Chin. Ann. of Math.
20B: 4(1999),479-488.

NONLINEAR STABILITY OF TWO-MODE SHOCK PROFILES FOR A RATE-TYPE VISCOELASTIC SYSTEM WITH RELAXATION

HSIAO LING* PAN RONGHUA*,**

Abstract

The authors study a 3×3 rate-type viscoelastic system, which is a relaxation approximation to a 2×2 quasi-linear hyperbolic system, including the well-known p-system. The nonlinear stability of two-mode shock waves in this relaxation approximation is proved.

Keywords Nonlinear stability, Two-mode shock profiles, Relaxation approximation
1991 MR Subject Classification 35L65, 76A10
Chinese Library Classification O175.27

§1. Introduction

In this paper, we study the following rate-type viscoelastic system, i.e.

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \quad x \in \mathbf{R}^1, \quad t > 0, \\ (p + Ev)_t = \frac{p_R(v) - p}{\tau}, \end{cases} \tag{1.1}$$

with the initial data

$$(v(x,0), u(x,0), p(x,0)) = (v_0(x), u_0(x), p_0(x)), \tag{1.2}$$

where v and $(-p)$ denote strain and stress respectively, u is related to the particle velocity, E is a positive constant, called the dynamic Young's modulus, $\tau > 0$ is a relaxation time.

This system was proposed in [14] to introduce a relaxation approximation to the following system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_R(v)_x = 0. \end{cases} \tag{1.3}$$

Since the system (1.3) can be obtained from (1.1) by an expansion procedure as the first order, it is natural to expect that the solution of (1.1) converges to that of (1.3) as $\tau \to 0$. However, the zero limit convergence has not been established yet, although some numerical experiments on (1.1) have been made[13] and certain effort on the L^2-estimates for the difference $|p - p_R(v)|$ of (1.1) have been done[2].

A tightly related problem is the nonlinear stability of waves for this relaxation approximation. As far as rarefaction waves (single or two) of (1.3) are concerned, the stability

Manuscript received September 8, 1997.
*Institute of Mathematics, Academia Sinica, Beijing 100080, China.
E-mail: hsiaol@sun.ihep.ac.cn
**Current address: SISSA, via Beirut n. 2–4, 34013, Trieste, Italy.

results have been proved in [4]. Another kind of elementary wave for (1.3) is shock waves. For any given single shock wave $(\sigma; v_-, u_-; v_+, u_+)$ of (1.3) satisfying the entropy and sub-characteristic conditions, it has been proved in [5] that (1.1) admits a smooth travelling wave solution $(\overline{v}, \overline{u}, \overline{p})(x,t) = (\overline{v}, \overline{u}, \overline{p})(\xi)$ with $\overline{v}(\pm\infty) = v^{\pm}$, $\overline{u}(\pm\infty) = u^{\pm}$, $\overline{p}(\pm\infty) = p^{\pm} \equiv p_R(v^{\pm})$, where $\xi = \frac{x-\sigma t}{\tau}$. The nonlinear stability of $(\overline{v}, \overline{u}, \overline{p})$ for (1.1) has been established by Hsiao and Luo in [5] under the following restriction

$$\int_{-\infty}^{+\infty} (v_0(x) - \overline{v}(x,0))dx = 0, \quad \int_{-\infty}^{+\infty} (u_0(x) - \overline{u}(x,0))dx = 0. \tag{1.4}$$

It says that this kind of perturbation of a shock profile produces only a translation. For 2×2 relaxation models, the stability of elementary waves has been obtained in [7], where the corresponding equilibrium equation is a scalar conservation law. Therefore a generic perturbation of a shock profile indeed produce only a translation. However, as mentioned in [8], [10], and [15], the equilibrium system for (1.1), i.e. (1.3) is a 2×2 system, and a generic perturbation of a single shock front will create not only a translation but also some new waves. By this observation, the stability of single shock front for a linearized system of (1.1) is proved in [10] without the restriction (1.4).

In the present paper, we investigate the asymptotic stability of two-mode shocks for this relaxation approximation and prove that, a generic perturbation does produce only translations of $1-$ and $2-$ shock profiles. Based on this fact, with the help of a careful analysis for the behavior of the travelling waves, we can establish the nonlinear stability results.

As far as the multi-dimensional case is concerned, we refer to [9] and [11] in which the stability for planar rarefaction waves and shock profiles are obtained respectively for a relaxation model on which the corresponding equilibrium equation is scalar. For the so-called reacting flow system, nonlinear stability results for single shock under the restriction (1.4) can be found in [12] and [16].

§2. Preliminaries—Travelling Waves

Consider the following Riemann problem

$$\begin{cases} v_t - u_x = 0, \\ u_t + (p_R(v))_x = 0, \end{cases} \tag{2.1}$$

$$(v(x,0), u(x,0)) = (v_0^r(x), u_0^r(x)), \tag{2.2}$$

where

$$(v_0^r(x), u_0^r(x)) = \begin{cases} (v_-, u_-), & x < 0, \\ (v_+, u_+), & x > 0, \end{cases}$$

(v_-, u_-) and (v_+, u_+) are two constant states.

