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THE FUNCTION $d_k(n)$ AT CONSECUTIVE INTEGERS**

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Abstract

Let $d_k(n)$ denote the k-fold iterated divisor function ($k \geq 2$). It is proved that for sufficiently large x, $d_k(n) = d_k(n+1)$ holds for $\gg x(\log\log x)^{-3}$ integers $n \leq x$.

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§1. Introduction

For all positive integers $k \geq 2$, we define the functions $d_k(n)$ by the identity

$$\Big(\sum_{n=1}^{\infty} \frac{1}{n^s}\Big)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \operatorname{Re} s > 1. \tag{1.1}$$

We know that $d_k(n)$ is a multiplicative function. If the standard form of n is $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, then we have the following formula

$$d_k(n) = \frac{(\alpha_1 + k - 1)!}{\alpha_1!(k-1)!} \cdots \frac{(\alpha_s + k - 1)!}{\alpha_s!(k-1)!}. \tag{1.2}$$

The above formulas can be found in [4]. In 1984, Heath-Brown[2] showed that for sufficiently large x,

$$\#\{n \leq x : d(n) = d(n+1)\} \gg x(\log x)^{-7}. \tag{1.3}$$

In 1987, Hildebrand[3] improved the lower bound to

$$\#\{n \leq x : d(n) = d(n+1)\} \gg x(\log\log x)^{-3}. \tag{1.4}$$

In this paper, we prove the following result.

Theorem. *Suppose $k(\geq 2)$ is a positive integer. For sufficiently large x,*

$$\#A_k(x) := \#\{n \leq x : d_k(n) = d_k(n+1)\} \gg x(\log\log x)^{-3}. \tag{1.5}$$

Here the constant $\gg$ is independent of k.

We find that for $k \geq 3$, $A_2(x) \not\subseteq A_k(x)$; for example, when $n = 3^5$, $n+1 = 61 \times 2^2$, $d(n) = d(n+1) = 6$,

$$d_k(n) = \frac{(k+4)!}{5!(k-1)!} > d_k(n+1) = \frac{k!(k+1)!}{2!(k-1)!(k-1)!}.$$

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The idea of this paper is to combine the methods of Heath-Brown with that of Hildebrand. Our construction of $\{q_i, 1 \le i \le 7\}$ is new and more powerful than that in [3]. For $k \ge 3$ we need the factorization formulas of the $\{a_i,\ i = 1, \cdots, 7\}$ in Lemma 2.1, Hildebrand's method for $k = 2$ was insufficient.

Let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity, $\omega(n)$ denote the number of different prime factors of n, and $\mu(n)$ denote the Möbius function; k is a positive integer greater than 1, and x denotes a sufficiently large real number.

§2. Lemmas

First of all, we need to generalize the "Key Lemma" of Heath-Brown in [2].

Lemma 2.1. *For any positive integer N, there exist N distinct natural numbers $a_1 < \cdots < a_N$ such that if $a_{m,n} = a_m - a_n$, then $a_{m,n} \mid (a_m, a_n)$ and*

$$d_k(a_m)d_k\Big(\frac{a_n}{a_{m,n}}\Big) = d_k(a_n)d_k\Big(\frac{a_m}{a_{m,n}}\Big), \qquad m > n. \tag{2.1}$$

Proof. We use the same symbols as that in the proof of the "Key Lemma" in [2] except for $a_{m,n}$ and $N = 2^r$. Now, we have $G = Z_2^r$, and write I for the zero element of G. Let $\sigma = (\sigma_1, \cdots, \sigma_r) \in G$ with $\sigma_i = 0$ or 1, and set $n(\sigma) = \sum\limits_{i=1}^{r} 2^{i-1}\sigma_i$. Thus $n(\sigma)$ gives a 1-1 correspondence between G and the set $\{0, 1, \cdots, N-1\}$. We also write

$$A_\sigma = a_{1+n(\sigma)}, \quad D_{\sigma\tau} = A_\sigma - A_\tau.$$

Then the conditions of Lemma 2.1 may be reformulated as

$$D_{\sigma\tau} \mid (A_\sigma, A_\tau), \quad \sigma \ne \tau, \tag{2.2}$$

$$d_k(A_\sigma)d_k\Big(\frac{A_\tau}{\mid D_{\sigma\tau}\mid}\Big) = d_k(A_\tau)d_k\Big(\frac{A_\sigma}{\mid D_{\sigma\tau}\mid}\Big), \ \ \sigma \ne \tau. \tag{2.3}$$

Let $P = \prod\limits_{\sigma\in G} p_\sigma$, $D_{\sigma\tau} = E_{\sigma\tau}F_{\sigma\tau}$, where $E_{\sigma\tau}$ is a product of powers of the primes p_π and $(F_{\sigma\tau}, P) = 1$.

