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ON THE NONLINEAR TIMOSHENKO-KIRCHHOFF BEAM EQUATION

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Abstract

When an elastic string with fixed ends is subjected to transverse vibrations, its length varies with the time: this introduces changes of the tension in the string. This induced Kirchhoff to propose a nonlinear correction of the classical D'Alembert equation. Later on, Woinowsky-Krieger (Nash & Modeer) incorporated this correction in the classical Euler-Bernoulli equation for the beam (plate) with hinged ends.

Here a new equation for the small transverse vibrations of a simply supported beam is proposed. Such equation takes into account Kirchhoff's correction, as well as the correction for rotary inertia of the cross section of the beam and the influence of shearing strains, already present in the Timoshenko beam equation (cf. the Mindlin-Timoshenko equation for the plate). The model is inspired by a remark of Rayleigh, and by a joint paper with Panizzi & Paoli. It looks more complicated than the one proposed by Sapir & Reiss, but as a matter of fact it is easier to study, if a suitable change of variables is performed.

The author proves the local well-posedness of the initial-boundary value problem in Sobolev spaces of order ≥ 2.5. The technique is abstract, i.e. the equation is rewritten as a fourth order evolution equation in Hilbert space (thus the results could be applied also to the formally analogous equation for the plate).

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§0. Introduction

The main results of this paper were presented in [4].

Let us consider the transversal vibrations $u(x,t)(0 \leq x \leq L, t \geq 0)$ of a homogeneous beam. In the following, the letters ρ, E, G (resp.S, I, k) with denote the usual physical (resp. geometrical) parameters of the beam. More precisely, ρ :=volume density, E:= Young modulus of elasticity, G:=shear modulus, S:=area of the cross section, I:=moment of inertia of the cross section, $R^2 := IS^{-1}$, k is a positive number ≤ 1 which depends upon the geometry of the cross section (see [62, 20]), e.g. for rectangular cross section it is $k = 5/6$.

The classical Euler-Bernoulli Equation

$$c_2\partial_{xxxx}u + R^{-2}\partial_{tt}u = 0, \tag{0.1}$$

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where $c_2 := E\rho^{-1}$, is non realistic in that it presents the phenomenon of infinite propagation speed. The Rayleigh-Love Equation[48,36]

$$-(\partial_{tt} - c_2\partial_{xx})\partial_{xx}u + R^{-2}\partial_{tt}u = 0 \tag{0.2}$$

is more accurate than Equation (0.1), since it takes into account the effect of the rotary inertia of the cross-sections of the beam, and has a bounded phase velocity. As a matter of fact uniqueness lacks for Equation (0.2) (cf.[25]), so a fortiori Equation (0.2) too has an infinite propagation speed, in the sence that the supports of the initial data may expand themselves at an infinite speed.

If, in addition to rotary inertia, one considers the effect of shearing strains, then one gets the so-called Timoshenko Equation[56] (cf. [14,24,46]) (see [38,52,29] for historical notes)

$$(\partial_{tt} - c_2\partial_{xx})c_1^{-1}(\partial_{tt} - c_1\partial_{xx})u + R^{-2}\partial_{tt}u = 0, \tag{0.3}$$

where $c_1 := kG\rho^{-1}$. ρ, E, G, S, I, k are positive constants, and so are R, c_2 and c_1. It is always $c_1 < c_2$(see Remark 2.1 below). We note that the standard method of letting some parametres go to zero, apparently does not permit to re-obtain Equation (0.2) from Equation (0.3) (if e.g. we let $c_1 \to \infty$, then a fortiori $c_2 \to \infty$ too, so we may get to the limit only Equation (0.1). Therefore Equation (0.2) can be considered by no means as an approximation of Equation (0.3). For the derivation of Equation (0.3), we refer the reader e.g. to [56,31,1,20,62], or to [9].

Equation (0.3) has a finite propagation speed in all senses, and indeed experimental results show that it is a more realistic model than Equations (0.1),(0.2) (see [57,1,3,31] and also the discussion in [44]). Due to this fact, Equation (0.3) is a model currently used in Control Theory (cf.[49,29,52,26,21]).

Inhomogeneous versions of Equation (0.3) were studied in [9]. Here we want to deal with a "mild" quasi-linear version of Equation (0.3), describing transversal vibrations in the presence of a state of axial tension or compression, which is in fact a nonlinear functional of the vertical displacement u itself.