We give the following hypotheses: for some constants c_1 and d_1 such that $-\infty < c_1 < v_-, v_+ < d_1 < +\infty$, it holds for $v \in [c_1, d_1]$ that

(H_1) $p_R'(v) < -a_1 < 0$, (H_2) $p_R''(v) > a_2 > 0$, for some positive constants a_1 and a_2;

(H_3) $|p_R'(v)| < E$, (H_4) $p_R(v)$, p_R', p_R'', p_R''' are bounded,

where (H_3) is so-called subcharacteristic condition (see [7]).

It is easy to know under (H_1)–(H_2) that, (2.1) is strictly hyperbolic and genuinely nonlinear, with eigenvalues

$$\lambda_1 = -(-p_R'(v))^{\frac{1}{2}} < 0 < (-p_R'(v))^{\frac{1}{2}} = \lambda_2. \tag{2.3}$$

Definition 2.1. *A discontinuity $(\sigma; v_l, u_l; v_r, u_r)$ of weak solutions for (2.1) is called a shock wave satisfying entropy condition if*

(i) *The Rankine-Hugoniot Condition is satisfied and the speed $\sigma(v_r, v_l)$ is defined, namely,*

$$\begin{aligned} \sigma(v_r - v_l) &= -(u_r - u_l), \\ \sigma(u_r - u_l) &= p_R(v_r) - p_R(v_l), \\ \sigma(v_r, v_l) &= \mp\Big[-\frac{p_R(v_r) - p_R(v_l)}{v_r - v_l}\Big]^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

(ii) *The entropy condition holds, namely, for any v between v_l and v_r,*

$$\begin{aligned} \sigma^2(v, v_r) \le \sigma^2(v_l, v_r) \le \sigma^2(v, v_l), \quad for \quad \sigma > 0, \\ \sigma^2(v, v_r) \ge \sigma^2(v_l, v_r) \ge \sigma^2(v, v_l), \quad for \quad \sigma < 0, \end{aligned} \tag{2.5}$$

where $\sigma^2(v, v_*) = -\frac{p_R(v) - p_R(v_*)}{v - v_*}$, $v_* = v_l$, *or* v_r.

A travelling wave solution of the rate-type viscoelastic system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \quad x \in \mathbf{R}^1, t > 0, \\ (p + Ev)_t = \frac{p_R(v) - p}{\tau} \end{cases} \tag{2.6}$$

corresponding to the shock $(\sigma; v_l, u_l; v_r, u_r)$ is the solution of the form

$$(v, u, p)(x, t) = (\overline{v}, \overline{u}, \overline{p})(\xi), \quad \xi = \frac{x - \sigma t}{\tau} \tag{2.7}$$

satisfying

$$\begin{aligned} (\overline{v}, \overline{u}, \overline{p})(-\infty) &= (v_l, u_l, p_l), \\ (\overline{v}, \overline{u}, \overline{p})(+\infty) &= (v_r, u_r, p_r), \end{aligned} \tag{2.8}$$

where $p_l \equiv p_R(v_l)$, $p_r \equiv p_R(v_r)$.

It is proved in [5] that (2.6) admits a smooth travelling wave solution , which is unique up to a shift of ξ. Since we are only interested in the large time behavior of solutions to (2.6) for fixed τ, we may assume that $\tau = 1$ without loss of generality. We give the properties of the travelling wave solution. For simplicity, we assume $\sigma > 0$. Then $v_l < v_r$. The case for $\sigma < 0$ can be treated similarly.

It is easy to know that $(\overline{v}, \overline{u}, \overline{p})$ satisfies

$$\begin{cases} -\sigma\overline{v_\xi} - \overline{u_\xi} = 0, \\ -\sigma\overline{u_\xi} + \overline{p_\xi} = 0, \\ -\sigma(\overline{p} + E\overline{v})_\xi = p_R(\overline{v}) - \overline{p}. \end{cases} \tag{2.9}$$

Thus

$$\overline{v}_\xi = g(\overline{v}), \tag{2.10}$$

where $g(\overline{v}) = -\frac{(p_R(\overline{v}) - p_l) + \sigma^2(\overline{v} - v_l)}{\sigma(E - \sigma^2)}$. It is easy to see, due to the entropy and sub-characteristic conditions, that v_l and v_r are the only roots of $g(\overline{v}) = 0$ and

$$\overline{v}_\xi > 0 \text{ for } \overline{v} \in (v_l, v_r). \tag{2.11}$$

So one is able to check that $\int_{v_0}^{\overline{v}} \frac{d\eta}{g(\eta)}$ is finite and monotone with respect to $\overline{v}$ for any given

$\overline{v}, v_0 \in (v_l, v_r)$, and

$$\int_{v_0}^{v_r} \frac{d\eta}{g(\eta)} = +\infty, \quad \int_{v_0}^{v_l} \frac{d\eta}{g(\eta)} = -\infty, \ \forall v_0 \in (v_l, v_r). \tag{2.12}$$

Thus, by integrating (2.10), we obtain $\xi = \int_{v_0}^{\overline{v}(\xi)} \frac{d\eta}{g(\eta)}$. This gives an implicit formula for $\overline{v}(\xi)$ which is uniquely determined (up to the choice of v_0) due to the properties of g. To be definitive, we take $v_0 = \frac{1}{2}(v_l + v_r)$.

