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THE BLOWUP OF RADIALLY SYMMETRIC SOLUTIONS FOR 2-D QUASILINEAR WAVE EQUATIONS WITH CUBIC NONLINEARITY**

YIN Huicheng* ZHENG Qin*

Abstract

For a special class of quasilinear wave equations with small initial data which satisfy the nondegenerate assumption, the authors prove that the radially symmetric solution develops singularities in the second order derivatives in finite time while the first order derivatives and the solution itself remain continuous and small. More precisely, it turns out that this solution is a "geometric blowup solution of cusp type", according to the terminology posed by S. Alinhac[2].

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§1. Introduction

In this paper, we consider the following two dimensional quasilinear wave equations with the nonlinearity of cubic form:

$$\begin{cases} \partial_t^2 u - c^2(\partial_t u, \nabla u)\Delta u = f(\partial_t u, \nabla u), \\ u(x,0) = \varepsilon u_0(x), \quad \partial_t u(x,0) = \varepsilon u_1(x), \end{cases} \tag{1.1}$$

where $x = (x_1, x_2)$, $\varepsilon > 0$ is small enough,

$$c^2(\partial_t u, \nabla u) \equiv c^2(\partial_t u, \partial_r u) = 1 + a_1(\partial_t u)^2 + a_2\partial_t u\partial_r u + a_3(\partial_r u)^2 + O(|\partial_t u|^3 + |\partial_r u|^3),$$

$$f(\partial_t u, \nabla u) \equiv f(\partial_t u, \partial_r u) = b_1(\partial_t u)^3 + b_2(\partial_t u)^2\partial_r u + b_3\partial_t u(\partial_r u)^2 + b_4(\partial_r u)^3 + O(|\partial_t u|^4 + |\partial_r u|^4),$$

$$a_1 - a_2 + a_3 \neq 0,$$

$u_0(x), u_1(x)$ are C^∞ radial functions (that is, smooth functions of $|x|^2$)and supported in a fixed ball of radius M. Moreover $u_0(x) \not\equiv 0$ or $u_1(x) \not\equiv 0$.

Our aim is to study the lifespan T_ε of solutions to (1.1) and the breakdown mechanism when the solutions stop being smooth. In the special case of

$$f(\partial_t u, \partial_r u) = \frac{\partial_r u}{r} G(\partial_t u, \partial_r u), \quad G(\partial_t u, \partial_r u) = O(|\partial_t u|^2 + |\partial_r u|^2),$$

A. Hoshiga[1] has discussed the lifespan T_ε and studied the asymptotic behaviour of solutions. Recently, under the "generic" condition on the Cauchy data, S. Alinhac[2,3] proved the

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*Department of Mathematics, Nanjing University, Nanjing 210093, China.
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geometric blowup of small data solutions for the following two dimensional quasilinear wave equations

$$\Box u + \sum_{0\le i,j,k\le 2} g_{ij}^k \partial_k u \partial_{ij}^2 u = 0.$$

In this paper, we will use S. Alinhac's idea but some techniques are different (see [2, 3, 4]). We hope that this paper will help to study the general solutions (not radially symmetric) of (1.1).

Denote

$$R^{(1)}(\sigma) = \frac{1}{2\sqrt{2\pi}} \int_{s\ge\sigma} \frac{1}{\sqrt{s-\sigma}} [R(s,u_1) - \partial_s R(s,u_0)] ds,$$

where $R(s,v)$ is the Radon transform of the axisymmetric function $v(x)$. Set

$$h(\sigma) = (a_1 - a_2 + a_3)\partial_\sigma R^{(1)}(\sigma)\partial_\sigma^2 R^{(1)}(\sigma) - (b_1 - b_2 + b_3 - b_4)[\partial_\sigma R^{(1)}(\sigma)]^2.$$

Then $h(\sigma)$ has negative minimum as long as $u_0(x) \not\equiv 0$ or $u_1(x) \not\equiv 0$.

Our conclusions are as follows:

Theorem 1.1. *For the Equation* (1.1) *and the above assumptions, if $h(\sigma)$ has a unique minimum point σ_0 and $h'(\sigma_0) = 0, h''(\sigma_0) > 0$, then we have*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln T_\varepsilon = \tau_0 = -\frac{1}{h(\sigma_0)}. \tag{1.2}$$

Moreover there is a circle $M_\varepsilon = (x_\varepsilon, T_\varepsilon)$, $|x_\varepsilon| = r_\varepsilon$, such that

(i) $u \in C^1(\mathbb{R}^2 \times \{t \le T_\varepsilon\})$ *and* $|u|_{C^1(\mathbb{R}^2\times\{t\le T_\varepsilon\})} \le C\varepsilon$.

(ii) *u is of class C^2 away from M_ε, and satisfies*

$$\|\nabla_{x,t}^2 u\| = \sup_x |\nabla_{x,t}^2 u(x,t)| \le \frac{C\sqrt{t}}{\varepsilon(T_\varepsilon - t)}, \text{ and } \|\partial_t^2 u\|,\ \|\partial_t \nabla_x u\|,\ \|\nabla_x^2 u\| \ge \frac{1}{C}\frac{\sqrt{t}}{\varepsilon(T_\varepsilon - t)}.$$

Therefore

$$\lim_{t\to T_\varepsilon} \|\partial_t^2 u\| = \lim_{t\to T_\varepsilon} \|\partial_t \nabla_x u\| = \lim_{t\to T_\varepsilon} \|\nabla_x^2 u\| = +\infty.$$

Close to M_ε, we have a much better description of u, given by the following theorem.

Theorem 1.2. *Suppose $0 < \tau_1 < \tau_0$. There exist a domain*

$$D = \{(s,\tau) : -C_0 \le s \le M, \tau_1 \le \tau \le \tau_\varepsilon = \varepsilon^2 \ln T_\varepsilon\},$$

a point $m_\varepsilon = (s_\varepsilon, \tau_\varepsilon)$ and functions φ, w and $v \in C^3(D)$ with the following properties:

(i) *φ satisfies*

$$\begin{aligned} &\partial_s\varphi \ge 0, \quad \partial_s\varphi(s,\tau) = 0 \Longleftrightarrow (s,\tau) = m_\varepsilon, \\ &\partial_{s\tau}^2\varphi(m_\varepsilon) < 0, \quad \partial_s(\partial_s\varphi)(m_\varepsilon) = 0, \quad \partial_s^2(\partial_s\varphi)(m_\varepsilon) > 0. \end{aligned} \tag{1.3}$$

(ii) $\varphi(s,\tau_1) = s, s \ge M \Longrightarrow \varphi \equiv s$.

(iii) $\partial_s w = \partial_s\varphi v$, *and* $\partial_s v(m_\varepsilon) \ne 0$.

(iv) $\operatorname{supp} w, \operatorname{supp} v$ *are in* $s \le M$. *Introduce the mapping $\Phi(s,\tau) = (\sigma = \varphi(s,\tau), \tau)$. The function $G(\sigma,\tau)$ is determined by $G(\Phi) = w(s,\tau)$. Then in the domain $\Phi(D)$, $u = \frac{\varepsilon}{r^{\frac{1}{2}}} G(r - t, \varepsilon^2 \ln t)$ satisfies the Equation* (1.1), *moreover $\varphi, w, v \in C^k(D)$ for $0 < \varepsilon \le \varepsilon_k$.*

§2. The Lower Bound of T_ε

Now we look for an approximate solution to equation (1.1).

Set $\Box u_1(r,t)=0, u_1(r,0)=u_0(x)=u_0(r), \partial_t u_1(r,0)=u_1(x)=u_1(r)$. Let $U(\sigma,\tau)$ satisfy

$$\begin{cases} \partial^2_{\sigma\tau}U+\frac{a_1-a_2+a_3}{2}(\partial_\sigma U)^2\partial^2_\sigma U=\frac{b_1-b_2+b_3-b_4}{2}(\partial_\sigma U)^3, \\ U(\sigma,0)=R^{(1)}(\sigma). \end{cases} \tag{2.1}$$

By Lemma 2.3.1 in [5], we know τ_0 is the lifespan of solution to (2.1). Choosing a cut-off function $\chi\in C^\infty(\mathbb{R})$ equal to 1 in $(-\infty,1)$ and 0 in $(2,\infty)$, we define the approximate solution $w(x,t)$ by

$$w(x,t)=w(r,t)=\varepsilon\chi(\varepsilon t)u_1(r,t)+\varepsilon(1-\chi(\varepsilon t))\chi(-3\varepsilon(r-t))r^{-\frac12}U(r-t,\varepsilon^2\ln t). \tag{2.2}$$

Lemma 2.1. *For $t\le e^{\frac{\bar\tau}{\varepsilon^2}}$, $0<\bar\tau<\tau_0$, we have*

$$|Z^\alpha w(x,t)|\le C_{\alpha\bar\tau}\varepsilon(1+t)^{-\frac12}$$
$$\|Z^\alpha J(\cdot,t)\|_{L^2(\mathbb{R}^2)}\le C_{\alpha\bar\tau}[\varepsilon^{\frac54}(1+t)^{-\frac54}+\varepsilon^{\frac72}(1+t)^{-1}],$$

where

$$J(x,t)=J(r,t)=\partial_t^2 w-c^2(\partial_t w,\partial_r w)\Delta w-f(\partial_t w,\partial_r w),$$
$$Z\in\{\partial_t,\partial_{x_1},\partial_{x_2},x_1\partial_{x_2}-x_2\partial_{x_1},t\partial_t+x_1\partial_{x_1}+x_2\partial_{x_2},t\partial_{x_1}+x_1\partial_t,t\partial_{x_2}+x_2\partial_t\}.$$

Proof. From (2.1) and [5], we have

$$(\partial_\sigma U)(\sigma,\tau)=\frac{\partial_\sigma R^{(1)}(\rho)}{\sqrt{1-b[\partial_\sigma R^{(1)}(\rho)]^2\tau}},$$

where ρ satisfies $\sigma=\rho-\frac{a}{2b}\ln[1-b(\partial_\sigma R^{(1)}(\rho))^2\tau]$, $a=a_1-a_2+a_3$, $b=b_1-b_2+b_3-b_4$. Hence

$$U(\sigma,\tau)=\int_M^\rho\frac{\partial_\sigma R^{(1)}(\rho)[1+a\tau\partial_\sigma R^{(1)}(\rho)\partial^2_\sigma R^{(1)}(\rho)-b\tau(\partial_\sigma R^{(1)}(\rho))^2]}{[1-b(\partial_\sigma R^{(1)}(\rho))^2\tau]^{\frac32}}d\rho,$$
$$\partial_\tau U(\sigma,\tau)=\frac32\int_M^\sigma\frac{b[\partial_\sigma R^{(1)}(\rho)]^3[1+a\tau\partial_\sigma R^{(1)}(\rho)\partial^2_\sigma R^{(1)}(\rho)-b\tau(\partial_\sigma R^{(1)}(\rho))^2]}{[1-b(\partial_\sigma R^{(1)}(\rho))^2\tau]^{\frac52}}d\rho$$
$$+\int_M^\sigma\frac{[\partial_\sigma R^{(1)}(\rho)]^2[a\partial^2_\sigma R^{(1)}(\rho)-b\partial_\sigma R^{(1)}(\rho)]}{[1-b(\partial_\sigma R^{(1)}(\rho))^2\tau]^{\frac32}}d\rho.$$

By the inductive argument, as in [1] we easily get the following result

$$|\partial^l_\sigma\partial^m_\tau\partial_\sigma U(\sigma,\tau)|\le C_{lm\bar\tau}(1+|\sigma|)^{-\frac32-l-3m} \tag{2.3}$$
$$|\partial^\alpha_{\sigma\tau}U(\sigma,\tau)|\le C_{\alpha\bar\tau}. \tag{2.4}$$

To prove Lemma 2.1, as in [1] or [5], we distinguish three different cases.