To be more precise, let us first examine how the presence of the axial tension T was included in Equations (0.1) and (0.2). According to B. de Saint-Venant, it was A. Clebsch[19] who first introduced this correction. Clebsch proposed the following equation:

$$-(\partial_{tt} - c_2\partial_{xx})\partial_{xx}u + R^{-2}(\partial_{tt} - c_0(T)\partial_{xx})u = 0, \tag{0.4}$$

where $c_0(T) := T\rho^{-1}$.

Later, Rayleigh[48] observed:

"When the bar, whose lateral vibrations are to be considered, is subject to longitudinal tension, the potential energy of any configuration is composed of two parts, the first depending on the stiffness by which the bending is directly opposed, and the second on the reaction against the extension, which is a necessary accompaniment of the bending, when the ends are nodes. The second part is similar to the potential energy of a deflected string; the first $[\cdots]$ is not entirely independent of the permanent tension."

Consequently, Rayleigh corrected Equation (0.4) as follows:

$$-(\partial_{tt} - c_2^*(T)\partial_{xx})\partial_{xx}u + R^{-2}(\partial_{tt} - c_0(T)\partial_{xx})u = 0, \tag{0.5}$$

where $c_2^*(T) := E^*(T)\rho^{-1}$, with $E^*(T) := E + T$. As a consequence, it is

$$c_2^*(T) = c_2 + c_0(T). \tag{0.6}$$

Following the variational appoach of K. Washizu[62], it seems reasonable (cf.[9]) to propose as a corrected version of Equation (0.3), in order to consider the effect of a tension T which possibly varies with time, the following one:

$$\begin{aligned}&(\partial_{tt} - c_2^*(T)\partial_{xx})(c_1^*(T))^{-1}(\partial_{tt} - (c_1^*(T) + c_0(T))\partial_{xx})u\\ &+ R^{-2}(\partial_{tt} - c_0(T)\partial_{xx})u = 0,\end{aligned} \tag{0.7}$$

where $c_1^*(T) := kG^*(T)\rho^{-1}$, with $G^*(T)$ denoting the shear modulus of the beam subjected to the axial tension T. If we apply the classical Cauchy's relations for an elastic body subjected to initial stresses (cf.[36]), it is possible to show that

the shear modulus $G^*(T)$ is independent of the axial tension T.

Therefore $c_1^*(T) = c_1$, so, taking into account (0.6), Equation (0.7) reduces to

$$(\partial_{tt} - (c_2 + c_0(T))\partial_{xx})(\partial_{tt} - (c_1 + c_0(T))\partial_{xx})u + a(\partial_{tt} - c_0(T)\partial_{xx})u = 0, \tag{0.3'}$$

where $a := c_1R^{-2} = kGS(\rho I)^{-1}$.

In a non-linear analysis, the tension T depends upon the unknown u. To perform a "mild" non-linear analysis, based on the assumption that some strains are very small, one can get the inspiration from Kirchhoff's [30] (cf.[28]) "mild" quasi-linear equation for the transversal vibration of the clamped string

$$\begin{cases}(\partial_{tt} - \gamma_0(u(\cdot,t))\partial_{xx})u = 0,\\ u = 0 \quad \text{for} \quad x = 0,\ x = L,\end{cases} \tag{0.8}$$

with

$$\gamma_0(v) := \Big(T_0 + E\,⨍(\partial_x v)^2dx/2\Big)\rho^{-1}, \tag{0.9}$$

where T_0 denotes the tension in the string in the rest position $u \equiv 0$, and $⨍(\cdot)dx$ denotes the mean value on $[0, L]$. Equation (0.8) takes into account the fact that the length of the string, and consequently the tension, will change during the motion. G. F. Carrier[17] and R. Narasimha[39] recovered Equation (0.8), without quoting Kirchhoff. For experiments see [45] (cf.[47]). From a purely mathematical point of view, Equation (0.8) was firstly studied by S. Bernstein[12]: assuming that $T_0 > 0$, he established local well-posedness in Sobolev spaces of any order ≥ 1.5, and global well-posedness in the space of analytic functions. For surveys on Equation (0.8) we refer the reader to [5, 54].

A "mild" quasi-linear version for the small transversal vibrations of the (0.1)-beam with hinged ends was proposed by S. Woinowsky- Krieger[63] and D. Burgreen[16] (use (0.9)):

$$\begin{cases}c_2\partial_{xxxx}u + R^{-2}(\partial_{tt} - \gamma_0(u(\cdot,t))\partial_{xx})u = 0,\\ u = \partial_{xx}u = 0 \quad \text{for} \quad x = 0, \quad x = L.\end{cases} \tag{0.10}$$

The above model is called the extensible (0.1)-beam. See also [13,23,28,43] and the references quoted there. From the mathematical point of view, the Cauchy problem for Equation (0.10) was studied first by [22, 10], and then in many other papers. An analogous model for the (0.2)-beam was studied, from the mathematical point of view, in [15].

