{3.3}$$

$$\overline{z}_{k-1}(t) = \sum_{m=1}^{n-1}\Big\{\frac{t^{m-1}}{(m-1)!} + \frac{1}{(n-1)!}\sum_{i=m}^{n-1}\sum_{j=0}^{i-m}(-1)^j(n-1)(n-2)$$

$$\cdots(n-i+m+j)c_j^{i-m}\cdot t^{n-i+m+j-1}a_i^{(j)}(0)\Big\}u_{m-1} + \frac{t^{n-1}}{(n-1)!}u_{n-1}$$

$$+\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\Big[f(s,w_{k-1}(s),w'_{k-1}(s),\cdots,w_{k-1}^{(n-1)}(s),(Tw_{k-1})(s))$$

$$+\sum_{i=0}^{n-1}a_i(s)w_{k-1}^{(i)}(s) + b(s)(Tw_{k-1})(s)\Big]ds, \tag{3.4}$$

and $h_i(t,s)$ $(i=1,2,\cdots)$ *are given by* (2.14) *and* (2.15). *Then* $\{v_k^{(i)}(t)\}$ *and* $\{w_k^{(i)}(t)\}$ *converge uniformly on* $J_r=[0,r]$ *(for any* $r>0$*) to* $\overline{u}^{(i)}(t)$ *and* $(u^*)^{(i)}(t)$ *respectively* $(i=0,1,\cdots,n-1)$. *Moreover, we have*

$$v_0^{(i)}(t)\le v_1^{(i)}(t)\le\cdots\le v_k^{(i)}(t)\le\cdots\le\overline{u}^{(i)}(t)\le u^{(i)}(t)\le(u^*)^{(i)}(t)\le\cdots$$

$$\le w_k^{(i)}(t)\le\cdots\le w_1^{(i)}(t)\le w_0^{(i)}(t),\ \forall\, t\in J\ (i=0,1,\cdots,n-1), \tag{3.5}$$

where $u(t)$ *is any solution of* IVP (1.1) *in* $[v_0,w_0]$.

Proof. For any $\eta\in[v_0,w_0]$, consider the linear IVP (2.10) with

$$y(t)=f(t,\eta(t),\eta'(t),\cdots,\eta^{(n-1)}(t),(T\eta)(t))+\sum_{i=0}^{n-1}a_i(t)\eta^{(i)}(t)+b(t)(T\eta)(t). \tag{3.6}$$

By Lemma 2.2, IVP (2.10) has a unique solution $u\in C^n[J,E]$ which is the unique solution of Equation (2.11) in $C^{n-1}[J,E]$ given by (2.12). Let $u=A\eta$. Then operator $A:\ [v_0,w_0]\to C^n[J,E]$, and we shall show that

(a) $v_0^{(i)}(t)\le(Av_0)^{(i)}(t)$ and $(Aw_0)^{(i)}(t)\le w_0^{(i)}(t)$ for $t\in J$ $(i=0,1,\cdots,n-1)$ and

(b) $\eta_1,\eta_2\in[v_0,w_0]$ and $\eta_1^{(i)}(t)\le\eta_2^{(i)}(t)$ $(t\in J,\ i=0,1,\cdots,n-1)$ imply

$$(A\eta_1)^{(i)}(t)\le(A\eta_2)^{(i)}(t)\text{ for }t\in J\ (i=0,1,\cdots,n-1).$$

To prove (a), we set $v_1=Av_0$ and $p=v_0-v_1$. By (2.10) and (3.6), we have

$$\begin{cases} v_1^{(n)}(t)=\sum\limits_{i=0}^{n-1}a_i(t)[v_0^{(i)}(t)-v_1^{(i)}(t)]+b(t)[(Tv_0)(t)-(Tv_1)(t)] \\ \qquad\qquad +f(t,v_0(t),v_0'(t),\cdots,v_0^{(n-1)}(t),(Tv_0)(t)),\ \forall\, t\in J, \\ v_1^{(i)}(0)=u_i,\quad i=0,1,\cdots,n-1. \end{cases}$$

