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THE OCCUPATION DENSITY FIELD FOR THE CATALYTIC SUPER-BROWNIAN MOTION**

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Abstract

It is proved that the occupation time of the catalytic super-Brownian motion is absolutely continuous for $d = 1$, and the occupation density field is jointly continuous and jointly Hölder continuous.

Keywords Branching rate functional, Brownian collision local time, Catalytic super-Brownian motion, Occupation time, Occupation density field

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§1. Introduction

Let $W = \Big[w_t, \Pi_{s,a}, s,t \geq 0, a \in R^d\Big]$ denote a standard Brownian motion in R^d with semigroup $\{P_t, t \geq 0\}$, $C(R^d)$ denote the Banach space of continuous bounded functions on R^d equipped with the sup norm. $\phi_p(a) := (1+|a|^2)^{-p/2}$, $a \in R^d$, $C_p(R^d) := \{f \in C(R^d),\ |f(x)| \leq C_f\phi_p(x)\}$ with some constant C_f, $M_p(R^d) := \{\mu$ is Radon measure on R^d and $\int(1+|x|^p)^{-1}\mu(dx) < \infty\}$ and M_p is endowed with the p-vague topology, $p > d$. $\langle \mu, f\rangle := \int f(x)\mu(dx)$, λ is Lebesgue measure.

Given the ordinary M_P-valued super-Brownian motion $\varrho := [\varrho_t, \Omega_1, P_{s,\mu}, t \geq s \geq 0, \mu \in M_p]$ as the catalytic medium, Dawson and Fleischmann[1] proved the existence of the Brownian collision local time (BCLT) of ϱ for $d \leq 3$, $L_{[w,\varrho]}(dr)$, which is an additive functional of W, for $P_{0,\lambda}$-a.s.ϱ. And for $f \in C_p(R^d)^+$,

$$\Pi_{s,a}\int_s^t L_{[w,\varrho]}(dr)f(w_r) = \int_s^t dr\int \varrho_r(db)p(r-s,a,b)f(b). \tag{1.1}$$

Furthermore, it has the branching rate functional property. The catalytic super-Brownian motion (CSBM) $X^\varrho := [X_t^\varrho, \Omega_2, P_{s,\mu}^\varrho, t \geq s \geq 0, \mu \in M_p]$ is the $M_p(R^d)$-valued SBM with the BCLT as its branching rate functional, for $d \leq 3$ and $P_{0,\lambda}$-a.s.ϱ.

Let $Y_{[s,t]}^\varrho := \int_s^t X_r^\varrho dr$ be the occupation time of X^ϱ, i.e., for $\psi(r,\cdot) \in C_p(R^d),\ r \geq 0$,

$$\langle Y^\varrho, \psi\rangle_{[s,t]} := \int_s^t \langle X_r^\varrho, \psi(r,\cdot)\rangle dr.$$

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Its Laplace transition functional is (see [1])

$$P^{\varrho}_{s,\mu}\exp[-\langle Y^{\varrho},\theta\psi\rangle_{[s,t]}]=\exp\langle\mu,-v(s,t,\cdot)\rangle,\ 0\le s\le t,\theta\ge 0, \tag{1.2}$$

where $v(\cdot,t,\cdot)$ satisfies the following equation

$$v(s,t,a)=\theta f(s,a)-\Pi_{s,a}\int_s^t L_{[w,\varrho]}(dr)v^2(r,t,w_r),\ a\in R^d, \tag{1.3}$$

with

$$f(s,a):=\Pi_{s,a}\int_s^t\psi(r,w_r)dr=\int_s^t p(r-s,a,b)\psi(r,b)dr. \tag{1.4}$$

For details about this model, we refer to [1].

In the present paper, using the moment estimation metheod, we prove that the occupation time of the CSBM is absolutely continuous, and the density field is jointly continuous; moreover it is (jointly) Hölder continuous even with some space uniformity.