(1) If $t\le\frac1\varepsilon$, then $w(x,t)=\varepsilon u_1(r,t)$ and

$$J(x,t)=-\varepsilon^3\{[a_1(\partial_t u_1)^2+a_2\partial_t u_1\partial_r u_1+a_3(\partial_r u_1)^2]\Delta u_1+[b_1(\partial_t u_1)^3+b_2(\partial_t u_1)^2\partial_r u_1$$
$$+b_3\partial_t u_1(\partial_r u_1)^2+b_4(\partial_r u_1)^3]+\varepsilon O(|\nabla_{r,t}u_1|^3|\partial_r^2 u_1|+|\nabla_{r,t}u_1|^4)\}.$$

Because $|Z^\alpha u_1|\le\frac{C_\alpha}{(1+t)^{\frac12}}$, we have $|Z^\alpha J|\le\frac{C_\alpha\varepsilon^3}{(1+t)^{\frac32}}$. Hence

$$\|Z^\alpha J(\cdot,t)\|_{L^2(\mathbb{R}^2)}\le\frac{C_\alpha\varepsilon^3}{(1+t)^{\frac12}}\le\frac{C_\alpha\varepsilon^{\frac94}}{(1+t)^{\frac54}}.$$

(2) If $\frac1\varepsilon\le t\le\frac2\varepsilon$, then the estimate similar to (1) still holds for the nonlinear term

$$-[a_1(\partial_t w)^2+a_2\partial_t w\partial_r w+a_3(\partial_r w)^2]\Delta w-[b_1(\partial_t w)^3+b_2(\partial_t w)^2\partial_r w$$
$$+b_3\partial_t w(\partial_r w)^2+b_4(\partial_r w)^3]+\varepsilon O(|\partial_r^2 u_1|\,|\nabla_{r,t}u_1|^3+|\nabla_{r,t}u_1|^4).$$

Hence we only examine

$$\begin{aligned}\Box w &= \varepsilon\Box[(1-\chi(\varepsilon t))(\chi(-3\varepsilon(r-t))-1)u_1] + \varepsilon\Box\Big[(1-\chi(\varepsilon t))\chi(-3\varepsilon(r-t))\Big(\frac{R^{(1)}(r-t)}{r^{\frac12}} \\ &\quad - u_1\Big)\Big] + \varepsilon\Box\{(1-\chi(\varepsilon t))\chi(-3\varepsilon(r-t))r^{-\frac12}[U(r-t,\varepsilon^2\ln t)-R^{(1)}(r-t)]\} \\ &= J_1+J_2+J_3.\end{aligned}$$

In the support of $\chi(-3\varepsilon(r-t))-1$, we have $r\le \frac56 t$. So it is easy to obtain

$$\|Z^\alpha J_1(.,t)\|_{L^2(\mathbb{R}^2)} \le \frac{C_\alpha\varepsilon^2}{1+t} \le \frac{C_\alpha \varepsilon^{\frac74}}{(1+t)^{\frac54}}.$$

If we note the following facts

$$\|S_m(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{\frac12}, \quad \text{if } m<-\frac12,$$
$$\|S_{-\frac12}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{\frac12}(\ln(e+t))^{\frac12},$$
$$\|S_m(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{m+1}, \quad \text{if } m>-\frac12,$$

where $S_m(r,t)$ satisfies $|S_m|\le C(1+|r-t|)^m$. Then for small ε we can get

$$\|Z^\alpha J_2(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C_\alpha\Big(\varepsilon^3+\varepsilon^3\sqrt{\ln\frac1\varepsilon}\Big) \le C_\alpha\frac{\varepsilon^{\frac54}}{(1+t)^{\frac54}}.$$

Since

$$\begin{aligned}J_3 &= \frac{\varepsilon^2}{r^{\frac12}}\Big\{\varepsilon(R^{(1)}-U)[6\chi'(\varepsilon t)\chi'(-3\varepsilon(r-t))+\chi''(\varepsilon t)\chi(-3\varepsilon(r-t))] - \chi'(\varepsilon t)\chi(-3\varepsilon(r-t)) \\ &\quad\times\Big[-2(\partial_\sigma U-\partial_\sigma R^{(1)})+\frac{\varepsilon^2}{t}\partial_\tau U\Big] + \frac{\varepsilon^2}{t}\partial_\tau U[6(1-\chi(\varepsilon t))\chi'(-3\varepsilon(r-t))-\chi'(\varepsilon t)] \\ &\quad -(1-\chi(\varepsilon t))\chi(-3\varepsilon(r-t))\frac{\varepsilon}{t^2}\partial_\tau U + (1-\chi(\varepsilon t))\chi(-3\varepsilon(r-t))\frac{\varepsilon}{t}\Big(-2\partial^2_{\sigma\tau}U+\frac{\varepsilon^2}{t}\partial^2_\tau U\Big)\Big\} \\ &\quad -\frac{\varepsilon}{4r^{\frac52}}(1-\chi(\varepsilon t))\chi(-3\varepsilon(r-t))[U(r-t,\varepsilon^2\ln t)-R^{(1)}(r-t)]\end{aligned}$$

and

$$\partial_\sigma U(\sigma,\tau)-\partial_\sigma R^{(1)}(\sigma) = \tau\int_0^1 \partial^2_{\sigma\tau}U(\sigma,\lambda\tau)d\lambda,$$

using (2.3), we have

$$|Z^\alpha[\partial_\sigma U(r-t,\varepsilon^2\ln t)-\partial_\sigma R^{(1)}(r-t)]| \le C_\alpha\varepsilon^2\ln\frac1\varepsilon(1+|r-t|)^{-\frac92}.$$

So we can get $|Z^\alpha J_3|\le C_\alpha\frac{\varepsilon^3}{r^{\frac12}}$.

It follows that

$$\|Z^\alpha J_3(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C_\alpha\varepsilon^{\frac52} \le C_\alpha\frac{\varepsilon^{\frac54}}{(1+t)^{\frac54}}.$$

(3) If $t\ge\frac2\varepsilon$, then $w(x,t)=\frac{\varepsilon}{r^{\frac12}}\chi(-3\varepsilon(r-t))U(r-t,\varepsilon^2\ln t)$. It follows that

$$\begin{aligned}J(r,t) &= -\frac{\varepsilon}{4r^{\frac52}}\chi(-3\varepsilon(r-t))U(r-t,\varepsilon^2\ln t) - \frac{\varepsilon^3}{r^{\frac12}t^2}\chi(-3\varepsilon(r-t))\partial_\tau U(r-t,\varepsilon^2\ln t) \\ &\quad + \frac{\varepsilon^5}{r^{\frac12}t^2}\chi\partial^2_\tau U(r-t,\varepsilon^2\ln t) + \frac{6\varepsilon^4}{r^{\frac12}t}\chi'(-3\varepsilon(r-t))\partial_\tau U(r-t,\varepsilon^2\ln t) - \frac{2\varepsilon^3}{r^{\frac12}t}\chi\partial^2_{\sigma\tau}U \\ &\quad -\frac{\varepsilon^3}{r^{\frac32}}\Big[a_1\Big(3\varepsilon\chi'U-\chi\partial_\sigma U+\chi\frac{\varepsilon^2}{t}\partial_\tau U\Big)^2 + a_2\Big(3\varepsilon\chi'U-\chi\partial_\sigma U+\chi\frac{\varepsilon^2}{t}\partial_\tau U\Big)\Big(-\frac{1}{2r}\chi U\end{aligned}$$

$$- 3\varepsilon\chi' U + \chi\partial_\sigma U\Big) + a_3\Big(-\frac{1}{2r}\chi U - 3\varepsilon\chi' U + \chi\partial_\sigma U\Big)^2\Big]\Big[9\varepsilon^2\chi'' U - 6\varepsilon\chi'\partial_\sigma U$$
$$+ \chi(\partial_\sigma U)^2 + \frac{1}{4r^2}\chi U\Big] - \frac{\varepsilon^3}{r^{\frac{3}{2}}}\Big[b_1\Big(3\varepsilon\chi' U - \chi\partial_\sigma U + \chi\frac{\varepsilon^2}{t}\partial_\tau U\Big)^3 + b_2\Big(3\varepsilon\chi' U - \chi\partial_\sigma U$$
$$+ \chi\frac{\varepsilon^2}{t}\partial_\tau U\Big)^2\cdot\Big(-\frac{1}{2r}\chi U - 3\varepsilon\chi' U + \chi\partial_\sigma U\Big) + b_3\Big(3\varepsilon\chi' U - \chi\partial_\sigma U + \chi\frac{\varepsilon^2}{t}\partial_\tau U\Big)$$
$$\cdot\Big(-\frac{1}{2r}\chi U - 3\varepsilon\chi' U + \chi\partial_\sigma U\Big)^2 + b_4\Big(-\frac{1}{2r}\chi U - 3\varepsilon\chi' U + \chi\partial_\sigma U\Big)^3\Big] + O\Big(\frac{\varepsilon^4}{r^2}\Big).$$

Noting that $t-\frac{2}{3\varepsilon} \le r \le t-\frac{1}{3\varepsilon}$ in the support of $\chi'(-3\varepsilon(r-t))$ or $\chi(-3\varepsilon(r-t))(1-\chi(-3\varepsilon(r-t)))$ and $\|S_m(.,t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{\frac{1}{2}}$, if $S_m(r,t)$ satisfies $|S_m| \le C(1+|r-t|)^m, m<-\frac{1}{2}$, then by Equation (2.1), we complete the proof of Lemma 2.1.

By the standard proof (see [1, 5, 6]), we can get the following conclusions.

Lemma 2.2. *Equation* (1.1) *has a* C^∞ *solution for* $0 \le t \le T$, *where* $\varepsilon^2 \ln T \le \bar{\tau} < \tau_0$. *If* $0<\varepsilon<\varepsilon_{\bar{\tau}}$, *then it follows that*

$$|Z^\alpha u(\cdot,t)| \le \frac{C_{\alpha\bar{\tau}}\varepsilon}{(1+t)^{\frac{1}{2}}}, \tag{2.5}$$

$$\|Z^\alpha\nabla_{x,t}(u-w)(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C_{\alpha\bar{\tau}}\varepsilon^{\frac{5}{4}}, \tag{2.6}$$

$$|Z^\alpha\nabla_{x,t}(u-w)| \le \frac{C_{\alpha\bar{\tau}}\varepsilon^{\frac{5}{4}}}{(1+t)^{\frac{1}{2}}(1+|r-t|)^{\frac{1}{2}}}, \tag{2.7}$$

where $\varepsilon_{\bar{\tau}}$ *and* $C_{\alpha\bar{\tau}}$ *are independent of* T *and* ε.