So, from (H_1), we find

$$\begin{cases} p^{(n)}(t) \le -\sum_{i=0}^{n-1} a_i(t)p^{(i)}(t) - b(t)(Tp)(t), & \forall\ t \in J, \\ p(0) \le \theta,\ p^{(i)}(0) \le p^{(i-1)}(0), & i = 1, 2, \cdots, n-1, \end{cases}$$

which implies by virtue of Lemma 2.1 that $p^{(i)}(t) \le \theta$ for $t \in J$ $(i = 0, 1, \cdots, n-1)$, i.e. $v_0^{(i)}(t) \le (Av_0)^{(i)}(t)$ for $t \in J$ $(i = 0, 1, \cdots, n-1)$. Similarly, we can show that $(Aw_0)^{(i)}(t) \le w_0^{(i)}(t)$ for $t \in J$ $(i = 0, 1, \cdots, n-1)$. To prove (b), let $p = A\eta_1 - A\eta_2$. It is easy to see from (2.10), (3.6) and (H_2) that

$$\begin{cases} p^{(n)}(t) = -\sum\limits_{i=0}^{n-1} a_i(t)p^{(i)}(t) - b(t)(Tp)(t) - \{f(t, \eta_2(t), \eta_2'(t), \cdots, \eta_2^{(n-1)}(t), (T\eta_2)(t)) \\ \qquad - f(t, \eta_1(t), \eta_1'(t), \cdots, \eta_1^{(n-1)}(t), (T\eta_1)(t)) + \sum\limits_{i=0}^{n-1} a_i(t)[\eta_2^{(i)}(t) - \eta_1^{(i)}(t)] \\ \qquad + b(t)[(T\eta_2)(t) - (T\eta_1)(t)]\} \le -\sum\limits_{i=0}^{n-1} a_i(t)p^{(i)}(t) - b(t)(Tp)(t),\ \forall\ t \in J, \\ p^{(i)}(0) = \theta, \quad i = 0, 1, \cdots, n-1, \end{cases}$$

so Lemma 2.1 implies that $p^{(i)}(t) \le \theta$ for $t \in J$ $(i = 0, 1, \cdots, n-1)$, i.e.

$$(A\eta_1)^{(i)}(t) \le (A\eta_2)^{(i)}(t) \text{ for } t \in J\ (i = 0, 1, \cdots, n-1),$$

and (b) is proved.

Now, let

$$v_k = Av_{k-1}, \quad w_k = Aw_{k-1}, \quad k = 1, 2, 3, \cdots. \tag{3.7}$$

By conclusions (a) and (b) just proved, we have

$$\begin{aligned} &v_0^{(i)}(t) \le v_1^{(i)}(t) \le \cdots \le v_k^{(i)}(t) \le \cdots \le w_k^{(i)}(t) \le \cdots \le w_1^{(i)}(t) \le w_0^{(i)}(t), \\ &\qquad \forall\ t \in J\ (i = 0, 1, \cdots, n-1). \end{aligned} \tag{3.8}$$

Let $r > 0$ be arbitrarily given. By the normality of P and (3.8) we see that $V_i = \{v_k^{(i)} : k = 0, 1, 2, \cdots\}$ $(i = 0, 1, \cdots, n-1)$ are bounded sets in $C[J_r, E]$. Since, in addition, (H_3) implies that $f(J_r, B_r, B_r, \cdots, B_r)$ is bounded, there exists a constant $\beta_r > 0$ such that

$$\begin{aligned} &\Big\| f(t, v_{k-1}(t), v_{k-1}'(t), \cdots, v_{k-1}^{(n-1)}(t), (Tv_{k-1})(t)) - \sum_{i=0}^{n-1} a_i(t)[v_k^{(i)}(t) - v_{k-1}^{(i)}(t)] \\ &- b(t)[(Tv_k)(t) - (Tv_{k-1})(t)] \Big\| \le \beta_r,\ \forall\ t \in J_r, \quad k = 1, 2, 3, \cdots. \end{aligned} \tag{3.9}$$