Then $\overline{u}(\xi)$ and $\overline{p}(\xi)$ can be easily determined. Therefore, we have proved

Theorem 2.1. *Under the entropy condition and the subcharacteristic condition, (2.6) has a smooth travelling wave solution which is unique up to a shift in ξ and satisfies $\sigma\overline{v}_\xi > 0$.*

It is also easy to show

Lamma 2.2. $|\overline{v}_\xi| \le C|v_r - v_l|$, $|\overline{u}_\xi| + |\overline{p}_\xi| \le C|\overline{v}_\xi|$, *and the same kind of estimates hold also for the second and the third derivatives of $\overline{v}, \overline{u}$ and $\overline{p}$, respectively.*

We need the following sharper estimates, which play a key role in our stability analysis.

Lemma 2.3.

$$|\overline{v}_\xi| < C_1(|v_r - v_l|)\exp(-C_2|\xi|), \tag{2.13}$$

where C_i $(i = 1, \ 2)$ is a positive constant. The same estimates hold for $|\overline{v}_{\xi\xi}|$ and $|\overline{v}_{\xi\xi\xi}|$.

Proof. From (2.10), $\overline{v}_\xi = -\frac{(p_R(\overline{v}) - p_l) + \sigma^2(\overline{v} - v_l)}{\sigma(E - \sigma^2)} > 0$. Thus $v_l < \overline{v} < v_r$. Due to the convexity of p_R, the entropy condition imply the Lax shock condition (see [1]), namely,

$$-p_R'(v_r) < \sigma^2 < -p_R'(v_l). \tag{2.14}$$

Then, the mean value theorem and the convexity of p_R imply that there exists a unique $\xi_0 \in \mathbf{R}$ such that $\sigma^2 = -p_R'(\overline{v}(\xi_0))$. Now we arrive at

$$\begin{cases} \sigma^2 + p_R'(\overline{v}(\xi)) < 0, \ \xi < \xi_0; \\ \sigma^2 + p_R'(\overline{v}(\xi)) > 0, \ \xi > \xi_0. \end{cases} \tag{2.15}$$

Moreover, $p_R'(\overline{v}(\xi))$ is strictly increasing with respect to ξ. We calculate

$$\overline{v}_{\xi\xi} = -\frac{(p_R'(\overline{v}(\xi)) + \sigma^2)\overline{v}_\xi}{\sigma(E - \sigma^2)}, \tag{2.16}$$

which implies

$$\overline{v}_\xi(\xi) = \overline{v}_\xi(\xi_0)\exp\Big(-\int_{\xi_0}^{\xi} \frac{(p_R'(\overline{v}(s)) + \sigma^2)}{\sigma(E - \sigma^2)} ds\Big). \tag{2.17}$$

Combining (2.15)-(2.17), we complete the proof.

Next we return to the Riemann problem (2.1) and (2.2). We are interested in the case that (v_-, u_-) and (v_+, u_+) can be connected by a 1-shock and a 2-shock successively. Namely, there exists a unique state (v_m, u_m) such that $(v_m, u_m) \in S_1(v_-, u_-)$ while $(v_+, u_+) \in S_2(v_m, u_m)$, where S_k denotes the k-shock curve in the phase plane. We denote the travelling wave solutions obtained in the above procedure corresponding to the k-shock by $(v_k, u_k, p_k)(x - \sigma_k t)$ $(k = 1, \ 2)$, where σ_k is the speed for this k-shock.

We know that $(v_k, u_k, p_k)(x - \sigma_k t)$ $(k = 1, \ 2)$ satisfy the above two lemmas. Furthermore, we have the following informations.

Lemma 2.4. *For any fixed $x_i \in \mathbf{R}$ $(i = 1, \ 2)$,*

$$|v_1(x + x_1 - \sigma_1 t) - v_m| \le O(1)|v_- - v_m|\exp[-C_3(t + |x|)], \ \text{in } \Omega_2,$$
$$|v_2(x + x_2 - \sigma_2 t) - v_m| \le O(1)|v_+ - v_m|\exp[-C_4(t + |x|)], \ \text{in } \Omega_1,$$

where $\Omega_1 \equiv \{(x,t)|t \geq 0, x \leq 0\}$, $\Omega_2 \equiv \{(x,t)|t \geq 0, x \geq 0\}$ *for some positive constants* $C_3,\ C_4$. *The same results hold for* u_k *and* p_k $(k=1,\ 2)$, *with* $p_m = p_R(v_m),\ p_\mp = p_R(v_\mp)$.

Proof. This lemma can be easily proved with the help of Lemma 2.3, so we omit the details.