From Lemma 2.2, we easily obtain

$$\liminf_{\varepsilon\to 0}\varepsilon^2\ln T_\varepsilon \ge \tau_0. \tag{2.8}$$

§3. The Properties of u

We change Equation (1.1) as follows:

$$\begin{cases} \Box u + \frac{1-c^2(u_t,u_r)}{c^2(u_t,u_r)}\partial_t^2 u = \frac{f(u_t,u_r)}{c^2(u_t,u_r)}, \\ u(x,0)=\varepsilon u_0(x), \quad \partial_t u(x,0)=\varepsilon u_1(x). \end{cases}$$

Without loss of generality and for simplicity, we restrict ourselves to the following equation ($a \ne 0$):

$$\begin{cases} \Box u - a(\partial_t u)^2\partial_t^2 u = b(\partial_t u)^3, \\ u(x,0)=\varepsilon u_0(x), \quad \partial_t u(x,0)=\varepsilon u_1(x). \end{cases} \tag{3.1}$$

Because we will prove that the blowup location of $\partial_t^2 u$ is near the forward light cone surface $|x|=t+M$, we mainly consider the problem in the exterior area that $-C_0 \le r-t \le M$ and r,t are large enough, where C_0 is a large constant and satisfies $C_0 > 2|\sigma_0|$.

Assume

$$u_a = \varepsilon u_1 + \varepsilon^3 u_3 + \cdots + \varepsilon^{2p+1}u_{2p+1}, \quad p\in\mathbb{N}, \quad R_a = \Box u_a - a(\partial_t u_a)^2\partial_t^2 u_a - b(\partial_t u_a)^3.$$

Then we have

$$R_a = \varepsilon\Box u_1 + \varepsilon^3[\Box u_3 - a(\partial_t u_1)^2\partial_t^2 u_1 - b(\partial_t u_1)^3] + \cdots + \varepsilon^{2p+1}[\Box u_{2p+1}$$
$$- \sum_{l_1+l_2+l_3=2p+1}\partial_t u_{l_1}\partial_t u_{l_2}(a\partial_t^2 u_{l_3} + b\partial_t u_{l_3})] + \sum_{\substack{p\le j_1+j_2+j_3\le 3p \\ 0\le j_1,j_2,j_3\le p}} c_{j_1j_2j_3}\varepsilon^{2j_1+2j_2+2j_3+3}$$
$$\times\partial_t u_{2j_1+1}\partial_t u_{2j_2+1}(a\partial_t^2 u_{2j_3+1} + b\partial_t u_{2j_3+1}).$$

In terms of the initial data in (3.1), we obtain the following linear wave equations

$$\begin{aligned}
&\Box u_1 = 0, \quad u_1(x,0) = u_0(x), \quad \partial_t u_1(x,0) = u_1(x),\\
&\Box u_3 - a(\partial_t u_1)^2 \partial_t^2 u_1 - b(\partial_t u_1)^3 = 0, \quad u_3(x,0) = 0, \quad \partial_t u_3(x,0) = 0,\\
&\quad \cdots\cdots\cdots\cdots,\\
&\Box u_{2p+1} - \sum_{l_1+l_2+l_3=2p+1} \partial_t u_{l_1} \partial_t u_{l_2} (a\partial_t^2 u_{l_3} + b\partial_t u_{l_3}) = 0,\\
&u_{2p+1}(x,0) = 0, \quad \partial_t u_{2p+1}(x,0) = 0.
\end{aligned}$$

Denote $\sigma = r - t$, $z = \frac{1}{r}$. We introduce the slow time variable $\tau = \varepsilon^2 \ln t$.

Lemma 3.1. *Assume $S(\sigma) \in C^\infty$, $\mu \in \mathbb{R}$, $k \in \mathbb{N}$, for any $N \in \mathbb{N}$. Then there exist functions $M_l(r-t, \frac{1}{r})(0 \le l \le k)$ and $L_l(r-t, \frac{1}{r})(0 \le l \le k+1)$ such that*

$$\Box \frac{1}{r^{\frac{1}{2}}} \Big\{ \sum_{0\le l\le k} t^{\mu+\frac{3}{2}} (\ln t)^l M_l + \sum_{0 \le l \le k+1} (\ln t)^l L_l \Big\} = t^\mu (\ln t)^k S(r-t) + O(z^N).$$

Proof. See [4, Lemma 3.1.1 (b)].

In terms of Lemma 3.1, we can obtain the structure and properties of u_a.

Lemma 3.2. *For any $N \in \mathbb{N}$, there exist C^∞ functions $L_k^l(\sigma, z)$ such that*

$$u_a = \frac{\varepsilon}{r^{\frac{1}{2}}} \sum_{0\le k\le p} \sum_{0\le l\le k} \varepsilon^{2(k-l)} \tau^l L_k^l\Big(r-t, \frac{1}{r}\Big) + O(z^N), \textit{ and } R_a = O\Big(\varepsilon^{2p+3} \frac{(\ln t)^{3p}}{r^{\frac{3}{2}}}\Big).$$

Proof. Since $u_0(x) = u_0(r)$, $u_1(x) = u_1(r)$, we have $u_1 = \frac{F_0(r-t,\frac{1}{r})}{r^{\frac{1}{2}}}$.

Because

$$\Box u_3 = a(\partial_t u_1)^2 \partial_t^2 u_1 + b(\partial_t u_1)^3 = a\frac{[(\partial_\sigma F_0)^2 \partial_\sigma^2 F_0](r-t,\frac{1}{r})}{r^{\frac{3}{2}}} - b\frac{[(\partial_\sigma F_0)^3](r-t,\frac{1}{r})}{r^{\frac{3}{2}}},$$

by Lemma 3.1 we have $u_3 = \frac{1}{r^{\frac{1}{2}}} \ln t L_1^{(1)}\Big(r-t, \frac{1}{r}\Big) + \frac{1}{r^{\frac{1}{2}}} \sum\limits_{0\le l\le N} t^{-l} M_1^{(l)}\Big(r-t,\frac{1}{r}\Big) + O(z^N)$. By induction and Lemma 3.1 and in away similar to the proof of Proposition 3.1 in [4], we get

$$u_{2k+1} = \frac{1}{r^{\frac{1}{2}}} \sum_{1\le l\le k} (\ln t)^l L_k^{(l)}\Big(r-t,\frac{1}{r}\Big) + \frac{1}{r^{\frac{1}{2}}} \sum_{0\le l\le N} t^{-l} M_k^{(l)}\Big(r-t,\frac{1}{r}\Big) + O(z^N), \ k \le p.$$

Hence, using $u_a = \sum\limits_{k=0}^{p} \varepsilon^{2k+1} u_{2k+1}$, we complete the proof.

Therefore, in $0 \le t \le \frac{2}{\varepsilon^{N_0}}$ (where N_0 is a sufficiently large constant), by Lemma 3.2 we have $R_a = O(\varepsilon^{2p+2})$. We define the time interval $0 \le t \le \frac{2}{\varepsilon^{N_0}}$ as " the first time interval". The approximate solution u_a in the first time interval is written as u_a^I. By the standard energy estimate, in the first time interval we easily obtain

$$|\partial_{x,t}^\alpha (u - u_a^I)| \le C_{\alpha p} \varepsilon^{2p+2-2N_0}. \tag{3.2}$$

We define the time interval $\frac{1}{\varepsilon^{N_0}} \le t \le e^{\frac{\bar\tau}{\varepsilon^2}}$ ($0 < \bar\tau < \tau_0$, where $\bar\tau$ is a constant) as "the second time interval". Now we analyze the properties of u in the second time interval. Assume $u = \frac{\varepsilon}{r^{\frac{1}{2}}} G(\sigma,\tau)$, for $\tau \ge -N_0 \varepsilon^2 \ln \varepsilon$. Then $G(\sigma,\tau)$ satisfies the following equation

$$\begin{aligned}
&-2[1 + \sigma e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}} (-\partial_\sigma G + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}} \partial_\tau G)^2] \partial_{\sigma\tau}^2 G + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}} [1 + \sigma e^{-\frac{\tau}{\varepsilon^2}}\\
&- a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}} (-\partial_\sigma G + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}} \partial_\tau G)^2] \partial_\tau^2 G - a(\partial_\sigma G - \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}} \partial_\tau G)^2 \partial_\sigma^2 G
\end{aligned}$$

$$- e^{-\frac{\tau}{\varepsilon^2}}[1+\sigma e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(-\partial_\sigma G + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G)^2]\partial_\tau G - \frac{e^{-\frac{\tau}{\varepsilon^2}}}{4\varepsilon^2(1+\sigma e^{-\frac{\tau}{\varepsilon^2}})}G$$
$$+ b(\partial_\sigma G - \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G)^3 = 0. \tag{3.3}$$

Now we prove the following conclusion.

Lemma 3.3. *For $0 < \bar\tau < \tau_0$ and sufficiently small ε, assume* (3.1) *has a C^∞ solution $u = \frac{\varepsilon}{r^{\frac12}}G(\sigma,\tau)$ in the domain*

$$\bar D = \{-N_0\varepsilon^2 \ln\varepsilon \le \tau \le \bar\tau, -C_0 + \frac{\tau}{\eta} \le r - t \le M\},$$

where $\eta \ge \frac{2\tau_0}{C_0}$. Then there exists a constant $\varepsilon_k > 0$ depending on $C_0, \bar\tau$ such that for $0<\varepsilon\le\varepsilon_k$, $\sup\limits_{\bar D}\sum\limits_{|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G(\sigma,\tau)| \le C_{k\bar\tau}$.

Proof. Choose $p = 2(N_0+1)k$ in (3.2), $N_0 \ge 4k, k \ge 5$. By Lemma 3.2 and (3.2), we have

$$G(\sigma,\tau)|_{\tau=-N_0\varepsilon^2\ln\varepsilon} \equiv G_0(\sigma) = R^{(1)}(\sigma) + O(\varepsilon^2\ln\varepsilon),$$
$$\partial^\alpha_{\sigma,\tau}G(\sigma,\tau)|_{\tau=-N_0\varepsilon^2\ln\varepsilon} = \partial^\alpha_{\sigma,\tau}\Big[\Big(\frac{r^{\frac12}}{\varepsilon}u^I_a\Big)\Big(\sigma, \frac{1}{\sigma+e^{\frac{\tau}{\varepsilon^2}}}, \tau\Big)\Big]\Big|_{\tau=-N_0\varepsilon^2\ln\varepsilon} + O(\varepsilon^2), \quad |\alpha|\le k.$$

It is easy to know

$$\Big|\partial^\alpha_{\sigma,\tau}\Big[\Big(\frac{r^{\frac12}}{\varepsilon}u^I_a\Big)\Big(\sigma, \frac{1}{\sigma+e^{\frac{\tau}{\varepsilon^2}}}, \tau\Big)\Big]\Big|_{\tau=-N_0\varepsilon^2\ln\varepsilon}\Big| \le C_\alpha \text{ for } |\alpha|\le k.$$

Moreover the compatibility of traces of G is known because G exists in $\tau \le -2N_0\varepsilon^2\ln\varepsilon$ (that is, Equation (3.1) has a solution in $t \le \frac{2}{\varepsilon^{N_0}}$).