By (3.7) and Lemma 2.2(a), we have

$$\begin{aligned} v_k(t) = u_0 + tu_1 + \cdots + \frac{t^{n-1}}{(n-1)!}u_{n-1} + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\Big\{ f(s, v_{k-1}(s), v_{k-1}'(s), \cdots, \\ v_{k-1}^{(n-1)}(s), (Tv_{k-1})(s)) - \sum_{i=0}^{n-1} a_i(s)[v_k^{(i)}(s) - v_{k-1}^{(i)}(s)] \\ - b(s)[(Tv_k)(s) - (Tv_{k-1})(s)]\Big\} ds,\ \forall\ t \in J,\ k = 1, 2, 3, \cdots. \end{aligned} \tag{3.10}$$

Differentiation of (3.10) gives

$$v_k^{(i)}(t) = u_i + tu_{i+1} + \cdots + \frac{t^{n-i-1}}{(n-i-1)!}u_{n-1} + \frac{1}{(n-i-1)!}\int_0^t (t-s)^{n-i-1}\Big\{f(s, v_{k-1}(s), v'_{k-1}(s), \cdots, v_{k-1}^{(n-1)}(s), (Tv_{k-1})(s)) - \sum_{i=0}^{n-1} a_i(s)[v_k^{(i)}(s) - v_{k-1}^{(i)}(s)] - b(s)[(Tv_k)(s) - (Tv_{k-1})(s)]\Big\}ds,$$
$$\forall\, t \in J,\ i = 0, 1, \cdots, n-1;\ k = 1, 2, 3, \cdots. \tag{3.11}$$

It follows from (3.9)–(3.11) that V_i $(i = 0, 1, \cdots, n-1)$ are equicontinuous on J_r, and so, functions $\alpha(V_i(t))$ $(i = 0, 1, \cdots, n-1)$ are continuous on J_r, where $V_i(t) = \{v_k^{(i)}(t) :\ k = 0, 1, 2, \cdots\}$. By using [1, Theorem 1.2.2 and Corollary 1.2.1] to (3.11), we find

$$\alpha(V_i(t)) \le \frac{2r^{n-i-1}}{(n-i-1)!}\int_0^t \Big[\alpha(f(s, V_0(s), V_1(s), \cdots, V_{n-1}(s), (TV_0)(s))) + \sum_{i=0}^{n-1} a_{ir}\alpha(V_i(s)) + b_r\alpha((TV_0)(s))\Big]ds, \quad \forall\, t \in J_r,\ i = 0, 1, \cdots, n-1, \tag{3.12}$$

where $(TV_0)(t) = \{(Tv_k)(t) :\ k = 0, 1, 2, \cdots\}$, $a_{ir} = \max\{a_i(t) :\ t \in J_r\}$ $(i = 0, 1, \cdots, n-1)$ and $b_r = \max\{b(t) :\ t \in J_r\}$. On the other hand, (H_3) implies that there exist constants $c_{ir} \ge 0$ $(i = 0, 1, \cdots, n)$ such that

$$\alpha(f(t, V_0(t), V_1(t), \cdots, V_{n-1}(t), (TV_0)(t))) \le \sum_{i=0}^{n-1} c_{ir}\alpha(V_i(t)) + c_{nr}\alpha((TV_0)(t)), \quad \forall t \in J_r. \tag{3.13}$$

In addition, [1, Theorem 1.2.2] implies that

$$\alpha((TV_0)(t)) \le k_r \int_0^t \alpha(V_0(s))ds,\ \forall\, t \in J_r, \tag{3.14}$$

where $k_r = \max\{k(t, s) :\ (t, s) \in J_r \times J_r,\ t \ge s\}$. Let $m(t) = \max\{\alpha(V_i(t)) :\ i = 0, 1, \cdots, n-1\}$. Then $m(t)$ is continuous on J_r. It is easy to see from (3.12)-(3.14) that