§2. Moment Estimation

Let $u=u_\theta:=\theta f-v, u^{(k)}:=\frac{\partial^k u}{\partial\theta^k}|_{\theta=0+}$. From (1.3), by simple calculation we have

$$\left.\begin{aligned}
&u^{(1)}(s,t,a)=f-v^{(1)}=0,\\
&u^{(2)}(s,t,a)=2\Pi_{s,a}\int_s^t L_{[w,\varrho]}(dr)f^2(r,w_r),\\
&u^{(k)}(s,t,a)=-2\Pi_{s,a}\int_s^t L_{[w,\varrho]}(dr)\cdot kf(r,w_r)u^{(k-1)}(r,t,w_r)\\
&\qquad+\sum_{2\le j\le k-2}C_k^j\Pi_{s,a}\int_s^t L_{[w,\varrho]}(dr)[u^{(k-j)}u^{(j)}](r,t,w_r).
\end{aligned}\right\} \tag{2.1}$$

From (1.2) and (1.3), for $P_{s,\mu}$-a.s.ϱ,

$$P^{\varrho}_{s,\mu}[\langle Y^{\varrho},\psi\rangle_{[s,t]}]=\langle\mu,f(s,\cdot)\rangle. \tag{2.2}$$

Consider the centered process

$$Z=Z^{\varrho}:=\langle\mu,f(s,\cdot)\rangle-\langle Y^{\varrho},\psi\rangle_{[s,t]}. \tag{2.3}$$

It has finite moments of all orders[1]

$$P^{\varrho}_{s,\mu}Z^k=\langle\mu,u^{(k)}(s,t,\cdot)\rangle+\sum_{2\le j\le k-2}C^j_{k-1}\langle\mu,u^{(k-1)}(s,t,\cdot)\rangle P^{\varrho}_{s,\mu}Z^j,\quad k\ge 2 \tag{2.4}$$

Let $p(r,a,b)=(2\pi r)^{-d/2}\exp\Big\{-\frac{|b-a|^2}{2r}\Big\},\ r>0,\ a,b\in R^d,\ q(s,t,a,b):=\int_s^t p(r,a,b)dr$, $\mu*q(s,t,b):=\int\mu(da)q(s,t,a,b).\ \varrho_0=\lambda.$

Lemma 2.1. *Fixed* $N>0,\ d=1$, *for*

$$f(s,a)=q((t-s)_++\varepsilon,t+h+\varepsilon-s,a,z),$$

$s\in[0,t+h],\ a\in R^d,\ 0<\varepsilon\le N,\ 0\le t\le t+h\le N,\ |z|\le N,\ \xi\in(0,1/2)$, *then there are constants* $c_k>0,\ k\ge 2$, $\limsup_{k\to\infty}c_k^{1/k}<\infty$ *such that for* $k\ge 2$, $P_{0,\lambda}$-a.s.ϱ

$$|u^{(k)}(s,a)|\le k!c_k(h+\varepsilon)^{(k-1)\xi}q((t-s)_+,t+2(h+\varepsilon)-s,a,z). \tag{2.5}$$

Proof. We prove (2.5) by induction.

(1) $k = 2$. Trivially $f(s,a) \le q((t-s)_+, t+2(h+\varepsilon)-s, a, z)$. Then from (1.1) and (2.1) we have

$$u^{(2)}(s,a) \le 2\Pi_{s,a} \int_s^{t+h} L_{[w,\varrho]}(dr) q^2((t-r)_+, t+2(h+\varepsilon)-r, w_r, z)$$
$$= 2\int_s^{t+h} dr \int \varrho_r(db) p(r-s,a,b) q^2(t-r)_+, t+2(h+\varepsilon)-r, b, z). \tag{2.6}$$