In order to prove Lemma 3.3, we first prove the local existence of solution to (3.3). We use the following iteration scheme

$$\begin{cases} \partial^2_{\sigma\tau}G_{n+1} + \frac{a(\partial_\sigma G_n - \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G_n)^2}{2[1+\sigma e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(-\partial_\sigma G_n + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G_n)^2]}\partial^2_\sigma G_{n+1} - \frac12\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial^2_\tau G_{n+1} \\ +\frac12 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G_n + \frac{e^{-\frac{\tau}{\varepsilon^2}}}{8\varepsilon^2(1+\sigma e^{-\frac{\tau}{\varepsilon^2}})[1+\sigma e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(-\partial_\sigma G_n + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G_n)^2]}G_n \\ -\frac{b(\partial_\sigma G_n - \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G_n)^3}{2[1+\sigma e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(-\partial_\sigma G_n + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G_n)^2]} = 0, \\ G_{n+1}|_{\tau=-N_0\varepsilon^2\ln\varepsilon} = G(\sigma,\tau)|_{\tau=-N_0\varepsilon^2\ln\varepsilon}, \quad \partial^\alpha_{\sigma,\tau}G_{n+1}(\sigma,\tau)|_{\sigma=M} = 0, \quad |\alpha|\le k, \\ \partial_\tau G_{n+1}|_{\tau=-N_0\varepsilon^2\ln\varepsilon} = \partial_\tau G|_{\tau=-N_0\varepsilon^2\ln\varepsilon}, \end{cases} \tag{3.4}$$

where

$$G_0(\sigma,\tau) = \sum_{i=0}^k \frac{1}{i!}(\tau + N_0\varepsilon^2\ln\varepsilon)^i \partial^i_\tau G(\sigma, -N_0\varepsilon^2\ln\varepsilon), \quad G_n(\sigma,\tau) \in H^\infty(\bar D).$$

From (3.3), we easily know

$$\partial^\alpha_{\sigma,\tau}G_{n+1}(\sigma,\tau)|_{\tau=-N_0\varepsilon^2\ln\varepsilon} = \partial^\alpha_{\sigma,\tau}G(\sigma,\tau)|_{\tau=-N_0\varepsilon^2\ln\varepsilon} \text{ for } |\alpha|\le k.$$

Assume $m, h > 0$ are large constants, and $\bar\tau(m,h) > 0$ is a constant depending on m, h such that

$$\sup_{\bar D_0}\sum_{|\alpha|\le 3}|\partial^\alpha_{\sigma,\tau}G_n| \le 2\sup_{\sigma\in[-C_0,M]}\sum_{|\alpha|\le 3}|\partial^\alpha_{\sigma,\tau}G(\sigma, -N_0\varepsilon^2\ln\varepsilon)|,$$
$$\iint_{\bar D_0}\sum_{|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G_n|^2 e^{h(\sigma - m\tau)}d\sigma d\tau \le 2\int_{\{\tau=-N_0\varepsilon^2\ln\varepsilon\}}\sum_{|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G|^2 e^{h(\sigma-m\tau)}d\sigma,$$

where

$$\bar{D}_0=\Big\{-N_0\varepsilon^2\ln\varepsilon\le\tau\le\bar{\tau}(m,h),-C_0+\frac{\tau}{\eta}\le\sigma\le M\Big\}.$$

Without loss of generality, we assume $\bar{\tau}(m,h)\le 1$.

At present, for sufficiently large m,h and small ε, we will prove

$$\sup_{\bar{D}_0}\sum_{|\alpha|\le 3}|\partial^\alpha_{\sigma,\tau}G_{n+1}|\le 2\sup_{\sigma\in[-C_0,M]}\sum_{|\alpha|\le 3}|\partial^\alpha_{\sigma,\tau}G(\sigma,-N_0\varepsilon^2\ln\varepsilon)|,$$

$$\iint_{\bar{D}_0}\sum_{|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G_{n+1}|^2e^{h(\sigma-m\tau)}d\sigma d\tau\le 2\int_{\{\tau=-N_0\varepsilon^2\ln\varepsilon\}}\sum_{|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G|^2e^{h(\sigma-m\tau)}d\sigma.$$

Denote $p_n(\sigma,\tau)=\frac{a(\partial_\sigma G_n-\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G_n)^2}{2[1+\sigma e^{-\frac{\tau}{\varepsilon^2}}-a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(-\partial_\sigma G_n+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G_n)^2]}$. From (3.4) and the expression of $G_0(\sigma,\tau)$, we know $\partial^\alpha_{\sigma,\tau}G_{n+1}|_{\tau=-N_0\varepsilon^2\ln\varepsilon}=\partial^\alpha_{\sigma,\tau}G(\sigma,\tau)|_{\tau=-N_0\varepsilon^2\ln\varepsilon}$ for $|\alpha|\le k$.

We should keep in mind the fact $|\partial^l_\tau e^{-\frac{\tau}{\varepsilon^2}}|\le\varepsilon^{N_0-2l}$ for $\tau\ge -N_0\varepsilon^2\ln\varepsilon$.

Choose the multiplier $MG_{n+1}=a'e^{h(\sigma-m\tau)}\partial_\sigma G_{n+1}-b'e^{h(\sigma-m\tau)}\partial_\tau G_{n+1}$, where a', $b'>0$ are appropriate constants. By the integrations by parts, we have

$$\begin{aligned}&\iint_{\bar{D}_0}\Big[\partial^2_{\sigma\tau}G_{n+1}+p_n(\sigma,\tau)\partial^2_\sigma G_{n+1}-\frac12\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial^2_\tau G_{n+1}\Big]MG_{n+1}d\sigma d\tau\\&=\iint_{\bar{D}_0}[K_0(\partial_\sigma G_{n+1})^2+K_1(\partial_\tau G_{n+1})^2+K_2\partial_\sigma G_{n+1}\partial_\tau G_{n+1}]d\sigma d\tau+I_1-I_0+J,\end{aligned}$$

where

$$\begin{aligned}2K_0&=(a'mh-a'hp_n-a'\partial_\sigma p_n-b'\partial_\tau p_n+b'mhp_n)e^{h(\sigma-m\tau)},\\2K_1&=\Big(b'h-\frac{a'}{2}h\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}+\frac{b'}{2}mh\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}+\frac{b'}{2}e^{-\frac{\tau}{\varepsilon^2}}\Big)e^{h(\sigma-m\tau)},\\K_2&=\Big(b'hp_n+b'\partial_\sigma p_n-\frac{a'}{2}mh\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}-\frac{a'}{2}e^{-\frac{\tau}{\varepsilon^2}}\Big)e^{h(\sigma-m\tau)}\\I_1&=\int_{\{\tau=\bar{\tau}(m,h)\}}\Big[\Big(\frac{a'}{2}+\frac{b'}{2}p_n\Big)(\partial_\sigma G_{n+1})^2-\frac{a'}{2}\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\sigma G_{n+1}\partial_\tau G_{n+1}\\&\quad+\frac{b'}{4}\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau G_{n+1})^2\Big]e^{h(\sigma-m\tau)}d\sigma,\\I_0&=\int_{\{\tau=-N_0\varepsilon^2\ln\varepsilon\}}\Big[\Big(\frac{a'}{2}+\frac{b'}{2}p_n\Big)(\partial_\sigma G_{n+1})^2-\frac{a'}{2}\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\sigma G_{n+1}\partial_\tau G_{n+1}\\&\quad+\frac{b'}{4}\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau G_{n+1})^2\Big]e^{h(\sigma-m\tau)}d\sigma,\\J&=\int_{-N_0\varepsilon^2\ln\varepsilon}^{\bar{\tau}(m,h)}\Big\{\Big[\Big(\frac{b'}{2}-\frac{a'}{4}\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\Big)+\frac{b'}{4\eta}\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\Big](\partial_\tau G_{n+1})^2\\&\quad+\Big(b'p_n-\frac{a'}{2\eta}\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\Big)\partial_\sigma G_{n+1}\partial_\tau G_{n+1}\\&\quad+\Big[\frac{1}{\eta}\Big(\frac{a'}{2}+\frac{b'}{2}p_n\Big)-\frac{a'}{2}p_n\Big](\partial_\sigma G_{n+1})^2\Big\}e^{h(\sigma-m\tau)}d\tau.\end{aligned}$$

We assume $\sum_{|\alpha|\le 2}|\partial^\alpha_{\sigma,\tau}p_n|\le A$, where A depends on the bound of at most third order derivatives of G_n. Let $b'\ge 1$, $a'\ge 216b'(1+A)^2$, $m>4A+4$, $h\ge\min\{4A,4\}$, $\varepsilon\le\min\{\frac{3b'}{2a'},\frac{1}{a'},\frac{1}{h}\}$, and choose $\eta\le\frac{1}{1+A}$ (this can be done if we enlarge C_0. Additionally, $\partial^\alpha_{\sigma,\tau}G(\sigma,\tau)|_{\tau=-N_0\varepsilon^2\ln\varepsilon}(|\alpha|\le k)$ are uniformly bounded on σ when $\varepsilon\le\varepsilon_{C_0}$). Then we get

$$\begin{aligned}&\iint_{\bar D_0}[K_0(\partial_\sigma G_{n+1})^2+K_1(\partial_\tau G_{n+1})^2+K_2\partial_\sigma G_{n+1}\partial_\tau G_{n+1}]d\sigma d\tau\\ \ge\ &\frac{h}{8}\iint_{\bar D_0}[a'm|\partial_\sigma G_{n+1}|^2+b'|\partial_\tau G_{n+1}|^2]e^{h(\sigma-m\tau)}d\sigma d\tau,\\ &I_1\ge 0,\quad J\ge 0,\quad I_0\le\int_{\{\tau=-N_0\varepsilon^2\ln\varepsilon\}}[a'(\partial_\sigma G_{n+1})^2+(\partial_\tau G_{n+1})^2]e^{h(\sigma-m\tau)}d\sigma.\end{aligned}$$

Using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}&\frac{h}{16}\iint_{\bar D_0}[a'm|\partial_\sigma G_{n+1}|^2+b'|\partial_\tau G_{n+1}|^2]e^{h(\sigma-m\tau)}d\sigma d\tau\\ \le\ &\varepsilon\iint_{\bar D_0}[(\partial_\tau G_n)^2+(G_n)^2]e^{h(\sigma-m\tau)}d\sigma d\tau\\ &+\int_{\{\tau=-N_0\varepsilon^2\ln\varepsilon\}}[a'(\partial_\sigma G_{n+1})^2+(\partial_\tau G_{n+1})^2]e^{h(\sigma-m\tau)}d\sigma.\end{aligned}\tag{3.5}$$