Let us introduce

$$\begin{cases} V(x,t;x_1,x_2) \equiv (v_1(x+x_1-\sigma_1 t) + v_2(x+x_2-\sigma_2 t) - v_m), \\ U(x,t;x_1,x_2) \equiv (u_1(x+x_1-\sigma_1 t) + u_2(x+x_2-\sigma_2 t) - u_m), \\ P(x,t;x_1,x_2) \equiv (p_1(x+x_1-\sigma_1 t) + p_2(x+x_2-\sigma_2 t) - p_m). \end{cases} \tag{2.18}$$

In view of Lemma 2.4, it follows

$$V(x,t;x_1,x_2) = \begin{cases} v_1 + F_1(x,t), \text{ in } \Omega_1, \\ v_2 + F_2(x,t), \text{ in } \Omega_2, \end{cases} \tag{2.19}$$

where
$$F_i(x,t) \leq O(1)\delta \exp[-\alpha_i(t+|x|)],\ i=1,\ 2 \tag{2.20}$$

for some positive constants α_i $(i=1,\ 2)$, and $\delta \equiv |v_- - v_m| + |v_+ - v_m|$. Similar results hold for $U(x,t;x_1,x_2)$ and $P(x,t;x_1,x_2)$. From this view point, we know that (V,U,P) is not the exact solution of (2.6), but it satisfies (2.6) approximately with an exponential decay error, namely,

$$\begin{cases} V_t - U_x = 0, \\ U_t + P_x = 0, \\ (P+EV)_t = P - p_R(V) + G(V) \end{cases} \tag{2.21}$$

with $G(V) = p_R(V) - p_R(v_1) - p_R(v_2) + p_R(v_m)$. It follows from Lemma 2.4, (2.19) and Taylor's theorem that

$$|G(V)| \leq O(1)\delta \exp[-\alpha_3(t+|x|)], \tag{2.22}$$

$$\Big|\frac{\partial^j}{\partial x^j} G(V)\Big| \leq O(1)\delta \exp[-\alpha_4(t+|x|)]. \tag{2.23}$$

§3. Stability Analysis

Consider

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (p+Ev)_t = p_R(v) - p, \end{cases} \tag{3.1}$$

with initial data

$$(v(x,0), u(x,0), p(x,0)) = (v_0(x), u_0(x), p_0(x)), \tag{3.2}$$

where $(v_0,u_0,p_0)(x)$ should be a generic perturbation of

$$(V_0,U_0,P_0)(x;0,0) \equiv (V,U,P)(x,0;0,0)$$

in the following sence

$$\int_{-\infty}^{+\infty} (v_0(x) - V_0(x;0,0), u_0(x) - U_0(x;0,0))\ dx = (\delta_1, \delta_2) \tag{3.3}$$

for suitable small numbers $\delta_1,\ \delta_2$.

Instead of $(V,U,P)(x,t;0,0)$, we should expect $(V,U,P)(x,t;x_1,x_2)$ to be the asymptotic state of $(v,u,p)(x,t)$ for some suitable $x_1,\ x_2$. A simple calculation shows that

$$\begin{aligned} &\int_{-\infty}^{+\infty} (V(x,t;x_1,x_2) - V(x,t;0,0), U(x,t;x_1,x_2) - U(x,t;0,0))\ dx \\ &= x_1(v_m - v_-, u_m - u_-) + x_2(v_+ - v_m, u_+ - u_m), \end{aligned} \tag{3.4}$$

since $(v_m - v_-, u_m - u_-)$ and $(v_+ - v_m, u_+ - u_m)$ are linearly independent if δ is suitable small. Moreover, x_1 and x_2 can be determined uniquely by the following relation

$$x_1(v_m - v_-, u_m - u_-) + x_2(v_+ - v_m, u_+ - u_m) = (\delta_1, \delta_2), \tag{3.5}$$

and therefore we have

$$\int_{-\infty}^{+\infty} (v_0(x) - V(x, 0; x_1, x_2), u_0(x) - U(x, 0; x_1, x_2))\, dx = 0. \tag{3.6}$$

Hereafter, we fix x_1 and x_2 determined above. For simplicity, $(V, U, P)(x, t; x_1, x_2)$ will be denoted by $(V, U, P)(x, t)$.

The purpose in this section is to show that, if the shock waves are weak (i.e. δ is small), then (V, U, P) is a global attractor for (3.1). For this purpose we also require that

$$\begin{cases} \int_{-\infty}^{+\infty} (1 + x^2)(v_0(x) - V(x, 0))^2\, dx < \delta_3, \\ \int_{-\infty}^{+\infty} (1 + x^2)(u_0(x) - U(x, 0))^2\, dx < \delta_4 \end{cases} \tag{3.7}$$

for some suitable small positive constants δ_3 and δ_4. Then we can establish the following stability theorem

Theorem 3.1. *Under* (H_1)-(H_4) *and* (3.7), *there exist positive constants* δ_0 *and* ε_0, *such that if* $\delta < \delta_0$ *and* $\|(v_0 - V(x, 0), u_0 - U(x, 0), p_0 - P(x, 0))\|_{H^1} + \delta_3 + \delta_4 \le \varepsilon_0$, *then the problem* (3.1)–(3.2) *has a unique smooth global solution* (v, u, p), *which tends to the wave* (V, U, P) *uniformly in* x *as* $t \to +\infty$. *Hereafter we use the following notation for simplicity*

$$\|(f_1, f_2, \dots, f_l)\|_{H^m}^2 \equiv \sum_{i=1}^{l} \|f_i\|_{H^m}^2,$$

with $l \ge 1$, $m \ge 0$ *and* $H^0 = L^2$.

