Chin. Ann. of Math.
20B: 4(1999),489-494.

PSEUDO ALMOST PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT

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Abstract

The authors discuss the existence of pseudo almost periodic solutions of differential equations with piecewise constant argument by means of introducing new concept, pseudo almost periodic sequence.

Keywords Pseudo almost periodic functions, Pseudo almost periodic sequences, Piecewise constant arguments, Ergodic perturbation
1991 MR Subject Classification 34K15, 34K20
Chinese Library Classification O175.12, O175.13

§1. Introduction

Differential equations with piecewise constant arguments (DEPCA) can describe hybrid dynamical systems and therefore combine properties of both differential equations and difference equations. They have applications in certain biomedical models[1]. In [2,3] and the references therein there have been a lot of results concerning DEPCA. Nevertheless, all of them dealt with the stability, periodicity, and oscillation etc. of solutions of DEPCA. Recently, [4] disscused the existence of almost periodic solutions of DEPCA. In this paper, we define pseudo almost periodic sequence, and by the existence of pseudo almost periodic sequence solutions of a difference equation we investigate the existance of pseudo almost periodic solutions to DEPCA. Pseudo almost periodic function is an extention of almost periodic function. It was defined in [5]. In what follows, we denote by [ ] the greatest–integer function.

We consider the inhomogeneous DEPCA of the form

$$\dot{x}(t) = ax(t) + bx([t]) + f(t), \quad t \in \mathbf{R} \tag{1.1}$$

and nonlinear DEPCA

$$\dot{x}(t) = ax(t) + bx([t]) + F(t, x), \quad t \in \mathbf{R}, \tag{1.2}$$

where a, b are constants. $f : \mathbf{R} \to \mathbf{R}$ and $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are continuous. We say that a function $x : \mathbf{R} \to \mathbf{R}$ is a solution of (1.1) (or (1.2)), if the following conditions hold:

Manuscript received July 2, 1997. Revised April 16, 1999.

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(i) x is continuous on $\mathbf{R}$;

(ii) The first derivative $\dot{x}$ of x exists on $\mathbf{R}$, except possibly at the points $t = n, n \in \mathbf{Z} = \{\cdots -1, 0, 1, \cdots\}$, where one-sided derivatives exist, and

(iii) x satisfies (1.1) (or (1.2)) in every interval $(n, n+1), \forall n \in \mathbf{Z}$.

Now we state some definitions.

Denote the set of all almost periodic functions on $\mathbf{R}$ by $\mathcal{A}P(\mathbf{R})$ (see [6,7]).

Definition 1.1.[6,7] *Suppose Ω is an open set in $\mathbf{R}$. A function $F : \mathbf{R} \times \Omega \to \mathbf{R}$ is called an almost periodic function for t uniformly on Ω, if for any compact subset $\mathbf{W} \subset \Omega$, the ϵ–translation set of F*

$$\mathcal{T}(F, \epsilon, \mathbf{W}) = \{\tau \in \mathbf{R} : |F(t+\tau, x) - F(t, x)| < \epsilon, \forall (t, x) \in \mathbf{R} \times \mathbf{W}\}$$

is a relatively dense set in $\mathbf{R}$. Denote by $\mathcal{AP}(\mathbf{R} \times \Omega)$ the set of all such functions.

We also denote by $\mathcal{PAP}_0(\mathbf{R})$ the set

$$\left\{\varphi \in \mathcal{C}(\mathbf{R}) : \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} |\varphi(t)| ds = 0\right\},$$

and by $\mathcal{PAP}_0(\mathbf{R} \times \Omega)$ the set

$$\left\{\varphi \in \mathcal{C}(\mathbf{R} \times \Omega) : \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} |\varphi(s, x)| ds = 0, \text{ uniformly for } x \in \Omega\right\}.$$

Definition 1.2.[5] *A bounded function $f : \mathbf{R} \to \mathbf{R}$ is called pseudo almost periodic if $f = g + \varphi$, where $g \in \mathcal{AP}(\mathbf{R})$, $\varphi \in \mathcal{PAP}_0(\mathbf{R})$. g and φ are called the almost periodic component and the ergodic perturbation, respectively, of f. Denote by $\mathcal{PAP}(\mathbf{R})$ the set of all such functions.*