By [1] or [6], the occupation time of ϱ is absolutely continuous for $d \le 3$. Let $y_\delta := \{y_{[\delta,\delta+t]}(b); t \ge 0, b \in R^d\}$ denote the occupation density field. Then (see (4.29) in [1])

$$\sup_{b \in R^d} y_{[\delta+s,\delta+s+\varepsilon]}(b)\phi_p(b) \le C \cdot \varepsilon^{\xi} \tag{2.7}$$

for $P_{0,\lambda}$-a.s. C is a constant depending only on N, it may have different values in different lines. Since $y_{[\delta,\delta+t]}(b)$ is nondecreasing in t, for each fixed b, it determines a locally finite random measure $\lambda^b(dt)$ on R_+. Then (2.6) could continue

$$= 2\int_s^{t+h} \lambda^b(dr) \int dbp(r-s,a,b) q^2((t-r)_+, t+2(h+\varepsilon)-r, b, z)$$
$$= 2\int_s^{t} \lambda^b(dr) \int dbp(r-s,a,b) q(t-r, t+2(h+\varepsilon)-r, b, z) \int_{t-r}^{t+2(h+\varepsilon)-r} dlp(l,b,z)$$
$$+ 2\int_{t\vee s}^{t+h} \lambda^b(dr) \int dbp(r-s,a,b) q(0, t+2(h+\varepsilon)-r, b, z) \int_0^{t+2(h+\varepsilon)-r} dlp(l,b,z). \tag{2.8}$$

From the proof of Lemma 9 in [1], we get

$$p(l,b,z) \le \text{const.} \cdot l^{-d/2}(1+l)^{p/2}\phi_p(b)/\phi_p(z) = C \cdot l^{-d/2}\phi_p(b). \tag{2.9}$$

Combining (2.7), (2.8) with (2.9), and interchanging the order of the integration, we obtain

$$\int_{t\vee s}^{t+h} \lambda^b(dr) \int_0^{t+2(h+\varepsilon)-r} dlp(l,b,z)$$
$$= \int_0^{t+2(h+\varepsilon)-s} dl \int_s^{t+2(h+\varepsilon)} \lambda^b(dr) p(l,b,z)$$
$$\le C \int_0^{t+2(h+\varepsilon)-s} l^{-1/2} dl \cdot y_{[s,t+2(h+\varepsilon)]}(b)\phi_p(b)$$
$$\le C(h+\varepsilon)^{\xi}. \tag{2.10}$$

Similarly, we get

$$\int_s^t \lambda^b(dr) \int_{t-r}^{t+2(h+\varepsilon)-r} dlp(l,b,z) \le C(h+\varepsilon)^{\xi}. \tag{2.11}$$

Then from (2.8), together with (2.10), (2.11), we obtain

$$|u^{(2)}(s,a)| \le 2C(h+\varepsilon)^{\xi} q((t-s)_+, t+2(h+\varepsilon)-s, a, z), \tag{2.12}$$

which is (2.5) for $k = 2$.

(2) If (2.5) is satisfied with $2 \le r \le k-1$, we consider $u^{(k)}$. Define

$$c_1 = 1, \quad c_k = C \cdot \sum_{1 \le i \le k-1} c_{k-i} \cdot c_i, \quad k \ge 2. \tag{2.13}$$

From Lemma 3 in [1], we know that $\{c_k; k \geq 1\}$ satisfy the conditions. By (2.1) and (2.13), we have

$$\begin{aligned}
|u^{(k)}| &\leq \Pi_{s,a} \int_s^{t+h} L_{[w,\varrho]}(dr) \Big[\sum_{i=1}^{k-1} C_k^i \cdot (k-i)! \cdot c_{k-i} \cdot (h+\varepsilon)^{(k-i-1)\xi} \\
&\quad \cdot i! \cdot c_i \cdot (h+\varepsilon)^{(i-1)\xi} \Big] \cdot q^2((t-r)_+, t+2(h+\varepsilon)-r, w_r, z) \\
&= k! \cdot \Big(\sum_{i=1}^{k-1} c_{k-i} \cdot c_i \Big) \cdot (h+\varepsilon)^{(k-2)\xi} \\
&\quad \cdot \Pi_{s,a} \int_s^{t+h} L_{[w,\varrho]}(dr) q^2((t-r)_+, t+2(h+\varepsilon)-r, w_r, z) \\
&\leq k! \cdot c_k (h+\varepsilon)^{(k-1)\xi} \cdot q((t-s)_+, t+2(h+\varepsilon)-s, a, z).
\end{aligned}$$

This complete the proof.