$$m(t) \le \tau_r \int_0^t m(s)ds,\ \forall\, t \in J_r, \tag{3.15}$$

where τ_r is a nonnegative constant depending on r only. By a known result (see [3, Theorem 1.9.1]), (3.15) implies that $m(t) = 0$ for $t \in J_r$. Consequently, by virtue of the Ascoli-Arzela theorem (see [1, Theorem 1.2.5]), V_i $(i = 0, 1, \cdots, n-1)$ are relatively compact in $C[J_r, E]$. Since P is normal and $\{v_k^{(i)}\}$ $(i = 0, 1, \cdots, n-1)$ are nondecreasing on account of (3.8), we see that $\{v_k^{(i)}\}$ converge uniformly on J_r to some $\overline{u}_i \in C[J_r, E]$ $(i = 0, 1, \cdots, n-1)$ respectively. Hence $\overline{u}_0 \in C^{n-1}[J_r, E]$ and $\overline{u}_0^{(i)}(t) = \overline{u}_i(t)$ for $t \in J_r$ $(i = 1, 2, \cdots, n-1)$. Write $\overline{u}_0 = \overline{u}$. We have

$$f(t, v_{k-1}(t), v'_{k-1}(t), \cdots, v_{k-1}^{(n-1)}(t), (Tv_{k-1})(t)) - \sum_{i=0}^{n-1} a_i(t)[v_k^{(i)}(t) - v_{k-1}^{(i)}(t)] - b(t)[(Tv_k)(t) - (Tv_{k-1})(t)]$$
$$\to f(t, \overline{u}(t), \overline{u}'(t), \cdots, \overline{u}^{(n-1)}(t), (T\overline{u})(t)) \text{ as } k \to \infty,\ \ \forall\, t \in J_r, \tag{3.16}$$

and, by (3.9),

$$\Big\|f(t,v_{k-1}(t),v'_{k-1}(t),\cdots,v_{k-1}^{(n-1)}(t),(Tv_{k-1})(t))-\sum_{i=0}^{n-1}a_i(t)[v_k^{(i)}(t)-v_{k-1}^{(i)}(t)]$$
$$-b(t)[(Tv_k)(t)-(Tv_{k-1})(t)]-f(t,\overline{u}(t),\overline{u}'(t),\cdots,\overline{u}^{(n-1)}(t),(T\overline{u})(t))\Big\|$$
$$\le 2\beta_r,\quad \forall\ t\in J_r,\quad k=1,2,3,\cdots. \tag{3.17}$$

Noticing (3.16), (3.17) and taking limits as $k\to\infty$ in (3.10), we get

$$\overline{u}(t)=u_0+tu_1+\cdots+\frac{t^{n-1}}{(n-1)!}u_{n-1}+\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}f(s,\overline{u}(s),\overline{u}'(s),$$
$$\cdots,\overline{u}^{(n-1)}(s),(T\overline{u})(s))ds,\ \forall\ t\in J_r. \tag{3.18}$$

Since $r>0$ is arbitrary, we see that $\overline{u}\in C^{n-1}[J,E]$ and (3.18) holds for all $t\in J$. Hence, Lemma 2.2 (a) implies that $\overline{u}\in C^n[J,E]$ and $\overline{u}(t)$ is a solution of IVP (1.1).

In the same way, we can show that $\{w_k\}$ converges to some $u^*\in C^n[J,E]$ uniformly on J_r for any $r>0$, and $u^*(t)$ is a solution of IVP (1.1); moreover, $\{w_k^{(i)}\}$ converge to $(u^*)^{(i)}$ uniformly on J_r for any $r>0$ $(i=1,2,\cdots,n-1)$ respectively.

Let $u(t)$ be any solution of IVP (1.1) in $[v_0,w_0]$. Then $v_0^{(i)}(t)\le u^{(i)}(t)\le w_0^{(i)}(t)$ for $t\in J$ $(i=0,1,\cdots,n-1)$. Assume that $v_{k-1}^{(i)}(t)\le u^{(i)}(t)\le w_{k-1}^{(i)}(t)$ for $t\in J$ $(i=0,1,\cdots,n-1)$, and let $p(t)=v_k(t)-u(t)$. We have, by (H_2),