Definition 1.3.[5] *A bounded function $F : \mathbf{R} \times \Omega \to \mathbf{R}$ is called uniformly pseudo almost periodic if $F = G + \Phi$, where $G \in \mathcal{AP}(\mathbf{R} \times \Omega), \Phi \in \mathcal{PAP}_0(\mathbf{R} \times \Omega)$. Denote by $\mathcal{PAP}(\mathbf{R} \times \Omega)$ the set of all such functions.*

Denote by $\mathcal{AP}(\mathbf{Z})$ the set of all almost periodic sequences on $\mathbf{Z}$ (see [6,8]). We denote by $\mathcal{PAP}_0(\mathbf{Z})$ the set

$$\left\{\varphi(n) : \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} |\varphi(n)| = 0\right\}.$$

Definition 1.4. *A bounded sequence $x : \mathbf{Z} \to \mathbf{R}$ is called pseudo almost periodic sequence, if $x = g + \varphi$ and $g \in \mathcal{AP}(\mathbf{Z}), \varphi \in \mathcal{PAP}_0(\mathbf{Z})$.*

If $x : \mathbf{R} \to \mathbf{R}$ is a solution of (1.1), then we easily get

$$x(t) = \Big(\Big(1 + \frac{b}{a}\Big)e^{a(t-n)} - \frac{b}{a}\Big)C_n + \int_n^t e^{a(t-s)} f(s) ds, \quad n \le t < n+1,$$

where $C_n = x(n)$. In view of continuity of solution, we have the following difference equation

$$C_{n+1} = PC_n + h_n, \quad n \in \mathbf{Z}, \tag{1.4}$$

where

$$P = \Big(1 + \frac{b}{a}\Big)e^a - \frac{b}{a}, \quad h_n = \int_n^{n+1} e^{a(n+1-s)} f(s) ds.$$

Now we state our main results.

Theorem 1.1. *If $|P| < 1$, then Equation (1.1) has a unique pseudo almost periodic solutions for every $f \in \mathcal{PAP}(\mathbf{R})$.*

Theorem 1.2. *Suppose $F \in \mathcal{PAP}(\mathbf{R} \times \Omega)$ and it satisfies Lipschitz condition*

$$|F(t, x_1) - F(t, x_2)| < L|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbf{R}, \tag{1.5}$$

where

$$L < \Big(\frac{M}{1-|P|}e^{|a|} + e^{|a|}\Big)^{-1}, \quad M = \max_{t\in[0,1]}\Big|\Big(1+\frac{b}{a}\Big)e^{at} - \frac{b}{a}\Big|.$$

Then (1.2) *has a unique pseudo almost periodic solution.*

§2. Proofs of Main Results

Lemma 2.1. $\Big\{h_n = \int_n^{n+1} e^{a(n+1-s)} f(s)ds\Big\}_{n\in Z}$ *is pseudo almost periodic sequence.*

Proof. Since

$$f = g + \varphi, \quad g \in \mathcal{AP}(\mathbf{R}), \quad \varphi \in \mathcal{PAP}_0(\mathbf{R}),$$

$$\int_n^{n+1} e^{a(n+1-s)} f(s)ds = \int_n^{n+1} e^{a(n+1-s)} g(s)ds + \int_n^{n+1} e^{a(n+1-s)} \varphi(s)ds,$$

$$\frac{1}{2N}\sum_{n=-N}^{N}\Big|\int_n^{n+1} e^{a(n+1-s)}\varphi(s)ds\Big| \le \frac{e^{|a|}}{2N}\int_{-N}^{N+1}|\varphi(s)|ds$$

$$\le \frac{2(N+1)e^{|a|}}{2N}\cdot\frac{1}{2(N+1)}\int_{-(N+1)}^{N+1}|\varphi(s)|ds \to 0 \quad (\text{as } N\to\infty),$$

from [4. Lemma 3], we know

$$\Big\{\int_n^{n+1} e^{a(n+1-s)} g(s)ds\Big\}_{n\in Z} \in \mathcal{AP}(\mathbf{Z}).$$

So, $\{h_n\}_{n\in Z} \in \mathcal{PAP}(\mathbf{Z})$.