Lemma 2.2. *Fixed* $N > 0,\ d = 1,\ 0 \leq \delta \leq N,\ 0 < \alpha < 1$ *for*

$$f(s,a) = q(\delta+\varepsilon, t+\delta+\varepsilon-s, a, z_1) - q(\delta+\varepsilon, t+\delta+\varepsilon-s, a, z_2),$$

$s \in [0, t+\delta],\ a \in R^d,\ 0 < \varepsilon \leq N,\ 0 \leq t \leq t+\delta \leq N,\ |z_1|, |z_2| \leq N,$ *then there are constants* $c_k > 0,\ k \geq 2,\ \limsup\limits_{k\to\infty} c_k^{1/k} < \infty$ *such that for* $k \geq 2$, $P_{0,\lambda}$-a.s.ϱ,

$$|u^{(k)}(s,a)| \leq k! \cdot c_k |z_1 - z_2|^{k\alpha/2} Q(s,a), \tag{2.14}$$

where $Q(s,a) = \sum\limits_{i=1}^{2} q(\delta+\varepsilon, t+\delta+\varepsilon-s, a, z_i)$.

Proof. Firstly, note that (see (3.44) in [6])

$$|p(r,a) - p(r,b)| \leq c_\alpha r^{-\alpha/2} |a-b|^\alpha [p(2r,a) + p(2r,b)]. \tag{2.15}$$

Similarly, we prove (2.14) by induction.

(1) Initial step. $k = 2$. By (2.1)

$$\begin{aligned}
u^{(2)} &\leq 2\Pi_{s,a} \int_s^t L_{[w,\varrho]}(dr) \int_{[\varepsilon+\delta, \varepsilon+\delta+t-r]^2} d[s_1, s_2] \cdot \prod_{i=1}^{2} |p(s_i, w_r, z_1) - p(s_i, w_r, z_2)| \\
&\leq 2 \int_s^{t+\delta} \lambda^b(dr) \int db p(r-s, a, b) \int_{\varepsilon+\delta}^{\varepsilon+\delta+t-r} ds_1 [p(s_1, b, z_1) + p(s_1, b, z_2)] \\
&\quad \cdot \int_{\varepsilon+\delta}^{\varepsilon+\delta+t-r} ds_2 |p(s_2, b, z_1) - p(s_2, b, z_2)|.
\end{aligned} \tag{2.16}$$

By (2.15), (2.7) and (2.9), we get

$$\begin{aligned}
&\int_s^{t+\delta} \lambda^b(dr) \int_{\varepsilon+\delta}^{\varepsilon+\delta+t-r} ds_2 |p(s_2, b, z_1) - p(s_2, b, z_2)| \\
&\leq c_\alpha |z_1 - z_2|^\alpha \cdot \int_s^{t+\delta} \lambda^b(dr) \int_{\varepsilon+\delta}^{\varepsilon+\delta+t-r} ds_2 \cdot [p(2s_2, b, z_1) + p(2s_2, b, z_2)] \\
&\leq C \cdot |z_1 - z_2|^\alpha.
\end{aligned} \tag{2.17}$$

Here the last inequality is by the same method as (2.10), where C is a constant depending only on α and N. Combining this with (2.16), we obtain

$$|u^{(2)}(s,a)| \leq 2 \cdot C |z_1 - z_2|^\alpha Q(s,a).$$

(2) Induction step. If (2.14) is satisfied with $2 \le r \le k-1$, we consider $u^{(k)}$. Define $\{c_k; k \ge 1\}$ as (2.13). By (2.1) and the initial step, (2.14) is still valid with $r = k$ (enlarging C where needed).