$$\begin{cases} p^{(n)}(t)=-\displaystyle\sum_{i=0}^{n-1}a_i(t)p^{(i)}(t)-b(t)(Tp)(t)-\Big\{f(t,u(t),u'(t),\cdots,u^{(n-1)}(t),(Tu)(t)) \\ \qquad -f(t,v_{k-1}(t),v'_{k-1}(t),\cdots,v_{k-1}^{(n-1)}(t),(Tv_{k-1})(t))+\displaystyle\sum_{i=0}^{n-1}a_i(t)[u(t)-v_{k-1}(t)] \\ \qquad +b(t)[(Tu)(t)-(Tv_{k-1})(t)]\Big\} \\ \quad\le-\displaystyle\sum_{i=0}^{n-1}a_i(t)p^{(i)}(t)-b(t)(Tp)(t),\ \forall\ t\in J, \\ p^{(i)}(0)=\theta,\quad i=0,1,\cdots,n-1, \end{cases}$$

which implies by virtue of Lemma 2.1 that $p^{(i)}(t)\le\theta$ for $t\in J$ $(i=0,1,\cdots,n-1)$, i.e. $v_k^{(i)}(t)\le u^{(i)}(t)$ for $t\in J$ $(i=0,1,\cdots,n-1)$. Similarly, we can show that $u^{(i)}(t)\le w_k^{(i)}(t)$ for $t\in J$ $(i=0,1,\cdots,n-1)$. Hence, by induction

$$v_k^{(i)}(t)\le u^{(i)}(t)\le w_k^{(i)}(t),\ \forall\ t\in J,\ i=0,1,\cdots,n-1;\ k=1,2,3,\cdots. \tag{3.19}$$

Taking limits as $k\to\infty$ in (3.19), we find $\overline{u}^{(i)}(t)\le u^{(i)}(t)\le(u^*)^{(i)}(t)$ for $t\in J$ $(i=0,1,\cdots,n-1)$.

Finally, (3.1)-(3.4) follow from (3.7), (3.6), (2.12)-(2.15), and (3.5) is obtained by (3.8) and (3.19).

Remark 3.1. In some cases, it is easy to find v_0 and w_0 satisfying (H_1). For example, let $f\in C[J\times P\times P\times\cdots\times P,P]$ and $u_0=u_1=\cdots=u_{n-1}=\theta$. If there is a $z\in P$ such that $f\big(t,ze^t,ze^t,\cdots,ze^t,z\int_0^t k(t,s)e^s ds\big)\le ze^t$, $\forall\ t\in J$, then $v_0(t)=\theta$ and $w_0(t)=ze^t$ satisfy (H_1). On the other hand, (H_2) is satisfied if

$$\frac{\partial f}{\partial x_i}\ge a_i(t),\ i=0,1,\cdots,n-1\ \text{ and }\ \frac{\partial f}{\partial x}\ge b(t)$$

for $t \in J,\ v_0^{(i)}(t) \le x_i \le w_0^{(i)}(t)\ (i = 0, 1, \cdots, n-1)$ and $(Tv_0)(t) \le x \le (Tw_0)(t)$. In addition, (H_3) is satisfied for $c_{ir} = 0\ (i = 0, 1, \cdots, n)$ if $f(J_r, B_r, B_r, \cdots, B_r)$ is relatively compact for any $r > 0$.

Theorem 3.2. *Let cone P be regular and Conditions* (H_1) *and* (H_2) *be satisfied. Assume that inequality* (2.2) *holds and $f(J_r, B_r, B_r, \cdots, B_r)$ is bounded for any $r > 0$, where $J_r = [0, r]$ and $B_r = \{x \in E : \ \|x\| \le r\}$. Then the conclusion of Theorem* 3.1 *holds.*

Proof. The proof is almost the same as that of Theorem 3.1. The only difference is that, instead of using Condition (H_3), the conclusion $\alpha(V_i(t)) = 0\ (t \in J_r,\ i = 0, 1, \cdots, n-1)$ is implied directly by (3.8) and the regularity of P.