Lemma 2.2. *Suppose $|P| < 1$. Then*

$$\Big\{C_n = \sum_{i=-\infty}^{n-1} P^{n-i-1}h_i\Big\}_{n\in Z}$$

is a pseudo almost periodic solution sequence of difference equation (1.4).

Proof. We can easily verify that

$$\Big\{C_n = \sum_{i=-\infty}^{n-1} P^{n-i-1}h_i\Big\}_{n\in Z}$$

is a solution sequence of (1.4). Let $h_i = g_i + \varphi_i$, $g_i \in \mathcal{PAP}(\mathbf{Z})$, $\varphi_i \in \mathcal{PAP}_0(\mathbf{Z})$. Therefore

$$\lim_{i\to\infty}\frac{1}{2N}\sum_{i=-N}^{N}|\varphi_i| = 0.$$

Let

$$G_n = \sum_{i=-\infty}^{n-1} P^{n-i-1}g_i, \quad \Phi_n = \sum_{i=-\infty}^{n-1} P^{n-i-1}\varphi_i.$$

Then $C_n = G_n + \Phi_n, n \in \mathbf{Z}$. From the proof of [4, Theorem 1], we know $\{G_n\}_{n\in\mathbf{Z}} \in \mathcal{A}P(\mathbf{Z})$.

Now we will prove $\{\Phi_n\}_{n\in\mathbf{Z}} \in \mathcal{PAP}_0(\mathbf{Z})$. In fact

$$\begin{aligned}
&\frac{1}{2N}\sum_{n=-N}^{N}|\Phi_n| \\
&= \frac{1}{2N}\left(\sum_{i=-\infty}^{-N-1}P^{-N-i-1}|\varphi_i| + \sum_{i=-\infty}^{-N}P^{-N-i}|\varphi_i| + \cdots + \sum_{i=\infty}^{N-1}P^{N-i-1}|\varphi_i|\right) \\
&= \frac{1}{2N}\big(|\varphi_{N-1}| + |\varphi_{N-2}|(1+|P|) + |\varphi_{N-3}|(1+|P|+|P|^2) \\
&\quad + |\varphi_{-N+1}|(1+|P|+\cdots+|P|^{2N-2}) + |\varphi_{-N}|(1+|P|+\cdots+|P|^{2N-1}) \\
&\quad + |\varphi_{-N-1}|(1+|P|+\cdots+|P|^{2N}) + |P||\varphi_{-N-2}|(1+|P|+\cdots+|P|^{2N}) \\
&\quad + |P|^2|\varphi_{-N-3}|(1+|P|+\cdots+|P|^{2N}) + |P|^3|\varphi_{-N-4}|(1+|P|+\cdots+|P|^{2N}) + \cdots\big) \\
&\le \frac{1}{2N(1-|P|)}(|\varphi_{N-1}| + |\varphi_{N-2}| + \cdots + |\varphi_{-N-1}|) + \frac{|P|}{2N(1-|P|)^2}\sup_{n\in\mathbf{Z}}|\varphi_n| \\
&\le \frac{1}{1-|P|}\frac{2(N+1)}{2N}\frac{1}{2(N+1)}\sum_{i=-N-1}^{N+1}|\varphi_i| + \frac{|P|}{1-|P|^2}\frac{1}{2N}\sup_{n\in\mathbf{Z}}|\varphi_n| \\
&\longrightarrow 0 \quad \text{as} \quad N\longrightarrow\infty.
\end{aligned}$$

So, $\{\Phi_n\}_{n\in\mathbf{Z}} \in \mathcal{PAP}_0(\mathbf{Z})$. The proof of Lemma 2.2 is completed.