By virtue of (3.5), we can introduce $(\phi_x, \psi_x, w) = (v, u, p) - (V, U, P)$, where

$$\phi(x, t) = \int_{-\infty}^{x} (v - V)(y, t)\, dy, \quad \psi(x, t) = \int_{-\infty}^{x} (u - U)(y, t)\, dy.$$

The weighted Poincare inequality (see [3, Theorem 328]) gives

$$\Big| \int_0^{+\infty} \Big| \int_y^{+\infty} (v_0(x) - V(x, 0))\, dx \Big|^2 dy \Big| \le 4 \int_0^{+\infty} (v_0(x) - V(x, 0))^2 x^2\, dx,$$

which implies that $\phi(x, 0) = \phi_0(x) \in H^2$ and $\|\phi_0(x)\|_{H^2}^2 \le C\varepsilon_0$. Similarly, $\psi(x, 0) = \psi_0(x) \in H^2$ and $\|\psi_0(x)\|_{H^2}^2 \le C\varepsilon_0$. In the following, we will use C to denote the generic constant independent of t.

It is easy to check that (ϕ, ψ, w) satisfies the following system

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + w = 0, \\ w_t + E\psi_{xx} + w + [p_R(V) - p_R(V + \phi_x)] = G(V), \end{cases} \tag{3.8}$$

where $G(V)$ is defined in (2.22).

We denote

$$L_1 \equiv \phi_t - \psi_x = 0, \tag{3.9}$$

$$L_2 \equiv \psi_{tt} - E\psi_{xx} + \psi_t - A\phi_x = G(V) + F(V, \phi_x)\phi_x^2, \tag{3.10}$$

with

$$A(V,\phi) = -p'_R(V), \tag{3.11}$$

$$F(V,\phi_x)\phi_x^2 = -(p_R(V+\phi_x) - p_R(V) - p'_R(V)\phi_x). \tag{3.12}$$

It is not difficult to know that (3.9)–(3.10) give a closed system for (ϕ,ψ). We consider (3.9)–(3.10) with initial data

$$\begin{cases} \phi(x,0) = \phi_0(x) = \int_{-\infty}^{x}(v_0(y) - V(y,0))dy \in H^2, \\ \psi(x,0) = \psi_0(x) = \int_{-\infty}^{x}(u_0(y) - U(y,0))dy \in H^2, \\ \psi_t(x,0) = \psi_1(x) = -w_0(x) = p_0(x) - P(x,0) \in H^1. \end{cases} \tag{3.13}$$

In order to show Theorem 3.1, one only needs to prove

Theorem 3.2. *Under* (H$_1$)-(H$_4$) *and* (3.7)*, there exist positive constants δ_0 and ε_0, such that if $\delta<\delta_0$ and $\|(\phi_0,\psi_0)\|_{H^2} + \|\psi_1\|_{H^1} \le \varepsilon_0$, then the problem* (3.9)–(3.10) *and* (3.13) *has a unique global solution such that (ϕ_x,ψ_x,ψ_t) tends to $(0,0,0)$ uniformly in x as $t\to+\infty$.*

We will solve the Cauchy problem (3.9)–(3.10) and (3.13) in the space

$$X(0,T) = \{(\phi,\psi)\in C^0(0,T;H^2),\ \psi_t \in C^0(0,T;H^1)\}$$

with the norm

$$N^2(t) = \sup_{0\le\tau\le t}(\|(\phi,\psi)(\tau)\|_{H^2}^2 + \|\psi_t(\tau)\|_{H^1}^2).$$

To prove Theorem 3.2, we need some a priori estimates. In the following, we always assume a priori that $(\phi,\psi)\in X(0,T)$ is the solution of (3.9)–(3.10) and (3.13) for some $T>0$. Furthermore, we will use the third derivatives of ϕ or ψ formally. This will not cause any trouble, since we may assume $(\phi,\psi)\in H^3$ first and use Fridrich's modifier then to deal with the original case.

Lemma 3.3. *Suppose the conditions in Theorem* 3.2 *are satisfied, $\delta<\delta_0$, and $N(T)\le\varepsilon$ for some suitably small δ_0 and ε. Then it holds*

$$\begin{aligned} &N^2(T) + \sup_{0\le\tau\le t}\|\psi_{tt}(\tau)\|^2 + \int_0^T \|\psi_{tt}(\tau)\|^2\ d\tau + \int_0^T \|(\phi_t,\phi_x,\psi_t,\psi_x)(t)\|_{H^1}^2 dt \\ &\le K^2(N^2(0)+\delta_0) \end{aligned}$$

for $(\phi,\psi)\in X(0,T)$, where $K>1$ is a positive constant which does not depend on T.