Example. Consider the infinite system for scalar third order integro- differential equations

$$\begin{cases} u_n''' = \dfrac{1}{50n^2(1+t+t^2)^6}[(t^3 - u_n)^2 + t^3 u_{n+1} + (u_{2n}')^3 + (6t - u_n'')^3] \\ \qquad + \dfrac{t}{3n^3(1+t)^8}\Big(t^3 - \displaystyle\int_0^t \frac{u_n(s)ds}{1+s+ts}\Big)^2, \ \forall\ 0 \le t < \infty, \\ u_n(0) = u_n'(0) = u_n''(0) = 0, \quad n = 1, 2, 3, \cdots. \end{cases} \tag{3.20}$$

Conclusion. System (3.20) has minimal and maximal C^3 solutions satisfying $0 \le u_n(t) \le \frac{t^3}{n^2},\ 0 \le u_n'(t) \le \frac{3t^2}{n^2},\ 0 \le u_n''(t) \le \frac{6t}{n^2}$ for $0 \le t < \infty,\ n = 1, 2, 3, \cdots$, and these solutions can be obtained by taking limits from some iterative sequences.

Proof. Let $E = l^1 = \Big\{u = (u_1, u_2, \cdots, u_n, \cdots) : \sum\limits_{n=1}^{\infty} |u_n| < \infty\Big\}$ with norm $\|u\| = \sum\limits_{n=1}^{\infty} |u_n|$ and $P = \{u = (u_1, u_2, \cdots, u_n, \cdots) \in l^1 : \ u_n \ge 0,\ n = 1, 2, 3, \cdots\}$. Then P is a normal cone in E. Since l^1 is weakly complete, we see that P is regular (see [1, Remark 1.2.4]). Now, system (3.20) can be regarded as an IVP of form (1.1) in E. In this situation, $u_0 = u_1 = u_2 = (0, 0, \cdots, 0, \cdots),\ k(t, s) = (1 + s + ts)^{-1},\ u = (u_1, u_2, \cdots, u_n, \cdots),\ v = (v_1, v_2, \cdots, v_n, \cdots),\ w = (w_1, w_2, \cdots, w_n, \cdots),\ \ z = (z_1, z_2, \cdots, z_n, \cdots),\ \ f = (f_1, f_2, \cdots, f_n, \cdots)$, in which

$$f_n(t, u, v, w, z) = \frac{1}{50n^2(1+t+t^2)^6}[(t^3 - u_n)^2 + t^3 u_{n+1} + (v_{2n})^3 + (6t - w_n)^3] \\ + \frac{t}{3n^3(1+t)^8}(t^3 - z_n)^2. \tag{3.21}$$

It is clear that $f \in C[J \times E \times E \times E \times E, E]$. Let $v_0(t) = (0, \cdots, 0, \cdots)$ and $w_0(t) = (t^3, \cdots, \frac{t^3}{n^2}, \cdots)$. Then $v_0, w_0 \in C^3[J, E],\ v_0(t) \le w_0(t)\ (t \in J)$ and

$$\begin{aligned} v_0'(t) &= (0, \cdots, 0 \cdots) \le \Big(3t^2, \cdots, \frac{3t^2}{n^2}, \cdots\Big) = w_0'(t), \ \forall\ t \in J, \\ v_0''(t) &= (0, \cdots, 0 \cdots) \le \Big(6t, \cdots, \frac{6t}{n^2}, \cdots\Big) = w_0''(t), \ \forall\ t \in J, \\ v_0(0) &= w_0(0) = (0, \cdots, 0 \cdots) = u_0, \\ v_0'(0) - v_0(0) &= w_0'(0) - w_0(0) = (0, \cdots, 0 \cdots) = u_1 - u_0, \\ v_0''(0) - v_0'(0) &= w_0''(0) - w_0'(0) = (0, \cdots, 0 \cdots) = u_2 - u_1, \\ v_0'''(t) &= (0, \cdots, 0 \cdots),\ w_0'''(t) = \Big(6, \cdots, \frac{6}{n^2}, \cdots\Big), \ \forall\ t \in J, \end{aligned}$$