Taking Equation (0.7) into account, we propose here the analogous "mild" quasi-linear version of Equation (0.3) for fixed ends. For simplicity's sake, we will consider only the case when $T_0 = 0$. Set

$$\begin{cases} \gamma_0(v) := E(2\rho)^{-1} ⨍ (\partial_x v)^2 dx, \\ \gamma_1(v) := c_1 + \gamma_0(v), \quad \gamma_2(v) := c_2 + \gamma_0(v), \end{cases} \tag{0.11}$$

and consider the equation

$$(\partial_{tt} - \gamma_2(u(\cdot,t))\partial_{xx})(\partial_{tt} - \gamma_1(u(\cdot,t))\partial_{xx})u + a(\partial_{tt} - \gamma_0(u(\cdot,t))\partial_{xx})u = 0, \tag{0.12}$$

where

$$\gamma_1(v) := c_1 + \gamma_0(v). \tag{0.13}$$

We will call Equation (0.12) the extensible (0.3)-beam (or Timoshenko- Kirchhoff beam equation). A model of this type was proposed by M. H. Sapir & E. L. Reiss [50, Equation A.17]. Their model looks similar to Equation (0.12) above, but with γ_1 replaced by $G\rho^{-1} + \gamma_0$ (i.e. formally $k = 1$). This would correspond to assuming that the shear stress is uniformly distributed over the cross section: "Unfortunately, things are not that simple"[20], and γ_2 replaced by c_2.

Equation (0.12) is more precise than that proposed in [50], since it is able to describe Rayleigh's correction to the characteristic speed c_2. In addition the surprising fact occurs that the equation is easier to handle with the term γ_2 in place of c_2.

The model (0.12) must also be compared with [18, 64, 42, 60] (which anyway are interested in particular solutions).

The aim of this paper is to study the well-posedness of the Cauchy problem for Equation (0.12) with the positions (0.11),(0.13), under the boundary conditions

$$u = \partial_{xx} u = 0 \ \text{ at } \ x = 0, \ \ x = L \ \ (\text{simply supported beam}). \tag{0.14}$$

More in general, we will study Equation (0.12) for a generic nonlinear relation stess-strain in the beam, i.e. when the tension in the beam is expressed by the formula $((\cdot)_x := \partial_x)$

$$T(u(\cdot,t)) := m\Big(⨍ u_x^2(x,t)dx \Big), \tag{0.15}$$

where $m(\cdot)$ is a generic continuously differentiable function (we remark that we need not assume that m is strictly increasing on its argument, so we are able to treat materials like low-carbon structural steel (see e.g.[40]), as well as possible new materials).

The constitutive relation (0.15) is non-local, but it is possible to treat Equation (0.12) in an abstract form.

Let H be a real Hilbert space, with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|_H$, and let $A : D(A) \subset H \to H$ be a self-adjoint positive definite operator. Set $\Box_{[r]} := \partial_{tt} + rA \ \ (r \in (-\infty, +\infty))$.

For $i = -1, 0, 1, 2$, let $m_i : [0, +\infty) \to (-\infty, +\infty)$ be continuously differentiable, and let functionals $\gamma_i : D(A) \to (-\infty, +\infty)$ be defined by $\gamma_i(v) := m_i((Av, v)) \ \ (v \in D(A))$.

Assume that $\gamma_{-1}(v) \neq 0 \ \ (v \in D(A))$, and that the following condition (of abstact hyperbolicity) holds true: $\gamma_2(v) \geq v > 0, \ \ \gamma_1(v) \geq v > 0 \ \ (v \in D(A))$. Consider the following equation

$$\big(\Box_{[\gamma_2(u)]}(\gamma_{-1}(u))^{-1}\Box_{[\gamma_1(u)]} + \Box_{[\gamma_0(u)]}\big)u = 0 \ \ (t > 0). \tag{0.16}$$

The abstract Timoshenko-Kirchhoff Equation (corresponding to Equation (0.12)) is obtained by imposing that $m_{-1}\equiv$constant $\neq 0, m_i\equiv c_i+m_0$ $(i=1,2)$, i.e., for $v\in D(A)$,

$$\gamma_{-1}(v):=\text{constant}\neq 0,\quad \gamma_0(v):=m_0((Av,v)),$$
$$\gamma_1(v):=c_1+\gamma_0(v),\quad \gamma_2(v):=c_2+\gamma_0(v).$$