In [2] Heath-Brown showed that there exist positive integers $\{A_\tau,\ \tau \in G\}$ which may be arranged such that

$$p_\sigma^{n(\sigma+\tau)} \parallel A_\tau, \tag{2.4}$$

$$p \mid F_{\sigma\tau} \ \text{ implies } \ p \parallel F_{\sigma\tau}, \ \ p \parallel A_\sigma, \ \ p \parallel A_\tau, \ \ \sigma \ne \tau. \tag{2.5}$$

From above we have $\mu^2(F_{\sigma\tau}) = 1$. In the following, we shall use (2.4) and (2.5) to show that (2.2) and (2.3) are valid.

It can be seen from the proof in [2] that, if (2.4) holds, we have

$$E_{\sigma\tau} = \prod_{\pi\in G} p_\pi^{\min\{n(\pi+\sigma), n(\pi+\tau)\}}, \quad \sigma \ne \tau. \tag{2.6}$$

Thus, $E_{\sigma\tau} \mid A_\sigma,\ \ \sigma \ne \tau$. Since (2.6) is symmetrical in σ and τ on the right-hand side, we deduce that

$$E_{\sigma\tau} \mid (A_\sigma, A_\tau) \quad (\sigma \ne \tau). \tag{2.7}$$

By (2.5), we have

$$F_{\sigma\tau} \mid (A_\sigma, A_\tau) \quad (\sigma \ne \tau). \tag{2.8}$$

From (2.7), (2.8) we obtain (2.2).

In the following, we shall prove (2.3).

Consider the contribution to the left-hand side of (2.3) arising from all the primes p_π.

According to (1.2), (2.4), and (2.6), this is

$$\prod_{\pi\in G}\frac{(n(\pi+\sigma)+k-1)!}{(n(\pi+\sigma))!(k-1)!}\prod_{\pi\in G}\frac{(n(\pi+\tau)-\min\{n(\pi+\sigma),n(\pi+\tau)\}+k-1)!}{(k-1)!(n(\pi+\tau)-\min\{n(\pi+\sigma),n(\pi+\tau)\})!}.$$

We substitute $\rho=\pi+\sigma$ and use $2\sigma=I$ to obtain

$$\prod_{\rho\in G}\frac{(n(\rho)+k-1)!}{n(\rho)!(k-1)!}\prod_{\rho\in G}\frac{(n(\rho+\tau+\sigma)-\min\{n(\rho),n(\rho+\sigma+\tau)\}+k-1)!}{(k-1)!(n(\rho+\tau+\sigma)-\min\{n(\rho),n(\rho+\sigma+\tau)\})!}.$$

Since this is symmetrical in σ and τ, the corresponding factors on two sides of (2.3) are the same.

For the remaining primes, as Heath-Brown has done in [2], we have the following two cases:

(i) if $p \mid (A_\sigma, A_\tau)$, then

$$p\parallel A_\sigma,\quad p\dagger\frac{A_\tau}{\mid D_{\sigma\tau}\mid},\quad p\parallel A_\tau,\quad p\dagger\frac{A_\sigma}{\mid D_{\sigma\tau}\mid},$$

or

(ii) if $p\dagger A_\sigma$, $p^e\parallel A_\tau$, $(e\ge 1)$, then

$$p\dagger A_\sigma,\quad p^e\parallel\frac{A_\tau}{\mid D_{\sigma\tau}\mid},\quad p^e\parallel A_\tau,\quad p\dagger\frac{A_\sigma}{\mid D_{\sigma\tau}\mid}.$$

In the case (i) the contributions to two sides of (2.3) arising from p are both $d_k(p)$; in the case (ii) the corresponding contributions on two sides are both $d_k(p^e)$, and then using the multiplication we obtain (2.3).

So that, we have proven that (2.2) and (2.3) are both valid. Arranging the $\{a_j\}$ increasingly, we complete the proof of Lemma 2.1.

Note that $\{a_i,\ \ i=1,\cdots,7\}$ do not depend on k.