To prove this lemma, we establish the following Lemmas 3.4–3.7 next.

By Sobolev embedding theorem, $H^{m+1}\hookrightarrow C^m$, $m\ge 0$. Thus if $N(T)\le\varepsilon$, then $\|(\phi,\psi)\|_{C^1}\le C\varepsilon$, $|\psi_t\|_{C^0}\le C\varepsilon$.

From these facts, we know that there are constants $-\infty<c<d<+\infty$ such that $c>c_1$ and $d<d_1$, and $v\in[c,d]$.

Lemma 3.4. *Suppose the conditions in Lemma* 3.3 *are satisfied, $\delta\le\delta_0$, and $N(T)\le\varepsilon$ for some suitably small δ_0 and ε. Then we have*

$$\begin{aligned} &\|(\phi,\psi,\psi_t,\psi_x)(t)\|^2 + \int_0^t \|(\psi_x,\psi_t)(\tau)\|^2 d\tau \\ &+ \int_0^t\int_{-\infty}^0 |\sigma_1 v_{1x}|\psi^2 dxd\tau + \int_0^t\int_0^{+\infty} |\sigma_2 v_{2x}|\psi^2 dxd\tau \\ &\le C(N^2(0)+\delta N(t)+\delta N^2(t)) + CN(t)\int_0^t \|\phi_x(\tau)\|^2 d\tau. \end{aligned} \tag{3.14}$$

Proof. We consider the equality

$$(\phi+\mu\psi_x)L_1+A^{-1}(\mu\psi_t+\psi)L_2=A^{-1}(\psi+\mu\psi_x)(G(V)+F\phi_x^2) \tag{3.15}$$

with a positive constant μ, which will be chosen later. The left hand side of (3.15) can be reduced to

$$\begin{aligned}&\Big[\frac{1}{2}\phi^2+\frac{1}{2}A^{-1}\psi^2+\mu\phi\psi_x+\frac{\mu}{2}A^{-1}\psi_t^2+A^{-1}\psi\psi_t+\frac{\mu E}{2}A^{-1}\psi_x^2\Big]_t\\&+\Big[(\mu-1)A^{-1}-\frac{1}{2}\mu A_t^{-1}\Big]\psi_t^2+\Big[EA^{-1}-\frac{1}{2}\mu EA_t^{-1}-\mu\Big]\psi_x^2\\&-\frac{1}{2}A_t^{-1}\psi^2-A_t^{-1}\psi\psi_t+EA_x^{-1}\psi\psi_x+E\mu A_x^{-1}\psi_t\psi_x+\{\cdots\}_x,\end{aligned}$$

where $\{\cdots\}_x$ denotes the terms which disappear after integrations with respect to x.

Taking

$$1<\mu=\frac{E+E_1}{2E_1}<\frac{E}{E_1}, \tag{3.16}$$

where $E_1=\sup\limits_{v\in[c,d]}|p_R'(v)|<E$. It is easy to see that

$$\begin{cases}b_1(\psi^2+\psi_t^2)\le[\frac{1}{2}A^{-1}\psi^2+\frac{\mu}{2}A^{-1}\psi_t^2+A^{-1}\psi\psi_t]\le b_2(\psi^2+\psi_t^2),\\ b_3(\phi^2+\psi_x^2)\le[\frac{1}{2}\phi^2+\mu\phi\psi_x+\frac{\mu E}{2}A^{-1}\psi_x^2]\le b_4(\phi^2+\psi_x^2),\\ [(\mu-1)A^{-1}-\frac{1}{2}\mu A_t^{-1}]\psi_t^2\ge b_5\psi_t^2,\\ [EA^{-1}-\frac{1}{2}\mu EA_t^{-1}-\mu]\psi_x^2\ge b_6\psi_x^2,\end{cases} \tag{3.17}$$

for some positive constants $b_i(i=1,\ \cdots,\ 6)$.

We also note that

$$-\frac{1}{2}A_t^{-1}=\begin{cases}\frac{1}{2}A^{-2}p_R''(\overline{v})\sigma_1v_{1x}+f_1, & \text{in } \Omega_1,\\ \frac{1}{2}A^{-2}p_R''(\overline{v})\sigma_2v_{2x}+f_2, & \text{in } \Omega_2\end{cases} \tag{3.18}$$

with $f_i=O(1)\delta\exp[-C(|x|+t)]$, for $i=1,\ 2$. Similar results hold for $|A_x|$. Then we have

$$\begin{aligned}&|A_t^{-1}\psi\psi_t|+|A_x^{-1}\psi\psi_x|\\&\le\begin{cases}\eta_1\sigma_1v_{1x}\psi^2+C(\eta_1)\delta\psi_x^2+C\delta^2\exp[-C(|x|+t)], & \text{in } \Omega_1,\\ \eta_2\sigma_2v_{2x}\psi^2+C(\eta_2)\delta\psi_x^2+C\delta^2\exp[-C(|x|+t)], & \text{in } \Omega_2,\end{cases}\end{aligned} \tag{3.19}$$

$$|A_x^{-1}\psi_t\psi_x|\le C\delta\psi_t^2+\delta\psi_x^2 \tag{3.20}$$

for any positive constants η_1 and η_2.