Proof of Theorem 1.1. We are going to show that

$$x(t) = \Big(\Big(1+\frac{b}{a}\Big)e^{a(t-n)} - \frac{b}{a}\Big)C_n + \int_n^t e^{a(t-s)}f(s)ds, \quad n\le t<n+1, \quad \forall n\in\mathbf{Z} \tag{2.1}$$

is a pseudo almost periodic function. Let

$$y(t) = \Big(\Big(1+\frac{b}{a}\Big)e^{a(t-n)} - \frac{b}{a}\Big)G_n + \int_n^t e^{a(t-s)}g(s)ds, \tag{2.2}$$

$$z(t) = \Big(\Big(1+\frac{b}{a}\Big)e^{a(t-n)} - \frac{b}{a}\Big)\phi_n + \int_n^t e^{a(t-s)}\varphi(s)ds. \tag{2.3}$$

Then $x(t) = y(t) + z(t)$. From the proof of [4, Theorem 1], we know $y\in\mathcal{AP}(\mathbf{R})$. Now let us test $z\in\mathcal{PAP}_0(\mathbf{R})$. It follows from (2.3) that

$$\begin{aligned}
\frac{1}{2t}\int_{-t}^{t}|z(s)|ds &\le \frac{1}{2t}\sum_{i=[-t]}^{[t]+1}\Big(\int_i^{i+1}\Big|\Big(1+\frac{b}{a}\Big)e^{a(s-i)} - \frac{b}{a}\Big||\Phi_i|ds \\
&\quad + \int_i^{i+1}\Big(\int_i^{i+1}e^{a(u-s)}|\varphi(s)|ds\Big)du\Big) \\
&\le \frac{M}{2t}\sum_{i=[-t]-1}^{[t]+1}|\phi_i| + \frac{e^{|a|}}{2t}\sum_{i=[-t]-1}^{[t]+1}\int_i^{i+1}|\varphi(s)|ds \\
&\le \frac{2M([t]+2)}{2t}\frac{1}{2([t]+2)}\sum_{i=-([t]+2)}^{[t]+2}|\Phi_i| \\
&\quad + \frac{2e^{|a|}([t]+2)}{2t}\frac{1}{2([t]+2)}\int_{-([t]+2)}^{[t]+2}|\varphi(s)|ds \quad \longrightarrow 0 \quad \text{as} \quad t\to\infty,
\end{aligned}$$

where $M = \max\limits_{t\in[0,1]}|(1+\frac{b}{a})e^{at} - \frac{b}{a}|$. So, $z\in\mathcal{PAP}_0(\mathbf{R})$, and therefore $x(t)\in\mathcal{PAP}(\mathbf{R})$.

If there was another bounded solution of Equation (1.1), denoted by $\overline{x}(t)$, then $\overline{x}(t)-x(t)$ is a solution of the corresponding homogeneous equation. Thus, $\{\overline{x}(n) - x(n)\}_{n\in z}$ is a solution of the homogeneous difference equation

$$C_{n+1} = PC_n. \tag{2.4}$$

Hence, there exists an $r \in \mathbf{R}$, such that

$$\overline{x}(n) - x(n) = rP^n. \tag{2.5}$$

If $r \neq 0$, then this means that $\{\overline{x}(n) - x(n)\}_{n\in \mathbf{Z}}$ is not bounded. This is a contradiction. So, $\overline{x}(n) - x(n) \equiv 0, n \in \mathbf{Z}$. This implies $\overline{x}(t) \equiv x(t), t \in \mathbf{R}$.

Proposition 2.1.[5] *If $f \in \mathcal{PAP}(\mathbf{R})$ and g is its almost periodic component, then we have $g(\mathbf{R}) \subset \overline{f(\mathbf{R})}$. Therefore*

$$\|f\| \geq \|g\| \geq \inf_{t\in\mathbf{R}} |g(t)| \geq \inf_{t\in\mathbf{R}} |f(t)|,$$

where $\|\phi\| = \sup\limits_{t\in\mathbf{R}} |\phi(t)|$.