Let

$$\begin{aligned} y^{\varrho,\varepsilon}_{[\delta,\delta+t]}(z) &= \langle Y^{\varrho}_{[\delta,\delta+t]}, p(\varepsilon,\cdot,z)\rangle, \\ Z^{\varrho,\varepsilon}_{\delta,t}(z) &= \mu * q(\varepsilon+\delta, \varepsilon+\delta+t, z) - y^{\varrho,\varepsilon}_{[\delta,\delta+t]}(z). \end{aligned} \tag{2.18}$$

Lemma 2.3. *Fixed $N > 0$, $d = 1$, then to each $k \ge 2$ there is a constant C_k such that*

$$\begin{aligned} &|P^{\varrho}_{0,\mu}[Z^{\varrho,\varepsilon}_{\delta,t+h}(z) - Z^{\varrho,\varepsilon}_{\delta,t}(z)]^k| \\ &\le C_k (h+\varepsilon)^{k\xi/2} \sum_{i=1}^{k-1} [\mu * q(\delta+t, \delta+t+2(h+\varepsilon), z)]^i, \end{aligned} \tag{2.19}$$

where $\mu \in M_P,\ 0 < \varepsilon \le N,\ z \in R^d$,

$$0 \le \delta \le \delta+t \le \delta+t+h \le N, \quad \xi \in (0, 1/2).$$

Proof. Consider (2.3) with $\psi(r,a) = I_{[\delta+t,\delta+t+h]}(r) p(\varepsilon,a,z)$ and

$$f = q(t+\delta+\varepsilon, t+\delta+\varepsilon+h, a, z).$$

(1) When $k = 2$, by (2.4) and (2.5)

$$\begin{aligned} |P^{\varrho}_{0,\mu}[Z^{\varrho,\varepsilon}_{\delta,t+h}(z) - Z^{\varrho,\varepsilon}_{\delta,t}(z)]^2| &= |\langle \mu, u^{(2)}(0,t,\cdot)\rangle| \\ &\le 2c_2(h+\varepsilon)^{\xi} \cdot [\mu * q(\delta+t, \delta+t+2(h+\varepsilon), z)]. \end{aligned}$$

(2) If (2.19) is satisfied for all $r \le k-1$, then by (2.4), (2.5) and (2.19) we obtain (enlarging C where needed)

$$\begin{aligned} &|P^{\varrho}_{0,\mu}[Z^{\varrho,\varepsilon}_{\delta,t+h}(z) - Z^{\varrho,\varepsilon}_{\delta,t}(z)]^k| \\ &\le C'_k (h+\varepsilon)^{(k-1)\xi} \cdot [\mu * q(\delta+t, \delta+t+2(h+\varepsilon), z)] \\ &\quad + C''_k \sum_{2 \le j \le k-2} (h+\varepsilon)^{(k-j-1)\xi} (\mu * q)(h+\varepsilon)^{j\xi/2} \sum_{i=1}^{j-1} (\mu * q)^i \\ &\le C_k (h+\varepsilon)^{k\xi/2} \sum_{i=1}^{k-1} [\mu * q(\delta+t, \delta+t+2(h+\varepsilon), z)]^i. \end{aligned}$$

The last step is because $k-1 \ge k/2$ and $(k-j-1) + j/2 \ge k/2$ for the considered j, k. This completes the proof by induction.