$$\begin{aligned}
&f_n(t, v_0(t), v_0'(t), v_0''(t), (Tv_0)(t))\\
&= \frac{1}{50n^2(1+t+t^2)^6}(t^6+216t^3) + \frac{t^7}{3n^3(1+t)^8} \geq 0, \quad \forall\, t \in J,\ n = 1,2,3,\cdots,\\
&f_n(t, w_0(t), w_0'(t), w_0''(t), (Tw_0)(t))\\
&\leq \frac{1}{50n^2(1+t+t^2)^6}\Big[\Big(1+\frac{1}{4}+\frac{27}{64}\Big)t^6 + 216t^3\Big] + \frac{t^7}{3n^3(1+t)^8}\\
&\leq \frac{218}{50n^2} + \frac{1}{3n^3} < \frac{6}{n^2},\ \forall\, t \in J,\ n = 1,2,3,\cdots.
\end{aligned}$$

So, v_0 and w_0 satisfy Condition (H$_1$). On the other hand, for $t \in J$, $v_0(t) \leq \overline{u} \leq u \leq w_0(t)$, $v_0'(t) \leq \overline{v} \leq v \leq w_0'(t)$, $v_0''(t) \leq \overline{w} \leq w \leq w_0''(t)$, and $(Tv_0)(t) \leq \overline{z} \leq z \leq (Tw_0)(t)$, we have

$$\begin{aligned}
&0 \leq \overline{u}_n \leq u_n \leq \frac{t^3}{n^2}, \quad 0 \leq \overline{v}_n \leq v_n \leq \frac{3t^2}{n^2}, \quad 0 \leq \overline{w}_n \leq w_n \leq \frac{6t}{n^2},\\
&0 \leq \overline{z}_n \leq z_n \leq \frac{t^3}{3n^2},\ n = 1,2,3,\cdots,
\end{aligned}$$

so, by (3.21),

$$\begin{aligned}
&f_n(t,u,v,w,z) - f_n(t,\overline{u},\overline{v},\overline{w},\overline{z})\\
&\geq \frac{1}{50n^2(1+t+t^2)^6}[(t^3-u_n)^2 - (t^3-\overline{u}_n)^2 + (6t-w_n)^3 - (6t-\overline{w}_n)^3]\\
&\quad + \frac{t}{3n^3(1+t)^8}[(t^3-z_n)^2 - (t^3-\overline{z}_n)^2]\\
&\geq -\frac{1}{50n^2(1+t+t^2)^6}[2t^3(u_n-\overline{u}_n) + 108t^2(w_n-\overline{w}_n)] - \frac{2t^4}{3n^3(1+t)^8}(z_n-\overline{z}_n)\\
&\geq -\frac{1}{25(1+t+t^2)^3}(u_n-\overline{u}_n) - \frac{54}{25(1+t+t^2)^4}(w_n-\overline{w}_n) - \frac{1}{3(1+t)^4}(z_n-\overline{z}_n),\\
&\qquad \forall\, t \in J,\ n = 1,2,3,\cdots.
\end{aligned}$$

Consequently, Condition (H$_2$) is satisfied for

$$a_0(t) = \frac{1}{25(1+t+t^2)^3}, \quad a_1(t) = 0, \quad a_2(t) = \frac{54}{25(1+t+t^2)^4}, \quad b(t) = \frac{1}{3(1+t)^4}.$$

Now, we have

$$\begin{aligned}
&\int_0^\infty \Big[\Big(1+t+\frac{t^2}{2}\Big)a_0(t) + (1+t)a_1(t) + a_2(t)\Big]dt\\
&\quad + \int_0^\infty b(t)dt\int_0^t \Big[\frac{t^2-(t-s)^2}{2} + 1 + s\Big]k(t,s)\Big)ds\\
&< \frac{1}{25}\int_0^\infty \frac{dt}{(1+t)^2} + \frac{54}{25}\int_0^\infty \frac{dt}{(1+t)^4} + \frac{1}{3}\int_0^\infty \frac{dt}{(1+t)^3} = \frac{139}{210} < 1.
\end{aligned}$$

So, inequality (2.2) is also satisfied. Hence, our conclusion follows from Theorem 3.2.

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