Lemma 2.2. *There exist positive constants δ_i, $i=1,2,3$, with the following property. Let a_i', b_i', $1\le i\le 7$ be integers satisfying*

$$\prod_{i=1}^{7}a_i'\prod_{1\le t<s\le 7}(a_t'b_s'-a_s'b_t')\ne 0,\tag{2.9}$$

and let $f(n)=\prod_{i=1}^{7}(a_i'n+b_i')$. Suppose that the polynomial $f(n)$ has no fixed prime divisor. Let

$$S(x):=\#\{n\le x:\Omega(f(n))\le 27;\quad \mu^2(f(n))=1;\quad p(f(n))>x^{\delta_2}\},$$

where $p(n)$ denotes the least prime factor of n. Then we have $S(x)\ge\delta_1 x(\log x)^{-7}$, provided x satisfies

$$2\max\{\mid a_i'\mid,\mid b_i'\mid: 1\le i\le 7\}\le x^{\delta_3}.$$

This is the Lemma 2 in [2], we also use the notes after it, the constants 7 and 27 are taken from [5].

§3. Proof of the Theorem

Let $a_1 < \cdots < a_7$ be fixed positive integers satisfying the conditions of Lemma 2.1. Suppose $\Omega(a_i) = g_i, \quad 1 \le i \le 7$. Further let z and δ be positive constants to be specified later and satisfying

$$z > a_7, \quad 0 < \delta < 1. \tag{3.1}$$

Now we construct the positive integers q_i, $1 \le i \le 7$ satisfying

$$d_k(q_i) = \frac{\prod_{i=1}^{7} d_k(a_i)}{d_k(a_i)}. \tag{3.2}$$

If the standard form of a_i is $a_i = p_{i1}^{\alpha_{i,1}} \cdots p_{il_i}^{\alpha_{i,l_i}}, \quad 1 \le i \le 7$, and let

$$B(a_i) = (\alpha_{i,1}, \cdots, \alpha_{i,l_i}), \quad 1 \le i \le 7,$$

then we have

$$g_i = \alpha_{i,1} + \cdots + \alpha_{i,l_i}, \quad \omega(a_i) = l_i \quad (1 \le i \le 7),$$

and write

$$l_1 + \cdots + l_7 = L = L_i + l_i \quad (1 \le i \le 7).$$

Suppose $z < p'_1 < \cdots < p'_{6L}$ are the first $6L$ primes exceeding z. We divide these $6L$ primes into seven disjointed sets the number of which are $L_1, \cdots, L_7$ respectively. We use these sets to construct $q_1, \cdots, q_7$ as follows:

$$\begin{cases} q_1 = p_{1,2,1}^{\alpha_{2,1}} p_{1,2,2}^{\alpha_{2,2}} \cdots p_{1,2,l_2}^{\alpha_{2,l_2}} p_{1,3,1}^{\alpha_{3,1}} \cdots p_{1,3,l_3}^{\alpha_{3,l_3}} \cdots p_{1,7,1}^{\alpha_{7,1}} \cdots p_{1,7,l_7}^{\alpha_{7,l_7}}, \\ q_2 = p_{2,1,1}^{\alpha_{1,1}} \cdots p_{2,1,l_1}^{\alpha_{1,l_1}} p_{2,3,1}^{\alpha_{3,1}} \cdots p_{2,3,l_3}^{\alpha_{3,l_3}} \cdots p_{2,7,1}^{\alpha_{7,1}} \cdots p_{2,7,l_7}^{\alpha_{7,l_7}}, \\ \cdots\cdots\cdots\cdots, \\ q_7 = p_{7,1,1}^{\alpha_{1,1}} \cdots p_{7,1,l_1}^{\alpha_{1,l_1}} \cdots p_{7,6,1}^{\alpha_{6,1}} \cdots p_{7,6,l_6}^{\alpha_{6,l_6}}. \end{cases}$$

These q_i, $1 \le i \le 7$ satisfy (3.2).

Let

$$x' = x^{\delta} / \max_{i=1}^{7} q_i, \quad z' = \max_{\substack{i,j=1,\cdots,7 \\ l \le l_j}} \{p_{i,j,l}\},$$

so that, when x is sufficiently large, $z' < x^{\delta}$. Suppose r_i, $i = 1, \cdots, 7$ satisfying

$$\begin{cases} r_i \le x', \quad p(r_i) > z', \quad 1 \le i \le 7, \\ \mu^2(r_1 \cdots r_7) = 1, \\ \Omega(r_1) = \cdots = \Omega(r_7), \end{cases} \tag{3.3}$$

then $m_i = q_i r_i$, $i = 1, \cdots, 7$ satisfying

$$\begin{cases} m_i \le x^{\delta}, \quad p(m_i) > z \quad (1 \le i \le 7), \\ (m_i, m_j) = 1 \quad (1 \le i < j \le 7), \\ d_k(m_i a_i) = d_k(m_j a_j) \quad (1 \le i < j \le 7). \end{cases} \tag{3.4}$$