Integrating (3.15) over $[0,t]\times(-\infty,+\infty)$, and taking η_1 and η_2 suitable small, we get (3.14) with the help of the above estimates and the following facts

$$\begin{cases}\int_0^t\int_{-\infty}^{+\infty}(|G(V)|+|f_1|+|f_2|)\ dx\ d\tau\le C\delta,\\ |(\psi,\phi,\psi_t,\psi_x)|\le CN(t),\\ |F(V,\phi_x)|\le C.\end{cases}$$

To estimate $\int_0^t\|\phi_x(\tau)\|^2\ d\tau$, we use the following lemma.

Lemma 3.5. *Suppose the conditions in Lemma* 3.3 *are satisfied,* $\delta\le\delta_0$, *and* $N(T)\le\varepsilon$ *for some suitably small* δ_0 *and* ε. *Then we have*

$$\int_0^t\|\phi_x(\tau)\|^2d\tau\le C(N^2(0)+\delta N(t)+\delta N^2(t)). \tag{3.21}$$

Proof. We investigate the following relation

$$(E\phi_x-\psi_t)\partial_xL_1-\phi_xL_2=-\phi_x(G(V)+F(V,\phi_x)\phi_x^2). \tag{3.22}$$

The left hand side of (3.22) can be reduced into

$$\Big[\frac{1}{2}E\phi_x^2 - \psi_t\phi_x - \frac{1}{2}\psi_x^2\Big]_t - \phi_x\psi_t + A\phi_x^2 + \{\cdots\}_x.$$

We know that $A > a_1 > 0$ from (H_1). Integrating (3.22) over $[0,t]\times(-\infty,+\infty)$, and using Young's inequality and Lemma 3.3, one can easily obtain (3.21).

Combining Lemmas 3.3 and 3.4, we arrive at

$$\|(\phi,\psi,\psi_t,\psi_x,\phi_x)(t)\|^2 + \int_0^t \|(\psi_t,\psi_x,\phi_x)(\tau)\|^2 d\tau \le C(N^2(0)+\delta N(t)). \tag{3.23}$$

Instead of (3.15) and (3.22) we study the following two equations

$$(\phi_x+\mu\psi_{xx})\partial_x L_1 + A^{-1}(\mu\psi_{xt}+\psi_x)\partial_x L_2 = A^{-1}(\psi_x+\mu\psi_{tx})(G(V)+F\phi_x^2)_x, \tag{3.24}$$

$$(E\phi_{xx}-\psi_{xt})\partial_{xx}L_1 - \phi_{xx}\partial_x L_2 = -\phi_{xx}(G(V)+F(V,\phi_x)\phi_x^2)_x. \tag{3.25}$$

Repeating the procedure in the proof of Lemmas 3.4 and 3.5, it is not difficult to show

Lemma 3.6. *Suppose the conditions in Lemma* 3.3 *are satisfied,* $\delta \le \delta_0$*, and* $N(T) \le \varepsilon$ *for some suitably small* δ_0 *and* ε*. Then we have*

$$\begin{aligned}&\|(\phi_{xx},\psi_{tx},\psi_{xx})(t)\|^2 + \int_0^t \|(\phi_{xx},\psi_{tx},\psi_{xx})(\tau)\|^2 d\tau\\ &\le C(N^2(0)+\delta N(t)).\end{aligned} \tag{3.26}$$

Lemma 3.7. *Suppose the conditions in Lemma* 3.3 *are satisfied,* $\delta \le \delta_0$*, and* $N(T) \le \varepsilon$ *for some suitably small* δ_0 *and* ε*. Then we have*

$$\|\psi_{tt}(t)\|^2 + \int_0^t \|\psi_{tt}(\tau)\|^2 d\tau \le C(N^2(0)+\delta_0). \tag{3.27}$$

Proof. We make use of the following equality

$$\psi_{tt}\partial_t L_2 = \psi_{tt}(G(V)+F\phi_x^2)_t, \tag{3.28}$$

which implies that

$$\frac{1}{2}\Big(\psi_{tt}^2 + \frac{E}{2}\psi_{tx}^2\Big)_t + \psi_{tt}^2 = A_t\psi_x\psi_{tt} + A\psi_{xx}\psi_{tt} + \{\cdots\}_x. \tag{3.29}$$

Integrating (3.29), with the help of Young's inequality and Lemmas 3.4–3.6, we obtain (3.27).