Similarly, for $2\le|\alpha|\le k$ and large h, we have

$$\begin{aligned}&\frac{hb'}{32}\iint_{\bar D_0}\sum_{2\le|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G_{n+1}|^2e^{h(\sigma-m\tau)}d\sigma d\tau\\ \le\ &\varepsilon\iint_{\bar D_0}\sum_{|\alpha|\le k}(\partial^\alpha_{\sigma,\tau}G_n)^2e^{h(\sigma-m\tau)}d\sigma d\tau\\ &+2a'\int_{\{\tau=-N_0\varepsilon^2\ln\varepsilon\}}\sum_{2\le|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G(\sigma,\tau)|^2e^{h(\sigma-m\tau)}d\sigma.\end{aligned}\tag{3.6}$$

Hence for large h we also have

$$\iint_{\bar D_0}\sum_{2\le|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G_{n+1}|^2e^{h(\sigma-m\tau)}d\sigma d\tau\le\int_{\{\tau=-N_0\varepsilon^2\ln\varepsilon\}}\sum_{|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G|^2e^{h(\sigma-m\tau)}d\sigma.$$

Since

$$\sum_{|\alpha|\le 3}\partial^\alpha_{\sigma,\tau}G_{n+1}=\sum_{|\alpha|\le 3}\partial^\alpha_{\sigma,\tau}G(\sigma,-N_0\varepsilon^2\ln\varepsilon)+\int_{-N_0\varepsilon^2\ln\varepsilon}^{\tau}\int_M^\sigma\sum_{|\alpha|\le 3}\partial^\alpha_{\sigma,\tau}(\partial^2_{\sigma\tau}G_{n+1})d\sigma d\tau,$$

we have

$$\begin{aligned}\sum_{|\alpha|\le 3}|\partial^\alpha_{\sigma,\tau}G_{n+1}|\le&\sum_{|\alpha|\le 3}|\partial^\alpha_{\sigma,\tau}G(\sigma,-N_0\varepsilon^2\ln\varepsilon)|+C\sum_{|\alpha|\le 5}\Big[\iint_{\bar D_0}|\partial^\alpha_{\sigma,\tau}G_{n+1}|^2e^{h(\sigma-m\tau)}d\sigma d\tau\Big]^{\frac12}\\ &\times e^{\frac{h}{4}(C_0+M+1)}\sqrt{\bar\tau(m,h)+N_0\varepsilon^2\ln\varepsilon}.\end{aligned}$$

Choosing again an appropriate constant $\bar\tau(m,h)$, we easily obtain

$$\begin{aligned}&\sup_{\bar D_0}\sum_{|\alpha|\le 3}|\partial^\alpha_{\sigma,\tau}G_{n+1}|\le 2\sup_{\sigma\in[-C_0,M]}\sum_{|\alpha|\le 3}|\partial^\alpha_{\sigma,\tau}G(\sigma,-N_0\varepsilon^2\ln\varepsilon)|,\quad\sum_{|\alpha|\le 2}|\partial^\alpha_{\sigma,\tau}p_{n+1}|\le A,\\ &\iint_{\bar D_0}\sum_{|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G_{n+1}|^2e^{h(\sigma-m\tau)}d\sigma d\tau\le 2\int_{\{\tau=-N_0\varepsilon^2\ln\varepsilon\}}\sum_{|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G|^2e^{h(\sigma-m\tau)}d\sigma.\end{aligned}$$

Moreover we can easily prove that $\{G_n\}$ is a Cauchy sequence in $H^{k-1}(\bar D_0)$, so Equation

(3.4) has a unique solution in $\bar{D}_0$ and

$$\sup_{\bar{D}_0}\sum_{|\alpha|\le k-2}|\partial^\alpha_{\sigma,\tau}G(\sigma,\tau)|\le C_k.$$

Now let $w(\sigma,\tau)$ satisfy the following equation:

$$\begin{cases}\partial^2_{\sigma\tau}w+\frac{a}{2}(\partial_\sigma w)^2\partial^2_\sigma w-\frac{b}{2}(\partial_\sigma w)^3=0,\\ w|_{\tau=-N_0\varepsilon^2\ln\varepsilon}=G|_{\tau=-N_0\varepsilon^2\ln\varepsilon},\quad \partial^\alpha_{\sigma,\tau}w|_{\sigma=M}=0,\quad |\alpha|\ge 0.\end{cases}\tag{3.7}$$

Then for any fixed $\bar{\tau}<\tau_0$ and small ε, we know that w exists in $[-N_0\varepsilon^2\ln\varepsilon,\bar{\tau}]$.

Set $V=G-w$. For small ε, if we assume

$$\sum_{|\alpha|\le k}\iint_{\bar{D}}|\partial^\alpha_{\sigma,\tau}V|^2d\sigma d\tau\le 1,$$

then by the energy estimate as above we can get

$$\sum_{|\alpha|\le k}\iint_{\bar{D}}|\partial^\alpha_{\sigma,\tau}V|^2d\sigma d\tau\le \frac{1}{2}.$$

Therefore, by the continuous induction argument, we complete the proof.

From Lemma 3.3, we easily get the following conclusion.

Theorem 3.1. *Under the assumptions of Theorem* 1.1, *for any fixed* τ_1,τ_2 *satisfying* $0<\tau_1<\tau_2<\tau_0$, *we assume that* (3.1) *has a* C^∞ *solution* $u=\frac{\varepsilon}{r^{\frac{1}{2}}}G(\sigma,\tau)$ *in the domain*

$$\{\tau_1\le\tau\le\tau_2,-C_0\le\sigma\le M\},\quad C_0>|\sigma_0|.$$

For $k\in\mathbb{N}$, *there exists a constant* $\varepsilon_k>0$ *depending on* k,τ_1,τ_2, *such that for* $0<\varepsilon\le\varepsilon_k$, $\sup_{|\alpha|\le k}|\partial^\alpha_{\sigma,\tau}G(\sigma,\tau)|\le C_{k\tau_1\tau_2}$.

§4. The Proof of Theorem 1.2

Below we study the solution u to (3.1) only in the exterior area $\bar{D}$:$\{0<\tau_1\le\tau=\varepsilon^2\ln t\le\tau_\varepsilon=\varepsilon^2\ln T_\varepsilon,-C_0\le\sigma=r-t\le M\}$. Assume $u(x,t)=\frac{\varepsilon}{r^{\frac{1}{2}}}G(\sigma,\tau)$ in the domain $\{\tau_1\le\tau\le\tau_2,-C_0\le r-t\le M\}$(where $\tau_2<\tau_0$). By Theorem 3.1 we know that $G(\sigma,\tau)$ is bounded in C^k(independently of ε) for $k\in\mathbb{N}$ and $0<\varepsilon\le\varepsilon_k$($\varepsilon_k$ depends on C_0,τ_1,τ_2). Denote

$$P(u)=\Box u-a(\partial_t u)^2\partial_t^2u-b(\partial_t u)^3.$$

Then G satisfies the following equation.

Lemma 4.1.

$$\begin{aligned}\frac{r^{\frac{3}{2}}}{\varepsilon^3}P(u)=&-2[1+\sigma e^{-\frac{\tau}{\varepsilon^2}}-a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(-\partial_\sigma G+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G)^2]\partial^2_{\sigma\tau}G+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}[1+\sigma e^{-\frac{\tau}{\varepsilon^2}}\\&-a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(-\partial_\sigma G+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G)^2]\partial^2_\tau G-a(\partial_\sigma G-\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G)^2\partial^2_\sigma G\\&-e^{-\frac{\tau}{\varepsilon^2}}[1+\sigma e^{-\frac{\tau}{\varepsilon^2}}-a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(-\partial_\sigma G+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G)^2]\partial_\tau G-\frac{e^{-\frac{\tau}{\varepsilon^2}}}{4\varepsilon^2(1+\sigma e^{-\frac{\tau}{\varepsilon^2}})}G\\&+b(\partial_\sigma G-\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau G)^3\equiv P(G).\end{aligned}$$

We want to solve $P(G)=0$ *in* $\bar{D}$ *with two trace conditions on* $\{\tau=\tau_1\}$ *corresponding to that for* u,G *supported in* $\{\sigma\le M\}$.

To blow up G in $\bar{D}$, we introduce the singular transform $\Phi = (\sigma = \varphi(s,\tau), \tau)$, where $\varphi(s,\tau)$ satisfies (1.3) in Theorem 1.2.

Denote $G(\Phi) = w(s,\tau), (\partial_\sigma G)(\Phi) = v(s,\tau)$. Then we have $P(G) = \frac{\partial_s v}{\partial_s \varphi} I_1 + I_2$, where

$$\begin{aligned} I_1 = & -a[v - \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]^2 + 2\{1 + \varphi e^{-\frac{\tau}{\varepsilon^2}} \\ & - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}[-v + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]^2\}\partial_\tau\varphi + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\{1 + \varphi e^{-\frac{\tau}{\varepsilon^2}} \\ & - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}[-v + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]^2\}(\partial_\tau\varphi)^2, \\ I_2 = & -2\{1 + \varphi e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}[-v + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]^2\}\partial_\tau v \\ & + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\{1 + \varphi e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}[-v + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]^2\}(\partial_\tau^2 w - v\partial_\tau^2\varphi \\ & - 2\partial_\tau v\partial_\tau\varphi) - e^{-\frac{\tau}{\varepsilon^2}}\{1 + \varphi e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}[-v + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]^2\}\times \\ & \times(\partial_\tau w - v\partial_\tau\varphi) - \frac{e^{-\frac{\tau}{\varepsilon^2}}}{4\varepsilon^2(1 + \varphi e^{-\frac{\tau}{\varepsilon^2}})} w + b[v - \varepsilon^3 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]^3. \end{aligned}$$

In order to solve the equation $P(G) = 0$, as in [3], we only solve the following blowup system on the unknown functions φ and w, v:

$$I_1 = 0, \quad I_2 = 0, \quad I_3 = \partial_s w - \partial_s\varphi v = 0. \tag{4.1}$$

We will use Nash-Moser iteration to solve the blowup system (4.1). This will be done in two steps.

Step 1. Construction of an approximate solution for the blowup system (4.1)

First, we show that the blowup system (4.1) can be locally solved. Since for small ε,

$$\frac{\partial I_1}{\partial(\partial_\tau\varphi)} = 2 + O(e^{-\frac{\tau}{\varepsilon^2}}) \neq 0,$$

by $I_1 = 0$ and the implicit function theorem we have

$$\partial_\tau\varphi = E(\tau, (\partial_\sigma G)(\varphi,\tau), (\partial_\tau G)(\varphi,\tau), \varphi). \tag{4.2}$$

The function G is in fact known and smooth in a small strip

$$S_1 = \{\tau_1 \le \tau \le \tau_1 + \eta, \eta > 0\}.$$

By Theorem 3.1 and the existence of solution for the ordinary differential equation, we can solve Equation (4.2) with the initial data $\varphi(s,\tau_1) = s$ for η small enough. Setting then

$$\bar{w}(s,\tau) = G(\varphi,\tau), \quad \bar{v}(s,\tau) = (\partial_\sigma G)(\varphi,\tau),$$

we obtain a smooth solution $(\bar{\varphi}, \bar{v}, \bar{w})$ of (4.1) in S_1. Moreover $\bar{w}, \bar{v}$ and $\bar{\varphi} - s$ are smooth and flat on $\{s = M\}$.