Lemma 2.4. *Fixed $d = 1$, $0 \le \delta \le N, 0 < \alpha < 1, 0 < \xi < 1/2$, then there exists a constant C_k such that*

$$\begin{aligned} &|P^{\varrho}_{0,\mu}[Z^{\varrho,\varepsilon}_{\delta,t}(z_1) - Z^{\varrho,\varepsilon}_{\delta,t}(z_2)]^k| \\ &\le C_k \cdot |z_1 - z_2|^{k\alpha/2} \sum_{i=1}^{k-1} \Big[\mu * \sum_{j=1}^{2} q(\delta, \delta+t+\varepsilon, z_j)\Big]^i, \end{aligned} \tag{2.20}$$

where $\mu \in M_P, 0 < \varepsilon \le N, z_1,\ z_2 \in R^d$,

$$|z_1|, |z_2| \le N, 0 \le t \le \delta+t \le N.$$

Proof. Consider (2.3) with $\psi(r,a) = I_{[\delta,\delta+t]}(r)[p(\varepsilon,a,z_1) - p(\varepsilon,a,z_2)]$ and

$$f = q(\delta+\varepsilon, t+\delta+\varepsilon, a, z_1) - q(\delta+\varepsilon, t+\delta+\varepsilon, a, z_1).$$

The proof is similar to that of Lemma 2.3, and the detail is omitted.

Lemma 2.5.[1] *Let* $d \geq 1$, $\mu \in M_p$, *and* $\delta > 0$. *Then* $\mu * q(\delta, \delta + r, z)$ *is locally Lipschitz continuous in* $[r, z] \in R_+ \times R^d$. *Moreover, the Lipschitz constants are proportional to* $||\mu|| = \langle \mu, \phi_p \rangle$.

§3. Main Results and Proofs

Theorem 3.1. *Let* $d = 1$. $\xi \in (0, 1/2), \alpha \in (0, 1)$, *Fix* $\mu \in M_p$ *and* $\delta \geq 0$. *If* $\delta = 0$, *assume additionally that*

$$[r, z] \to \mu * q(0, r, z) \text{ is continuous on } R_+ \times R. \tag{3.1}$$

Then for $P_{0,\lambda}$-a.s.ϱ,

(a) *the* $L^2(P^{\varrho}_{0,\mu})$*-limit of* $y^{\varrho,\varepsilon}_{[\delta,\delta+t]}(z)$ *as* $\varepsilon \to 0$ *exists and is denoted by* $y^{\varrho}_{[\delta,\delta+t]}(z)$, $t \geq 0, z \in R$;

(b) *with respect to* $P^{\varrho}_{0,\mu}$, *the random measure* $Y^{\varrho}_{[\delta,\delta+t]}(dz)$ *is absolutely continuous with density function* $y^{\varrho}_{[\delta,\delta+t]}(z)$:

$$P^{\varrho}_{0,\mu}(Y^{\varrho}_{[\delta,\delta+t]}(dz) = y^{\varrho}_{[\delta,\delta+t]}(z)dz) = 1;$$

(c) *there exists a modification of* $y^{\varrho}_{[\delta,\delta+t]}(z)$ (*still denoted by* $y^{\varrho}_{[\delta,\delta+t]}(z)$) *such that* $y^{\varrho}_{[\delta,\delta+t]}(z)$ *is* (*jointly*) *continuous in* t *and* z.

Proof. According to Proposition 5 in [1], to prove (a) and (b), it suffices to show that for $P_{0,\lambda}$-a.s.ϱ,

$$\Pi_{0,\mu} \int_0^{\delta+t} L_{[w,\varrho]}(dr) q^2(r', \varepsilon + r', w_r, z) \to 0 \quad \text{as } \varepsilon \to 0 \tag{3.2}$$

for $r' = (\delta - r)_+$ and $r' = \delta + t - r$.

$$\begin{aligned}\text{The l.h.s. of (3.2)} &= \int \mu(da) \int_0^{\delta+t} dr \int \varrho_r(db) q^2(r', r' + \varepsilon, b, z) \\ &= \int \mu(da) \int_0^{\delta+t} \lambda^b(dr) \int dbq^2(r', r' + \varepsilon, b, z). \end{aligned} \tag{3.3}$$