The abstract problem is thus

$$\big(\Box_{[\gamma_2(u)]}\Box_{[\gamma_1(u)]}+\gamma_{-1}\Box_{[\gamma_0(u)]}\big)u=0\quad (t>0), \tag{0.12}'$$

which can be treated in a very simple way by exploiting the property $\gamma_2-\gamma_1\equiv c_2-c_1=$ constant, which implies that the operators $\Box_{[\gamma_2(v)]}$ and $\Box_{[\gamma_1(v)]}$ commute. If the abstract strict hyperbolicity condition $c_1\neq c_2$ is satisfied, then one gets (Theorem 1.1) the local well-posedness of the Cauchy problem for Equation (0.12)' in the phase space $D(A^{\alpha/2})\times D(A^{(\alpha-1)/2})\times D(A^{(\alpha-2)/2})\times D(A^{(\alpha-3)/2})$ for any $\alpha\geq 1.5$.

§1. Local Well-Posedness of the Abstract Cauchy Problem for Equation (0.12)′

Let H be a real Hilbert space, with scalar product $(\cdot,\cdot)$, and let $A:D(A)\subset H\to H$ be a self-adjoint positive definite operator. For $\beta\geq 0$, $D(A^\beta)$ denotes the domain of the β-th power of the operator A (see e.g. [55]); $D(A^\beta)$ is made a Hilbert space under the norm $\|u\|_{D(A^\beta)}:=\|A^\beta u\|_H$. For $\beta<0, D(A^\beta)$ denotes the dual space of $D(A^{-\beta})$, endowed with the dual norm. For $r\in(-\infty,+\infty)$, we set $\Box_{[r]}:=\partial_{tt}+rA$.

Theorem 1.1(Main Result). *Let $m_0:[0,+\infty)\to(-\infty,+\infty)$ be continuously differentiable, and for $c_1,c_2\in(-\infty,+\infty)$, let the functionals $\gamma_i:D(A)\to(-\infty,+\infty)$ $(i=-1,0,1,2)$ be defined by*

$$\begin{cases}\gamma_{-1}(v):=\text{constant}\neq 0,\quad \gamma_0(v):=m_0((Av,v)),\\ \gamma_1(v):=c_1+\gamma_0(v),\quad \gamma_2(v):=c_2+\gamma_0(v).\end{cases} \tag{1.1}$$

Assume that the following conditions (abstract strict hyperbolicity) are satisfied

$$\gamma_1(v)\geq v>0,\quad \gamma_2(v)\geq v>0\quad (v\in D(A)), \tag{1.2}$$

$$c_1\neq c_2. \tag{1.3}$$

Then, for any $\alpha\geq 1.5$ the Cauchy problem

$$\begin{cases}\Box_{[\gamma_2(u)]}\Box_{[\gamma_1(u)]}u+\gamma_{-1}\Box_{[\gamma_0(u)]}u=0\quad (t>0),\\ u(0)=u_0,\quad \partial_t u(0)=u_1,\quad \partial_{tt}u(0)=u_2,\quad \partial_{ttt}u(0)=u_3\end{cases} \tag{1.4}$$

is uniquely solvable(*) in the phase space

$$D(A^{\alpha/2})\times D(A^{(\alpha-1)/2})\times D(A^{(\alpha-2)/2})\times D(A^{(\alpha-3)/2}). \tag{1.5}$$

Remark 1.1 Assumption(1.2) may be replaced by the weaker one:

$$\gamma_1(u_0)>0,\quad \gamma_2(u_0)>0. \tag{1.6}$$

Proof of Theorem 1.1 and Remark 1.1. We write down the proof in such a way that it could be possibly generalized to the more general situation of Equation (0.16). Following

(*)This means that for any choice of the initial data in the phase space (1.5), there exists $T>0$ such that the Cauchy problem (1.4) admits a unique solution u in $\bigcap_{j=0}^{3}C^j([0,T);D(A^{\alpha-j)/2}))$.