For $\varepsilon = 0$, the exact solution $\bar{\varphi}_0, \bar{v}_0, \bar{w}_0$ of the blowup system is

$$\begin{aligned} \bar{\varphi}_0(s,\tau) &= s - \frac{a}{2b}\ln[1 - b(\tau - \tau_1)(\partial_\sigma R^{(1)}(s,\tau_1))^2], \\ \bar{v}_0(s,\tau) &= \frac{(\partial_\sigma R^{(1)})(s,\tau_1)}{\sqrt{1 - b(\tau - \tau_1)(\partial_\sigma R^{(1)}(s,\tau_1))^2}}, \\ \bar{w}_0(s,\tau) &= \int_M^s \frac{\partial_\sigma R^{(1)}(s,\tau_1)\{1 + (\tau - \tau_1)[a\partial_\sigma R^{(1)}(s,\tau_1)\partial_\sigma^2 R^{(1)}(s,\tau_1) - b(\partial_\sigma R^{(1)}(s,\tau_1))^2]\}}{[1 - b(\tau - \tau_1)(\partial_\sigma R^{(1)}(s,\tau_1))^2]^{\frac{3}{2}}} ds \\ &\quad + \partial_\sigma R^{(1)}(s,\tau_1), \end{aligned}$$

where $R^{(1)}(\sigma,\tau)$ satisfies

$$\partial^2_{\tau\sigma}R+\frac{a}{2}\partial_\sigma R\partial^2_\sigma R=\frac{b}{2}[\partial_\sigma R]^3,\quad R(\sigma,0)=R^{(1)}(\sigma).$$

Now we verify that $\bar{\varphi}_0(s,\tau)$ satisfies (1.3).

It is easy to compute

$$\partial_s\bar{\varphi}_0=\frac{[1-b\tau_1(\partial_\sigma R^{(1)}(\sigma))^2][1+\tau(a\partial_\sigma R^{(1)}(\sigma)\partial^2_\sigma R^{(1)}(\sigma)-b(\partial_\sigma R^{(1)}(\sigma))^2)]}{[1-b\tau(\partial_\sigma R^{(1)}(\sigma))^2][1+\tau_1(a\partial_\sigma R^{(1)}(\sigma)\partial^2_\sigma R^{(1)}(\sigma)-b(\partial_\sigma R^{(1)}(\sigma))^2)]},$$

where $s=\sigma-\frac{a}{2b}\ln[1-b\tau_1(\partial_\sigma R^{(1)}(\sigma))^2]$.

Obviously,

$$1-b\tau_1(\partial_\sigma R^{(1)}(\sigma))^2>0,\quad 1+\tau_1(a\partial_\sigma R^{(1)}(\sigma)\partial^2_\sigma R^{(1)}(\sigma)-b(\partial_\sigma R^{(1)}(\sigma))^2)>0,$$
$$1-b\tau(\partial_\sigma R^{(1)}(\sigma))^2>0\ \text{ for } \tau_1\le\tau\le\tau_0.$$

Hence, by the assumptions on $h(\sigma)$ in Theorem 1.1, we know $\bar{\varphi}_0$ satisfies (1.3) at the point

$$(s,\tau)=\Big(\sigma_0-\frac{a}{2b}\ln[1-b\tau_0(\partial_\sigma R^{(1)}(\sigma_0))^2],\tau_0\Big).$$

As in [2, 3], gluing together the local true solution $(\bar{\varphi},\bar{v},\bar{w})$ with $(\bar{\varphi}_0,\bar{v}_0,\bar{w}_0)$, we can get an approximate solution $(\bar{\varphi}^{(0)},\bar{v}^{(0)},\bar{w}^{(0)})$. Moreover $I_1=f_1^{(0)},I_2=f_2^{(0)}$, $I_3=f_3^{(0)}$, where $f_1^{(0)}$, $f_2^{(0)}$ and $f_3^{(0)}$ are smooth, flat on $\{s=M\}$, zero near $\{\tau=\tau_1\}$ and vanish for $\varepsilon=0$. From the above properties on $\bar{\varphi}_0$, we easily verify that $\bar{\varphi}^{(0)}$ satisfies the condition (1.3) in Theorem 1.2 at a certain point $m=(s_0,\tau_0)$.

Step 2. Tame estimates for the linearized blowup system

As in [2] and [3], we perform a change of variables depending on the parameter λ close to zero

$$s=x,\quad \tau=[\tau_1+(\tau_0-\tau_1)t][1+\lambda(1-\chi(t))]\equiv\tau(t,\lambda),\tag{4.3}$$

where χ is one near zero and zero near one. We hope that one will not confuse these coordinates with the original coordinates. We will from now on work on a fixed domain

$$D_0=\{-C_0\le x\le M,0\le t\le 1\}.$$

We denote now by $\tilde{I}_i=\tilde{I}_i(\lambda,\varphi,v,w)$ the equation I_i transformed by the changes of variables (4.3). The transformed approximate solution for $\lambda=\lambda^{(0)}=0$ is written as

$$\begin{aligned}\varphi^{(0)}(x,t)&=\bar{\varphi}^{(0)}(x,\tau_1+(\tau_0-\tau_1)t),\\ v^{(0)}&=\bar{v}^{(0)}(x,\tau_1+(\tau_0-\tau_1)t),\\ w^{(0)}&=\bar{w}^{(0)}(x,\tau_1+(\tau_0-\tau_1)t)\end{aligned}\tag{4.4}$$

and set

$$\tilde{I}_i(\lambda^{(0)},\varphi^{(0)},v^{(0)},w^{(0)})=f_i^{(0)}=\bar{f}_i^{(0)}(x,\tau_1+(\tau_0-\tau_1)t).$$

We note that $\varphi^{(0)}$ satisfies (1.3) in D_0 for a point $m_0=(x_0,1)$. As in [2] and [3], it is enough to solve the transformed linearized system

$$\widetilde{(I'_j)}=\dot{f}_j,\quad j=1,2,3,\tag{4.5}$$

where $\widetilde{(I'_j)}$ denotes the linearized system from (4.1) in the original variables s,τ by the change of variables (4.3).

From (4.1),(4.3) and (4.5), we have

$$\begin{aligned}\widetilde{(I_1')} &= \Big\{ -2a[v-\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)](1+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi) + 2a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi(2+\\ &\quad + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)(1+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)[-v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]\Big\}\dot V\\ &\quad + 2\Big\{[1+\varphi e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi))^2](1+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)\frac{\partial t}{\partial\tau}\Big\}\partial_t\dot\Phi\\ &\quad + e^{-\frac{\tau}{\varepsilon^2}}\Big\{\partial_\tau\varphi(2+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)[1-2a\varepsilon^4 e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi))\partial_\tau v]\\ &\quad + 2a\varepsilon^2[v-\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]\partial_\tau v\Big\}\dot\Phi - 2a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\Big\{[1-e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi(2+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)\\ &\times(-v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi))]\frac{\partial t}{\partial\tau}\Big\}\partial_t\dot Z,\end{aligned}$$

$$\begin{aligned}\widetilde{(I_2')} &= -2\Big\{(1+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\varphi)[1+\varphi e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi))^2]\frac{\partial t}{\partial\tau}\Big\}\partial_t\dot V\\ &\quad + \Big\{e^{-\frac{\tau}{\varepsilon^2}}\big[2a\varepsilon^2(1+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)[-v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)][-2\partial_\tau v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau^2 w\\ &\quad - v\partial_\tau^2\varphi - 2\partial_\tau v\partial_\tau\varphi) - e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)] + [1+\varphi e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(-v+\\ &\quad \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi))^2]\partial_\tau\varphi\big] + 3b[v-\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]^2(1+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\varphi)\Big\}\dot V\\ &\quad + e^{-\frac{\tau}{\varepsilon^2}}\Big\{-2a\varepsilon^4 e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau v[-v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)][-2\partial_\tau v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\\ &\quad \times(\partial_\tau^2 w - v\partial_\tau^2\varphi - 2\partial_\tau v\partial_\tau\varphi) - e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi) - e^{-\frac{\tau}{\varepsilon^2}}(\varepsilon^2\partial_\tau^2 v - \partial_\tau v)(1+\varphi e^{-\frac{\tau}{\varepsilon^2}}\\ &\quad - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi))^2)] - \frac{1}{4\varepsilon^2(1+\varphi e^{-\frac{\tau}{\varepsilon^2}})}v + \frac{e^{-\frac{\tau}{\varepsilon^2}}}{4\varepsilon^2(1+\varphi e^{-\frac{\tau}{\varepsilon^2}})^2}w\\ &\quad - 3b\varepsilon^2[v-\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]^2\partial_\tau\varphi\Big\}\dot\Phi + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\Big\{[1+\varphi e^{-\frac{\tau}{\varepsilon^2}} - a\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(-v\\ &\quad + \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi))^2]\Big(\frac{\partial t}{\partial\tau}\Big)^2\Big\}\partial_t^2\dot Z + e^{-\frac{\tau}{\varepsilon^2}}\Big\{-2a\varepsilon^4 e^{-\frac{\tau}{\varepsilon^2}}[-v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w\\ &\quad - v\partial_\tau\varphi)][-2\partial_\tau v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau^2 w - v\partial_\tau^2\varphi - 2\partial_\tau v\partial_\tau\varphi) - e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]\frac{\partial t}{\partial\tau}\\ &\quad + \Big(\varepsilon^2\frac{\partial^2 t}{\partial\tau^2} - \frac{\partial t}{\partial\tau}\Big)[1+\varphi e^{-\frac{\tau}{\varepsilon^2}} - 2c''(0)\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi))^2]\\ &\quad - 3b\varepsilon^2[v-\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi)]^2\frac{\partial t}{\partial\tau}\Big\}\partial_t\dot Z - e^{-\frac{\tau}{\varepsilon^2}}\Big[\frac{1}{4\varepsilon^2(1+\varphi e^{-\frac{\tau}{\varepsilon^2}})}\Big]\dot Z,\end{aligned}$$

$$\widetilde{(I_3')} = \partial_x\dot Z + \partial_s v\dot\Phi - \partial_s\varphi\dot V,$$

where $\dot Z = \dot W - v\dot\Phi$, $t = t(\tau,\lambda)$ and $\tau = \tau(t,\lambda)$ are determined by (4.3).

Now we give the tame estimates for the system (4.5) with the trace conditions that $\dot\Phi, \dot V, \dot Z$ are flat on $\{t=0\}$ and $\{x=M\}$.