When $r' = (\delta - r)_+$, (3.3) continues

$$\begin{aligned} &= \int \mu(da) \int_0^{\delta} \lambda^b(dr) \int dbp(r, a, b) q(\delta - r, \varepsilon + \delta - r, b, z) \int_{\delta-r}^{\varepsilon+\delta-r} dlp(l, b, z) \\ &+ \int \mu(da) \int_{\delta}^{t+\delta} \lambda^b(dr) \int dbp(r - s, a, b) q(0, \varepsilon, b, z) \int_0^{\varepsilon} dlp(l, b, z). \end{aligned} \tag{3.4}$$

Similar to (2.10), interchanging the order of the integration, we get

$$\begin{aligned} \int_0^{\delta} \lambda^b(dr) \int_{\delta-r}^{\varepsilon+\delta-r} dlp(l, b, z) &\leq C_1 \cdot \varepsilon^{\xi}, \\ \int_{\delta}^{t+\delta} \lambda^b(dr) \int_0^{\varepsilon} dlp(l, b, z) &\leq C_2 \cdot \varepsilon^{1/2}. \end{aligned} \tag{3.5}$$

Combining (3.3), (3.4) with (3.5), we have

$$\text{the l.h.s. of (3.2)} \leq C_1 \cdot \mu * q(\delta, \varepsilon + \delta, z) \cdot \varepsilon^{\xi} + C_2 \cdot \mu * q(\delta, \delta + \varepsilon + t, z) \cdot \varepsilon^{1/2}.$$

Then (3.2) follows from Condition (3.1) and Lemma 2.5.

When $r' = \delta + t - r$, (3.3) continues

$$= \int \mu(da) \int_0^{\delta+t} \lambda^b(dr) \int dbp(r,a,b) q(\delta+t-r, \varepsilon+\delta+t-r, b, z) \int_{\delta+t-r}^{\varepsilon+\delta+t-r} dlp(l,b,z). \quad (3.6)$$

Similarly, we have

$$\int_0^{\delta+t} \lambda^b(dr) \int_{\delta+t-r}^{\varepsilon+\delta+t-r} dlp(l,b,z) \le C \cdot \varepsilon^{\xi}. \quad (3.7)$$

Then (3.2) follows from (3.6), (3.7) , (3.1) and Lemma 2.4.

(c) From (a), by simple calculation, we have

$$P^{\varrho}_{0,\mu} y^{\varrho}_{[\delta,\delta+t]}(z) = \mu * q(\delta, \delta+t, z),$$

which is continuous in t and z by Condition (3.1) and Lemma 2.4. So it suffices to consider the continuity of the centered field

$$Z^{\varrho}_{\delta,t}(z) := \mu * q(\delta, \delta+t, z) - y^{\varrho}_{[\delta,\delta+t]}(z). \quad (3.8)$$

Combining (a) with Lemma 2.3, we get

$$\begin{aligned} P^{\varrho}_{0,\mu}[Z^{\varrho}_{\delta,t+h}(z) - Z^{\varrho}_{\delta,t}(z)]^{2k} &\le \liminf_{\varepsilon\to 0} P^{\varrho}_{0,\mu}[Z^{\varrho,\varepsilon}_{\delta,t+h}(z) - Z^{\varrho,\varepsilon}_{\delta,t}(z)]^{2k} \\ &\le C_k h^{k\xi} \sum_{i=1}^{2k-1} [\mu * q(\delta+t, \delta+t+2h, z)]^i. \end{aligned} \quad (3.9)$$

Similarly, from Lemma 2.4 we have

$$P^{\varrho}_{0,\mu}[Z^{\varrho}_{\delta,t}(z_1) - Z^{\varrho}_{\delta,t}(z_2)]^{2k} \le C_k \cdot |z_1 - z_2|^{k\alpha} \sum_{i=1}^{2k-1} \Big[\mu * \sum_{j=1}^{2} q(\delta, \delta+t+\varepsilon, z_j)\Big]^i. \quad (3.10)$$

By (3.1) and Lemma 2.5, the sums at (3.9) and (3.10) are finite. Then choosing k large enough, by Kolmogorov's moment criterion, we see that there exists a jointly continuous version of $Z^{\varrho}_{\delta,t}(z)$. The proof is completes.