[9], we introduce the new variables

$$v_1 := (\gamma_{-1})^{-1}\Box_{[\gamma_2(u)]}u, \qquad v_2 := (\gamma_{-1})^{-1}\Box_{[\gamma_1(u)]}u, \tag{1.7}$$

and set

$$\gamma :\equiv (\gamma_{-1})^{-1}(\gamma_2 - \gamma_1). \tag{1.8}$$

By (1.1), we know that

$$\gamma \equiv \text{ constant}, \tag{1.9}$$

and by the strict hyperbolicity assumption (1.3), it is

$$\gamma \neq 0. \tag{1.10}$$

We can thus introduce the two numbers

$$\lambda :\equiv \gamma^{-1}(\gamma_2 - \gamma_0), \quad \mu :\equiv -\gamma^{-1}(\gamma_1 - \gamma_0) \tag{1.11}$$

(i.e. $\lambda = \gamma^{-1}c_2$ and $\mu = -\gamma^{-1}c_1$), for which

$$\lambda + \mu \equiv \gamma_{-1}, \quad \lambda\gamma_1 + \mu\gamma_2 \equiv \gamma_{-1}\gamma_0. \tag{1.12}$$

On the other hand, if we denote by $[\,\cdot\,,\,\cdot\,]$ the commutator, then we have

$$\gamma_{-1}[(\gamma_{-1})^{-1}\Box_{[\gamma_1(u)]}, (\gamma_{-1})^{-1}\Box_{[\gamma_2(u)]}] = \partial_t(\gamma(u))A\partial_t + \partial_t(\partial_t(\gamma(u))A\cdot). \tag{1.13}$$

By (1.12) and (1.13), the new variables v_i must satisfy the second order system

$$\begin{cases} \Box_{[\gamma_1(u)]}v_1 + \mu v_1 + \lambda v_2 - \partial_t(\gamma(u))A\partial_t u - \partial_t(\partial_t(\gamma(u))Au) = 0 & (t > 0), \\ \Box_{[\gamma_2(u)]}v_2 + \mu v_1 + \lambda v_2 = 0 & (t > 0). \end{cases} \tag{1.14}$$

By (1.9), the system (1.14) reduces to

$$\begin{cases} \Box_{[\gamma_1(u)]}v_1 + \mu v_1 + \lambda v_2 = 0 & (t > 0), \\ \Box_{[\gamma_2(u)]}v_2 + \mu v_1 + \lambda v_2 = 0 & (t > 0). \end{cases} \tag{1.15}$$

Let α be any number ≥ 1.5. By (1.10), we can define the isomorphism

$$S : D(A^{(\alpha-2)/2}) \to D(A^{\alpha/2}), \quad S : w \mapsto (\gamma A)^{-1}w. \tag{1.16}$$

By positions (1.7), (1.8) and (1.16), we can express

$$u = S(v_1 - v_2). \tag{1.17}$$

We claim that the local well-posedness of the Cauchy problem (1.4) in the phase space (1.5) is equivalent to the local well-posedness of the Cauchy problem for the system (1.15) in the phase space

$$(D(A^{(\alpha-2)/2}) \times D(A^{(\alpha-3)/2}))^2. \tag{1.18}$$

Indeed, by (1.7), one gets

$$\partial_{tt}u = \gamma^{-1}(\gamma_2(u)v_2 - \gamma_1(u)v_1) = \gamma^{-1}(\gamma_2(S(v_1 - v_2))v_2 - \gamma_1(S(v_1 - v_2))v_1), \tag{1.19}$$

and if $((v_1, \partial_t v_1), \ (v_2, \partial_t v_2))$ is a continuous function of the time into the phase space (1.18), then $\gamma_i(S(v_1 - v_2))$ is a C^1 function of the time $(i = 1, 2)$.

If we put $\mathcal{V} := (v_1, v_2)$, then the system (1.15) reads as

$$\partial_{tt}\mathcal{V} + \mathsf{g}(\mathcal{V})\mathcal{A}\mathcal{V} + \mathcal{B}_1\mathcal{V} = 0 \ \ (t > 0), \tag{1.20}$$

where

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \mathsf{g}(\mathcal{V}) := \begin{pmatrix} g_1(\mathcal{V}) & 0 \\ 0 & g_2(\mathcal{V}) \end{pmatrix}, \tag{1.21}$$

with

$$g_j(\mathcal{V}) := \gamma_j(S(v_1 - v_2)), \quad (j = 1, 2), \quad \mathcal{V} = (v_1, v_2), \tag{1.22}$$

and

$$\mathcal{B}_1 := \begin{pmatrix} \mu & \lambda \\ \mu & \lambda \end{pmatrix}.$$