Consider the system of congruences

$$\begin{cases} n_0 \equiv 0 \quad \left(\bmod\ 7! \prod_{i=1}^{7} a_i^2\right), \\ n_0 \equiv -a_i \pmod{m_i}, \qquad (1 \le i \le 7). \end{cases} \tag{3.5}$$

Let $P' = 7! \prod_{i=1}^{7} a_i^2 m_i$. Then the solutions of (3.5) have the form

$$n_0(t) = n_0 + tP' \ (t \in Z),$$

where n_0 is the least positive solution. Let

$$n_i(t) = n_0(t) + a_i = n_0 + a_i + tP' \quad (1 \le i \le 7).$$

As Hildebrand did in [3], we have

$$n_i(t) = a_i m_i f_i(t) = a_i m_i (P_i t + Q_i), \tag{3.6}$$

where

$$P_i = \frac{P'}{a_i m_i}, \quad Q_i = \frac{n_0 + a_i}{a_i m_i}.$$

If there exist some $t \ge 1$ and some $i < j$ satisfying

$$\begin{cases} p(f_i(t) f_j(t)) > x^{\delta}, \\ d_k(f_i(t)) = d_k(f_j(t)), \end{cases} \tag{3.7}$$

then we obtain, by (2.1) and (3.4),

$$\frac{d_k(\frac{n_j(t)}{a_{j,i}})}{d_k(\frac{n_i(t)}{a_{j,i}})} = \frac{d_k(\frac{a_j}{a_{j,i}}) d_k(m_j) d_k(f_j(t))}{d_k(\frac{a_i}{a_{j,i}}) d_k(m_i) d_k(f_i(t))} = 1,$$

and from $n_j(t) = n_i(t) + a_j - a_i = n_i(t) + a_{j,i}$, when taking $n = \frac{n_i(t)}{a_{j,i}}$, we have

$$d_k(n+1) = d_k(n). \tag{3.8}$$

Thus, for fixed $i < j$, every tuple $(\underline{m}, t) = (m_1, \cdots, m_7, t)$ satisfying (3.4) and (3.7) gives a solution to (3.8), and when $t m_1 \cdots m_7 \le cx$, we have $n \le x$, where c is a small constant.

As in [3], every such $n \le x$ arises at most once. So we deduce

$$\# A_k(x) \ge \sum_{(3.4)} T\Big(\underline{m}, \frac{cx}{m_1 \cdots m_7}\Big),$$

where the summation $\sum\limits_{(3.4)}$ is extended over all $\underline{m} = (m_1, \cdots, m_7)$ satisfying (3.4) and $T(\underline{m}, y)$ denotes the number of positive integers $t \le y$, for which (3.7) is satisfied for some pair $i < j$. In [3], $T(\underline{m}, y)$ has the estimation

$$T(\underline{m}, y) \gg y (\log y)^{-7} \quad (x \ge y \ge x^{1/2}). \tag{3.9}$$

The above estimation comes from Lemma 2.2, and t satisfys

$$\Omega(f(t)) \le 27, \quad \mu^2(f(t)) = 1, \quad p(f(t)) > y^{\delta_2}, \tag{3.10}$$

where

$$f(t) = \prod_{i=1}^{7} f_i(t).$$

By (3.10), there exists some pair $i < j$, satisfying $\Omega(f_i(t)) = \Omega(f_j(t))$. Now, since $\mu^2(f(t)) = 1$, we have

$$d_k(f_i(t)) = d_k(f_j(t)).$$

Choose δ very small such that

$$\delta < \frac{1}{15}, \quad \delta < \frac{\delta_2}{2},$$

thus we obtain (3.9) from Lemma 2.2.

Therefore we get

$$\#A_k(x) \gg \sum_{(3.4)} \frac{x(\log x)^{-7}}{m_1 \cdots m_7}$$
$$\gg x(\log x)^{-7} \frac{1}{q_1 \cdots q_7} \sum_{(3.3)} \frac{1}{r_1 \cdots r_7}. \tag{3.11}$$

In [3], Hildebrand used the method of Erdös-Pomerance-Sarközy[1] to obtain

$$\sum_{(3.3)} \frac{1}{r_1 \cdots r_7} \gg \frac{(\log x')^7}{(\log z')^7 (\log\log x')^3}, \tag{3.12}$$

provided z is large enough.

Using this formula and noting that $q_1, \cdots, q_7, z'$ depend on $a_1, \cdots, a_7$ but not on k, x, we have

$$\#A_k(x) \gg x(\log\log x)^{-3}. \tag{3.13}$$

So the proof of Theorem is now complete.

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