Remark. (a) and (b) are proved by Dawson and Fleschmann (see Theorem 7 in [2]) for the special case that μ is the Lebesgue measure λ. (c) realizes the conjecture noted in Remark 9 in [2]. Furthermore, the Hölder continuity for the occupation density field is obtained in the following theorem.

Theorem 3.2. *Let* $d = 1,\ \xi \in (0, 1/2), \eta \in (0, \xi/2), \alpha \in (0,1)$, *$k$ be the smallest natural number satisying* $k > \frac{d+1}{\xi - 2\eta}$. *Then for* $P_{0,\lambda}$-a.s.ϱ,

(a) *if* $X^{\varrho}_0 = \mu, \delta \ge 0$, *and when* $\delta = 0$ *assume additionally that*

$$\begin{aligned} &[r,z] \to \mu * q(0,r,z) \text{ is locally } \eta - \text{Hölder continuous on } R_+ \times R \\ &\text{with Hölder constants propotional to } ||\mu||_p = \langle \mu, \phi_p \rangle. \end{aligned} \quad (3.11)$$

Then with respect to $P^{\varrho}_{0,\mu}$, *there exists a modification of* $y^{\varrho}_{[\delta,\delta+t]}(z)$ *such that for* $N \ge 1$, *with* $P^{\varrho}_{0,\mu}$*-a.s.*

$$|y^{\varrho}_{[\delta,\delta+t_1]}(z_1) - y^{\varrho}_{[\delta,\delta+t_2]}(z_2)| \le C_{\eta,N,k} |[t_1, z_1] - [t_2, z_2]|^{\eta}, \quad (3.12)$$

where $[t_i, z_i] \in E_N := [0, N] \times [-N, N],\ i = 1, 2$. $C_{\eta,N,k}$ *is a random constant satisfying*

$$P^{\varrho}_{0,\mu} C_{\eta,N,k} \le \text{const.} \cdot (1 \vee ||\mu||^{2k}_p)$$

with the constant independent of μ.

(b) *If* $X_0^\varrho = \lambda$ *(*λ *is the Lebesgue measure), then with respect to* $P_{0,\lambda}^\varrho$ *, there is a modification of* $y_{[\delta,\delta+t]}^\varrho(z)$ *such that for each* $N \geq 1$*,*

$$\sup_{G_N} \frac{|y_{[\delta,\delta+t_1]}^\varrho(z_1)\phi_p(z_1) - y_{[\delta,\delta+t_2]}^\varrho(z_2)\phi_p(z_2)|}{|[t_1, z_1] - [t_2, z_2]|^\eta} \tag{3.13}$$

is finite $P_{0,\lambda}^\varrho$*-a.s., where*

$$G_N := \{[t_i, z_i] \in R_+ \times R,\ i = 1, 2,\ 0 \leq t_i \leq N,\ z_i \in R,\ [t_1, z_1] \neq [t_2, z_2]\}.$$

Proof. (a) Let $\alpha > \xi$. Then based on (3.9) and (3.10) in Theorem 2.1, we obtain that for $P_{0,\lambda}$-a.s.ϱ,

$$P_{0,\mu}^\varrho(C_k^{-\frac{1}{2k}}[Z_{\delta,t_1}^\varrho(z_1) - Z_{\delta,t_2}^\varrho(z_2)])^{2k} \leq |[t_1, z_1] - [t_2, z_2]|^{k\eta} \tag{3.14}$$

with $\eta \in (0, \xi)$. The remaining proof is similar to Theorem 2 in [1].

(b) As λ is (spacially) shift-invariant, based on (a), we can prove (b) similar to Theorem 3 in [1], the detail is ommited.

Remark. The results in this paper are partially announced in [5].

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