From $\widetilde{(I_2')} = \dot f_2$, we have

$$\begin{aligned}\dot V &= \varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\frac{A_4}{2A_1}\partial_t\dot Z + \Big[e^{-\frac{\tau}{\varepsilon^2}}\frac{A_5}{2A_1} - q\partial_t\Big(\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_4}{2A_1}\Big)\Big]\dot Z + q\int_0^t\Big\{\Big[e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_6}{2A_1}\\ &\quad - \partial_t\Big(e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_5}{2A_1}\Big) + \partial_t^2(\varepsilon^2 e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_4}{2A_1}\Big)\Big]\dot Z + e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_3}{2A_1}\dot\Phi - \frac{p\dot f_2}{2A_1}\Big\}dt, \qquad (4.6)\end{aligned}$$

where

$$p(x,t,\lambda) = \exp\Big\{-\int_0^t\frac{A_2(x,t,\lambda)}{2A_1(x,t,\lambda)}dt\Big\}, \quad q(x,t,\lambda) = \frac{1}{p(x,t,\lambda)}, \quad \tau = \tau(t,\lambda) \ge \tau_1,$$

$$A_1(x,t,\lambda)=\Big\{(1+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\varphi)[1+\varphi e^{-\frac{\tau}{\varepsilon^2}}-a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w - v\partial_\tau\varphi))^2]\frac{\partial t}{\partial\tau}\Big\}(x,t,\lambda),$$

$$A_2(x,t,\lambda)=\{e^{-\frac{\tau}{\varepsilon^2}}[2a\varepsilon^2(1+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)[-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)][-2\partial_\tau v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}} \times(\partial_\tau^2w-v\partial_\tau^2\varphi-2\partial_\tau v\partial_\tau\varphi)-e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)] + [1+\varphi e^{-\frac{\tau}{\varepsilon^2}}-a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\times(\partial_\tau w-v\partial_\tau\varphi))^2]\partial_\tau\varphi] + 3b[v-\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)]^2(1+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)\}(x,t,\lambda),$$

$$A_3(x,t,\lambda)=\Big\{-2a\varepsilon^4e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau v[-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)][-2\partial_\tau v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}} \times(\partial_\tau^2w-v\partial_\tau^2\varphi-2\partial_\tau v\partial_\tau\varphi)-e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)-e^{-\frac{\tau}{\varepsilon^2}}(\varepsilon^2\partial_\tau^2v-\partial_\tau v)(1 + \varphi e^{-\frac{\tau}{\varepsilon^2}}-a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi))^2)]-\frac{1}{4\varepsilon^2(1+\varphi e^{-\frac{\tau}{\varepsilon^2}})}v + \frac{e^{-\frac{\tau}{\varepsilon^2}}}{4\varepsilon^2(1+\varphi e^{-\frac{\tau}{\varepsilon^2}})^2}w-3b\varepsilon^2[v-\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)]^2\partial_\tau v\Big\}(x,t,\lambda),$$

$$A_4(x,t,\lambda)=\Big\{[1+\varphi e^{-\frac{\tau}{\varepsilon^2}}-a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi))^2]\Big(\frac{\partial t}{\partial\tau}\Big)^2\Big\}(x,t,\lambda),$$

$$A_5(x,t,\lambda)=\Big\{-2a\varepsilon^4e^{-\frac{\tau}{\varepsilon^2}}[-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)][-2\partial_\tau v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau^2w-v\partial_\tau^2\varphi - 2\partial_\tau v\partial_\tau\varphi)-e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)]\frac{\partial t}{\partial\tau}+\Big(\varepsilon^2\frac{\partial^2t}{\partial\tau^2}-\frac{\partial t}{\partial\tau}\Big) \times[1+\varphi e^{-\frac{\tau}{\varepsilon^2}}-a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi))^2] - 3b\varepsilon^2[v-\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)]^2\frac{\partial t}{\partial\tau}\Big\}(x,t,\lambda),$$

$$A_6(x,t,\lambda)=-\Big[\frac{1}{4\varepsilon^2(1+\varphi e^{-\frac{\tau}{\varepsilon^2}})}\Big](x,t,\lambda).$$

Especially, note that $A_1>0$ and $A_4>0$ for small ε.

Inserting (4.6) into $\widetilde{(I_1')}=\dot f_1$, we get

$$\dot\Phi=-\frac{\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}}{2B_2}\Big(\frac{B_1A_4}{2A_1}-2aB_4\Big)\dot Z-q_1\int_0^t\Big\{\frac{p_1B_1}{2B_2}\Big[e^{-\frac{\tau}{\varepsilon^2}}\frac{A_5}{2A_1}-q\partial_t\Big(\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_4}{2A_1}\Big)\Big] - \partial_t\Big[\frac{p_1}{2B_2}\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\Big(\frac{B_1A_4}{2A_1}-2aB_4\Big)\Big]\Big\}\dot Zdt+q_1\int_0^t\frac{p_1\dot f_1}{2B_2}dt - q_1\int_0^t\Big\{\frac{p_1B_1q}{2B_2}\int_0^t\Big\{\Big[e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_6}{2A_1}-\partial_t\Big(e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_5}{2A_1}\Big)+\partial_t^2\Big(\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_4}{2A_1}\Big)\Big]\dot Z + e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_3}{2A_1}\dot\Phi-\frac{p\dot f_2}{2A_1}\Big\}dt\Big\}dt, \tag{4.7}$$

where

$$p_1(x,t,\lambda)=\exp\Big(\int_0^te^{-\frac{\tau}{\varepsilon^2}}\frac{B_3}{2B_2}dt\Big),\quad q_1(x,t,\lambda)=\frac{1}{p_1(x,t,\lambda)},$$

$$B_1(x,t,\lambda)=\{-2a[v-\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)](1+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)+2a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi \times(2+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)(1+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)[-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)]\}(x,t,\lambda),$$

$$B_2(x,t,\lambda)=\Big\{[1+\varphi e^{-\frac{\tau}{\varepsilon^2}}-a\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi))^2](1+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)\frac{\partial t}{\partial\tau}\Big\}(x,t,\lambda),$$

$$B_3(x,t,\lambda)=\{\partial_\tau\varphi(2+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)[1-2a\varepsilon^4e^{-\frac{\tau}{\varepsilon^2}}(-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi))\partial_\tau v]+2a\varepsilon^2[v-\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi)]\partial_\tau v\}(x,t,\lambda),$$

$$B_4(x,t,\lambda)=\Big\{[1-e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi(2+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_\tau\varphi)(-v+\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}(\partial_\tau w-v\partial_\tau\varphi))]\frac{\partial t}{\partial\tau}\Big\}(x,t,\lambda).$$

Substituing (4.6) into $\widetilde{(I_3')}=\dot f_3$, we obtain

$$\begin{aligned}&\Big[\partial_x-\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_s\varphi\frac{A_4}{2A_1}\partial_t\Big]\dot Z-\partial_s\varphi\Big[e^{-\frac{\tau}{\varepsilon^2}}\frac{A_5}{2A_1}-q\partial_t\Big(\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_4}{2A_1}\Big)\Big]\dot Z+\partial_s v\dot\Phi\\&-q\partial_s\varphi\int_0^t\Big\{\Big[e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_6}{2A_1}-\partial_t\Big(e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_5}{2A_1}\Big)+\partial_t^2\Big(\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_4}{2A_1}\Big)\Big]\dot Z\\&+e^{-\frac{\tau}{\varepsilon^2}}\frac{pA_3}{2A_1}\dot\Phi-\frac{p\dot f_2}{2A_1}\Big\}dt=\dot f_3.\end{aligned}\tag{4.8}$$

Since $\partial_s\varphi\ge0$, $\frac{A_4}{2A_1}>0$, the first order equation

$$\Big[\partial_x-\varepsilon^2e^{-\frac{\tau}{\varepsilon^2}}\partial_s\varphi\frac{A_4}{2A_1}\partial_t\Big]Y+a(x,t)Y=f(x,t)$$

can be solved in the domain $\{-C_0\le x\le M, 0\le t\le1\}$ if $Y(s,\tau)$ has two traces on $\{t=0\}$ and $\{x=M\}$.

We choose the small constant η_0 such that $0<\eta_0<\frac14|v^{(0)}(s_0,\tau_1)|$, for the meaning of s_0 see the end in Step 1 (this can be done. Since $\partial_\sigma R^{(1)}(\sigma_0)\ne0$, we have $(\partial_\sigma R^{(1)})(s_0,\tau_0)\ne0$. If we choose τ_1 close to τ_0, then $v^{(0)}(s_0,\tau_1)\ne0$). If

$$|\varphi-\varphi^{(0)}|_{C^4(D_0)}+|v-v^{(0)}|_{C^4(D_0)}+|w-w^{(0)}|_{C^4(D_0)}\le\eta_0,$$

similar to the energy estimate for the first order equation with initial and boundary data, then there exists $\varepsilon_0>0$ such that for $0<\varepsilon\le\varepsilon_0$ and all s, there exists C_s such that for all smooth $\dot Z$ flat on $\{t=0\}$ and $\{x=M\}$, we have the following estimate

$$\begin{aligned}|\dot Z|_s\le C_s[&(1+|\varphi|_{s+4}+|w|_{s+4}+|v|_{s+4})|\dot f_2|_2+|\dot f_3|_s+|\dot f_2|_s\\&+(|\varphi|_{s+4}+|v|_{s+4}+|w|_{s+4})|\dot\Phi|_2]+\tilde C|\dot\Phi|_s,\end{aligned}\tag{4.9}$$

where $\tilde C>0$ is a constant, $|\dot Z|_s=\sum\limits_{|\alpha|\le s}|\partial_{x,t}^\alpha\dot Z|_{L^2(D_0)}$.

From (4.7), for small ε_0 we can easily get

$$|\dot\Phi|_s\le\frac{1}{2\tilde C}|\dot Z|_s+C_s[(1+|\varphi|_{s+4}+|v|_{s+4}+|w|_{s+4})(|\dot Z|_2+|\dot f_1|_2+|\dot f_2|_2)+|\dot f_1|_s+|\dot f_2|_s].\tag{4.10}$$

Moreover by (4.7) and (4.8), we easily obtain

$$|\dot Z|_2+|\dot\Phi|_2\le C(|\dot f_1|_2+|\dot f_2|_2+|\dot f_3|_2).\tag{4.11}$$

Substituing (4.10) and (4.11) into (4.9), we have

$$|\dot Z|_s\le C_s[|\dot f_1|_s+|\dot f_2|_s+|\dot f_3|_s+(1+|\varphi|_{s+4}+|v|_{s+4}+|w|_{s+4})(|\dot f_2|_2+|\dot f_3|_2)].\tag{4.12}$$

From (4.12) and (4.10), we have

$$|\dot\Phi|_s\le C_s[|\dot f_1|_s+|\dot f_2|_s+|\dot f_3|_s+(1+|\varphi|_{s+4}+|v|_{s+4}+|w|_{s+4})(|\dot f_1|_2+|\dot f_2|_2+|\dot f_3|_2)].\tag{4.13}$$

Substituing (4.12),(4.13) into (4.6), then one has

$$|\dot{V}|_s \le C_s[|\dot{f}_1|_{s+1} + |\dot{f}_2|_{s+1} + |\dot{f}_3|_{s+1} + (1 + |\varphi|_{s+5} + |v|_{s+5} + |w|_{s+5}) \times (|\dot{f}_1|_2 + |\dot{f}_2|_2 + |\dot{f}_3|_2)]. \tag{4.14}$$

By (4.12),(4.13) and (4.14), we obtain the tame estimate for the linearized system (4.5).

Therefore in terms of the standard Nash-Moser method (see [2, 3, 7]), there exists a solution (φ, v, w) to (4.1) in the domain

$$D_1 = \{-C_0 \le s \le M, \tau_1 \le \tau \le \tau_\varepsilon\}.$$

If $0 < \varepsilon \le \varepsilon_k$, this solution is of C^k.