Now we combine Lemmas 3.4–3.7 to get

$$\begin{aligned}&N^2(T) + \sup_{0\le\tau\le t}\|\psi_{tt}(\tau)\|^2 + \int_0^T \|\psi_{tt}(\tau)\|^2\, d\tau + \int_0^T \|(\phi_t,\phi_x,\psi_t,\psi_x)(t)\|_{H^1}^2 dt\\ &\le K^2(N^2(0)+\delta N(t)),\end{aligned}$$

which means that

$$\begin{aligned}&N^2(T) + \sup_{0\le\tau\le t}\|\psi_{tt}(\tau)\|^2 + \int_0^T \|\psi_{tt}(\tau)\|^2\, d\tau + \int_0^T \|(\phi_t,\phi_x,\psi_t,\psi_x)(t)\|_{H^1}^2 dt\\ &\le K^2(N^2(0)+\delta_0)\end{aligned} \tag{3.30}$$

if we use the Cauchy-Schwarz inequality with a suitable weight. Hence we have proved Lemma 3.3.

Since the local (in time) existence and uniqueness of the solution for initial value problem (3.9)–(3.10) and (3.13) can be obtained by a standard procedure in view of the a priori estimates in Lemma 3.3, it follows from Lemma 3.3 and a standard continuity argument (see [4, 5, 6, 10]) that the problem (3.5)–(3.6) and (3.9) has a unique global (in time) solution $(\phi,\psi)\in X(0,+\infty)$, satisfying (3.30) for any $t\geq 0$. Thus

$$\int_0^{+\infty}\Big(\|\phi_x(t)\|^2+\Big|\frac{d}{dt}\|\phi_x(t)\|^2\Big|\Big)dt<+\infty.$$

It follows then that $\lim\limits_{t\to+\infty}\|\phi_x(t)\|^2=0$. Similarly, we have

$$\lim_{t\to+\infty}\|\psi_x(t)\|^2=0,\quad \lim_{t\to+\infty}\|\psi_t(t)\|^2=0.$$

For any $(x,t)\in\mathbf{R}\times\mathbf{R}^+$, we have

$$\phi_x^2(x,t)=2\int_{-\infty}^x\phi_x(y,t)\phi_{xx}(y,t)dydt\leq C\|\phi_x(t)\|,$$

which implies that $\lim\limits_{t\to+\infty}\sup\limits_{x\in\mathbf{R}}|\phi_x(x,t)|=0$. Similarly, it can be proved that

$$\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}|\psi_x(x,t)|=0,\quad \lim_{t\to+\infty}\sup_{x\in\mathbf{R}}|\psi_t(x,t)|=0.$$

This completes the proof of Theorem 3.2.

References

[1] Chang, T. & Hsiao, L., The Riemann problem and interaction of waves in gas dynamics,, Longman Sci.& Tech., 1989.

[2] Faciu, C., Numerical aspects in modelling phase transition by rate-type constitutive equation, *Int. J Engng. Sci.,* **29**(1991), 1103–1119.

[3] Hardy, G. H., Littlewood, J. E. & Polya, G., Inequalities, Cambridge Univ. Press, 1934.

[4] Hsiao, L. & Pan, R. H., Nonliear stability of rarefaction waves for a rate-type viscoelastic system, *Chin. Ann. of Math.,* **20B:**2(1999), 223–232.

[5] Hsiao, L. & Luo, t., The stability of travelling wave solutions for a rate-type viscoelastic system, to appear.

[6] Kawashima, S. & Matsmura, A., Astmptitic stability to travelling wave solutions of systems for one-dimensional gas motion, *Comm. Math. Phys.,* **101**(1985), 97–127.

[7] Liu, T. P., Hyperbolic conservation laws with relaxation, *Comm. Math. Phys.,* **108**(1987), 153–175.

[8] Liu, T. P., Nonliear stability of shock waves for viscous conservation laws, *Memoirs of Amer. Math. Soc.,* **328**(1986).

[9] Luo, T., Asymptotic stability of planar rarefaction waves for the relaxation approximations of conservation laws in several dimensions, *J. Diff. Eqns.,* **133:**2(1997), 255–279.

[10] Luo, T. & Serre, D., Linear stability of shock profiles for a rate-type viscoelastic system, to appear.

[11] Luo, T. & Xin, Z. P., Asymptotic stability of planar shock profiles for the relaxation approximations of conservation laws in several dimensions, *J. Diff. Eqns.,* **139:**2(1997), 365–408.

[12] Pan, R. H., The nonliear stability of traveling wave solutions for a reacting flow model with source term, to appear.

[13] Pitman, R. B. & Ni, Y. G., Visco-elastic relaxation with a Van der Waals type stress, *Int. J. Engng. Sci.,* **32:**2(1994), 327–338.

[14] Suliciu, I., On modelling phase transitions by means of rate-type constitutive equations, shock wave structure, *Int. J. Engng. Sci.,* **28**(1990), 827–841.

[15] Szepessy, A. & Xin, Z. P., Nonliear stability of viscous shock waves, *Arch. Rational Mech. Anal.,* **122** (1993), 53–103.

[16] Yong, W. A., Existence and asymptotic stability of traveling wave solutions of a model system for reacting flow, *Nonlinear Analysis, TMA,* **20:**12(1996).