Propsition 2.2. *$\mathcal{PAP}(\mathbf{R})$ is a Banach space with the norm $\|\phi\| = \sup\limits_{t\in\mathbf{R}} |\phi(t)|$.*

Proof. Let a sequence $\{f^{(n)}\} \subset \mathcal{PAP}(\mathbf{R})$ be Cauchy, and

$$f^{(n)} = g^{(n)} + \varphi^{(n)}, \quad g^{(n)} \in \mathcal{AP}(\mathbf{R}), \quad \varphi^{(n)} \in \mathcal{PAP}_0(\mathbf{R}).$$

By Proposition 2.1, the sequence $\{g^{(n)}\}$ is Cauchy, so is $\{\varphi^{(n)}\}$. Since $\mathcal{AP}(\mathbf{R})$ is a Banach space (see [6, 10]), there is $g \in \mathcal{AP}(\mathbf{R})$ such that $\|g^{(n)} - g\| \to 0$. We know that the set of all bounded continuous functions is a Banach space with the same norm. Denote this set by $\mathcal{C}_B(\mathbf{R})$. Then there is $\varphi \in \mathcal{C}_B(\mathbf{R})$ such that for any $\epsilon > 0$, there is $K > 0$, such that if $n > K$, then $\|\varphi^{(n)} - \varphi\| < \epsilon$. Therefore $|\varphi(t) - \varphi^{(n)}(t)| < \epsilon$ and $|\varphi(t)| < |\varphi^{(n)}(t)| + \epsilon$. It follows that

$$\frac{1}{2t}\int_{-t}^{t} |\varphi(s)|ds < \frac{1}{2t}\int_{-t}^{t} |\varphi^{(n)}(t)|ds + \epsilon.$$

This implies $\varphi \in \mathcal{PAP}_0(\mathbf{R})$. Let $f = g + \varphi$. Then $f \in \mathcal{PAP}(\mathbf{R})$ and $\|f^{(n)} - f\| \to 0$ as $n \to \infty$. This completes the proof.

Proof of Theorem 1.2. For any $\phi \in \mathcal{PAP}(\mathbf{R})$, the following equation

$$\dot{x}(t) = ax(t) + bx([t]) + F(t, \varphi(t)), \qquad t \in \mathbf{R} \tag{2.6}$$

has a pseudo almost periodic solution $T\varphi$ by using Theorem 1.1. Thus, it follows that T is a mapping from $\mathcal{PAP}(\mathbf{R})$ into itself. For any $\phi, \psi \in \mathcal{PAP}(\mathbf{R})$, $T\phi - T\psi$ satisfies the following equation

$$\dot{\omega}(t) = a\omega(t) + b\omega([t]) + F(t, \phi(t)) - F(t, \psi(t)). \tag{2.7}$$

So we have

$$(T\phi)(n+1) - (T\psi)(n+1) = P((T\phi)(n) - (T\psi)(n)) + H_n, \tag{2.8}$$

where

$$H_n = \int_n^{n+1} e^{a(n+1-s)}(F(s, \phi(s)) - F(s, \psi(s))ds.$$

From the proof of Theorem 1.1, we know that $T\phi - T\psi$ is a unique pseudo almost periodic solution and

$$(T\phi)(n) - (T\psi)(n) = \sum_{i=-\infty}^{n-1} P^{n-i-1} H_n. \tag{2.9}$$

This implies

$$|(T\phi)(n) - (T\psi)(n)| \le \frac{1}{1-|P|} \sup_{n\in\mathbf{Z}} |H_n| \le \frac{e^{|a|}}{1-|P|} L\|\phi-\psi\|. \tag{2.10}$$

It follows that

$$|(T\phi)(t) - (T\psi)(t)| \le \Big(\frac{Me^{|a|}}{1-|P|} + e^{|a|}\Big) L\|\phi-\psi\|.$$

Since $L < \Big(\frac{Me^{|a|}}{1-|P|} + e^{|a|}\Big)^{-1}$, $T : \mathcal{PAP}(\mathbf{R}) \to \mathcal{PAP}(\mathbf{R})$ is a contract mapping. This implies that T has a unique fixed point in $\mathcal{PAP}(\mathbf{R})$. This completes the proof of Theorem 1.2.

Remark 2.1. If $f \in \mathcal{AP}(\mathbf{R})$ in Theroem 1.1, then (1.1) has a unique almost periodic solution.

Remark 2.2. If $|P| > 1$, then we can easily check that $C_n = -\sum\limits_{i=n}^{+\infty} P^{n-i-1} h_i$ is a pseudo almost periodic sequence solution of (1.4). Similarly one can discuss every problem mentioned above.

Acknowledgement. The authors thank the referee for his valuable suggestions.

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