Proof of Theorem 1.2. We only need to prove $\partial_s v(m_\varepsilon) \ne 0$.

In fact, if we differentiate $I_1 = 0$ with respect to s, then

$$\frac{\partial I_1}{\partial s} = -2av\partial_s v + 2\partial^2_{s\tau}\varphi + O\Big(\frac{e^{-\frac{\tau}{\varepsilon^2}}}{\varepsilon^2}\Big). \tag{4.15}$$

If we choose τ_1 close to τ_0, then

$$\begin{aligned}|v(m_\varepsilon)| &\ge |v^{(0)}(s_0,\tau_1)| - |v^{(0)}(m_0) - v^{(0)}(s_0,\tau_1)| - |v(m_0) - v^{(0)}(m_0)| - |v(m_\varepsilon) - v(m_0)| \\ &\ge \frac{1}{4}|v^{(0)}(s_0,\tau_1)| > 0\end{aligned}$$

for small ε. Set $(s,\tau) = m_\varepsilon$ in (4.15), and note that $\tau \ge \tau_1 > 0$, $v(m_\varepsilon) \ne 0$ and $\partial^2_{s\tau}\varphi(m_\varepsilon) \ne 0$, then for small ε, $\partial_s v(m_\varepsilon) \ne 0$. Hence Theorem 1.2 holds.

§5. The Proof of Theorem 1.1

In §4, we have obtained the construction of a piece of blowup solution $\bar{u}$ of (4.1) in the domain $\Phi(D)$. Completely in a way similar to the proof in [2] and [3], by a standard uniqueness argument for wave equations, we can get a true solution u which coincides with $\bar{u}$ in an appropriate domain $\tilde{D}$ (containing the blowup point).

To prove Theorem 1.1, first we will prove the following result.

Lemma 5.1. *Under the assumptions of Theorem* 1.1, *for* $k \in \mathbb{N}$, *there exist two constants* ε_k *and* C, *such that for each* $0 < \varepsilon \le \varepsilon_k$ *there is a sphere* $M_\varepsilon = (x_\varepsilon, T_\varepsilon)$, $|x_\varepsilon| = r_\varepsilon$, *such that in the small neighbourhood* $\tilde{D}$ *of* M_ε *we have*

(1) $u \in C^1(\tilde{D} \cap (\mathbb{R}^2 \times \{t \le T_\varepsilon\}))$ *and* $|u|_{C^1(\tilde{D}\cap(\mathbb{R}^2\times\{t\le T_\varepsilon\}))} \le C\varepsilon$.

(2) *In* $\tilde{D}$ *and* $t < T_\varepsilon$,

$$\|\nabla^2_{x,t}u\| = \sup_x |\nabla^2_{x,t}u(x,t)| \le \frac{C\sqrt{t}}{\varepsilon(T_\varepsilon - t)},$$

and $\|\partial_t^2 u\|$, $\|\partial_t \nabla_x u\|$, $\|\nabla_x^2 u\| \ge \frac{1}{C}\frac{\sqrt{t}}{\varepsilon(T_\varepsilon - t)}$.

Proof. (i) By the Condition (1.3) in Theorem 1.2, we easily prove the property (i) in Lemma 5.1. So we omit the proof.

(ii) In this part we study u in $t < T_\varepsilon$.

Since

$$\begin{aligned}\partial_t^2 u = \frac{\varepsilon}{r^{\frac{1}{2}}}\frac{1}{\partial_s\varphi}\Big\{&\partial_s v - \frac{2\varepsilon^2}{t}(\partial_\tau v\partial_s\varphi - \partial_s v\partial_\tau\varphi) + \frac{\varepsilon^4}{t^2}[\partial_s\varphi(\partial_\tau^2 w - v\partial_\tau^2\varphi - 2\partial_\tau v\partial_\tau\varphi) \\ &+ \partial_s v(\partial_\tau\varphi)^2] - \frac{\varepsilon^2}{t^2}(\partial_\tau w - v\partial_\tau\varphi)\partial_s\varphi\Big\},\end{aligned}$$

using $\partial_s v(m_\varepsilon)\neq 0$ and noting $\partial_s\varphi\geq 0$, in a small neighbourhood of m_ε we have

$$\|\partial_t^2 u\|\geq \frac{\varepsilon}{Cr^{\frac12}}\frac{1}{\tau_\varepsilon-\tau}=\frac{1}{C}\frac{1}{r^{\frac12}\varepsilon(\ln T_\varepsilon-\ln t)}.$$

Since $\ln T_\varepsilon-\ln t=\ln(1+\frac{T_\varepsilon-t}{t})\leq\frac{T_\varepsilon-t}{t}$, noting $r\sim t$ in the exterior area, we have

$$\|\partial_t^2 u\|\geq\frac{1}{C}\frac{\sqrt{t}}{\varepsilon(T_\varepsilon-t)}.$$

Additionally

$$|\partial_t^2 u|\leq\frac{C\varepsilon}{r^{\frac12}(\tau_\varepsilon-\tau)}\leq\frac{Ct^{\frac12}}{\varepsilon(T_\varepsilon-t)}.$$

Because

$$\partial^2_{tx_i}u=-\frac{\varepsilon x_i}{2r^{\frac52}}\Big[-v+\frac{\varepsilon^2}{t}(\partial_\tau w-v\partial_\tau\varphi)\Big]+\frac{\varepsilon x_i}{r^{\frac32}}\frac{1}{\partial_s\varphi}\Big[-\partial_s v+\frac{\varepsilon^2}{t}(\partial_s\varphi\partial_\tau v-\partial_s v\partial_\tau\varphi)\Big],$$

$$\partial^2_{ij}u=\Big(\frac{5\varepsilon x_ix_j}{4r^{\frac92}}-\frac{\varepsilon\delta_{ij}}{2r^{\frac52}}\Big)w-\frac{2\varepsilon x_ix_j}{r^{\frac72}}v+\frac{\varepsilon\delta_{ij}}{r^{\frac32}}v+\frac{\varepsilon x_ix_j}{r^{\frac52}}\frac{\partial_s v}{\partial_s\varphi},$$

by the similar proof for $\partial_t^2 u$, we can get Lemma 5.1.

Secondly, we prove that u blows up only at M_ε.

We recall the approximate solution constructed in §2, that is

$$u_a(x,t)=\varepsilon\chi(\varepsilon t)u_1(x,t)+\varepsilon(1-\chi(\varepsilon t))\chi(-3\varepsilon(r-t))r^{-\frac12}U(r-t,\varepsilon^2\ln t).$$

Denote

$$R(u_a)=\Box u_a-a(\partial_t u_a)^2\partial_t^2u_a-b(\partial_t u_a)^3.$$

Now we give the following result

Lemma 5.2. *There exists an approximate solution $\bar u_a$, such that in $\{r-t\leq -C_0, e^{\frac{\bar\tau}{\varepsilon^2}}\leq t\leq T_\varepsilon\}$ we have*

$$|Z^\alpha\bar u_a|\leq\frac{C_\alpha\varepsilon}{(1+t)^{\frac12}},\tag{5.1}$$

$$\|Z^\alpha R(\bar u_a)(\cdot,t)\|_{L^2(\mathbb{R}^2)}\leq C_{\alpha\bar\tau}\Big(\frac{\varepsilon^{\frac54}}{(1+t)^{\frac54}}+\frac{\varepsilon^{\frac72}}{1+t}\Big).\tag{5.2}$$

Moreover $u_a=u$ for $r-t\geq -C_0$.

Proof. As in [3] and [4], we consider G in a strip

$$-C_2\leq r-t\leq -C_1$$

where $C_0<C_1<C_2$. Then there is not a blowup point in this strip for G. By Lemma 4.1, we get

$$\partial^2_{\sigma\tau}G+\frac{a}{2}(\partial_\sigma G)^2\partial_\sigma^2G=\frac{b}{2}(\partial_\sigma G)^3+\frac{e^{-\frac{\tau}{\varepsilon^2}}}{\varepsilon^2}\Big[\frac{A}{\partial_s\varphi}+B\Big],\tag{5.3}$$

where A, B are the smooth expressions of $\varphi, v, w, \nabla_{s,\tau}\varphi, \nabla_{s,\tau}v, \nabla_{s,\tau}w, \nabla^2_{s,\tau}\varphi, \nabla^2_{s,\tau}v, \nabla^2_{s,\tau}w$. From Equations (2.1) and (3.7), we know

$$[G(\sigma,\tau)-U(\sigma,\tau)]|_{\tau=\bar\tau}=O(\varepsilon^2\ln\varepsilon)\quad\text{for}\quad\sigma\geq -2C_2.$$

Hence from (2.1) and (5.3), in the domain $\{-C_2\leq r-t\leq -C_1, e^{\frac{\bar\tau}{\varepsilon}}\leq t\leq T_\varepsilon\}$, we get

$$G(\sigma,\tau)-U(\sigma,\tau)=O(\varepsilon^2\ln\varepsilon).\tag{5.4}$$

Set

$$\bar{u}_a = \chi_1(r-t)\frac{\varepsilon}{r^{\frac{1}{2}}}G(r-t,\varepsilon^2 \ln t) + (1-\chi_1(r-t))(1-\chi(\varepsilon t))\chi(-3\varepsilon(r-t))\frac{\varepsilon}{r^{\frac{1}{2}}}U(r-t,\varepsilon^2 \ln t),$$

where $\chi_1(s)$ is one for $s \geq -C_1$ and zero for $s \leq -C_2$. By computation, we know Lemma 5.2 holds.

Assume $u = \bar{u}_a + \dot{u}$. By (2.6) we know

$$\|\nabla_{x,t} Z^\alpha \dot{u}(\cdot, e^{\frac{\bar{\tau}}{\varepsilon^2}})\|_{L^2(\mathbb{R}^2)} \leq C_{\alpha\bar{\tau}}\varepsilon^{\frac{5}{4}}.$$

By Lemma 5.2, using the standard energy estimate between $e^{\frac{\bar{\tau}}{\varepsilon^2}}$ and T_ε (for example, see [5],[6] or [1]), we can obtain $\dot{u} \in C^2$ for $e^{\frac{\bar{\tau}}{\varepsilon^2}} \leq t \leq T_\varepsilon$.

Combine the above conclusion with Lemma 5.1, we know Theorem 1.1 holds.

Remark. For three dimensional quasilinear wave equations

$$\partial_t^2 u - c^2(\partial_t u, \partial_r u)\Delta u = f(\partial_t u, \partial_r u),$$

where

$$c^2(\partial_t u, \partial_r u) = 1 + a_1\partial_t u + a_2\partial_r u + O(|\partial_t u|^2 + |\partial_r u|^2),$$
$$f(\partial_t u, \partial_r u) = b_1(\partial_t u)^2 + b_2\partial_t u\partial_r u + b_3(\partial_r u)^2 + O(|\partial_t u|^3 + |\partial_r u|^3),$$

if $a_1 \neq a_2$ we can prove the similar conclusion to Theorem 1.1 by our method in this paper. In the case of $a_1 = a_2$ and $b_1 - b_2 + b_3 = 0$, by [8] we know the above equations have global solutions.

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