If $\psi : [0, +\infty) \to (-\infty, +\infty)$ is any function such that $|m_0(r_1) - m_0(r_2)| \le \psi(R)|r_1 - r_2|$ $(r_1, r_2 \le R)$, and if c denotes any positive constant such that $(Av, v) \ge c^2\|v\|_H^2$ $(v \in D(A))$, then for $j = 1, 2$ the following bound holds [6]:

$$|g_j(\mathcal{V}_1) - g_j(\mathcal{V}_2)| \le 2\gamma^{-1}(2Rc)^{1/2}\psi(R) \cdot \|\mathcal{V}_1 - \mathcal{V}_2\|_{D(\mathcal{A}^{-3/4})}$$
$$(\|\mathcal{V}_i\|_{D(\mathcal{A}^{-1/4})} \le \gamma(Rc/2)^{1/2}, \quad i = 1, 2).$$

Now we have

Theorem 1.2. *Let* $\beta \in (-\infty, +\infty)$*, and let* $g : D(A^{\beta/2}) \to (-\infty, +\infty)$ *satisfy the condition*

$$\begin{cases} \exists \varphi : [0, +\infty) \to (-\infty, +\infty) \ \ \text{such that} \ \forall \ \|v_i\|_{D(A^{\beta/2})} \le R \ \ (i = 1, 2), \\ |g(v_1) - g(v_2)| \le \varphi(R)\|v_1 - v_2\|_{D(A^{(\beta-1)/2})}. \end{cases} \tag{1.23}$$

For $u_0 \in D(A^{\beta/2})$ *and* $u_1 \in D(A^{(\beta-1)/2})$*, we consider the Cauchy problem*

$$\begin{cases} \partial_{tt}u + g(u)Au = 0 \ \ (t > 0), \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases} \tag{1.24}$$

Assume that the following condition holds true (*local abstract strict hyperbolicity*) : $g(u_0) > 0$. *Then, for any* $\alpha \ge \beta$*, the Cauchy problem* (1.24) *is uniquely solvable in the phase space* $D(A^{\alpha/2}) \times D(A^{(\alpha-1)/2})$.

Theorem 1.2 above generalizes the result of Theorem 1 of [6]. The proof is omitted for reasons of space, however it is similar to the one of [6]: it consists in finding the solution as a fixed point in a set $C^0([0, T]; B\text{-weak})$, B being a convenient ball in the space $D(A^{\beta/2}) \times D(A^{(\beta-1)/2})$, by application of the contraction principle with respect to the supremum norm in $D(A^{(\beta-1)/2}) \times D(A^{(\beta-2)/2})$.

Thanks to the hyperbolicity condition (1.2) (or (1.6)), it is easy to check that the technique may be adapted to solve the Cauchy problem for Equation (1.20). In this way one gets, for any $\alpha \ge 3/2$, the local well-posedness of the Cauchy problem for Equation (1.20) in the phase space

$$D(\mathcal{A}^{(\alpha-2)/2}) \times D(\mathcal{A}^{(\alpha-3)/2}), \tag{1.25}$$

i.e. the local well-posedness of the Cauchy problem for system (1.15) in the phase space (1.18), which in turn is equivalent to the thesis.

Remark 1.2. The same proof of Theorem 1.1 may be easily adapted to treat the more general case of Equation (1.4) when the functionals $\gamma_i : D(A^{\beta/2}) \to (-\infty, +\infty)$ $(i = -1, 0, 1, 2)$ are not of the type (1.1), but the functional γ, defined by (1.8), still satisfies (1.9). In this case the strict hyperbolicity condition (1.3) must be replaced by (1.10). Then if the γ_i's satisfy (1.2) and (1.23), the Cauchy problem (1.4) is uniquely solvable in the phase space (1.5) for any $\alpha \ge \beta$.

Remark 1.3. In the proof of Theorem 1.1 we changed variable and then linearized the resulting second order equation. In a different (more standard) approach, one can study the fourth order linearized equation, and then achieve the solution as a fixed point of a contraction map. This is done in [7] (cf.[8]), however under stronger assumptions (m is assumed to be thrice differentiable, and $\alpha \geq 2.5$).

§2. Applications to the Concrete Timoshenko-Kirchhoff Beam Equation (0.12)

We apply the abstract theory of §1 to the concrete Equation (0.12), with the positions (0.11), (0.13), under the boundary condition (0.14). We remember that it describes the transversal vibrations of a simply supported beam under the assumption that the shear modulus G of the beam is not affected by changes in the axial tension.

Theorem 2.1. *let L, ρ, E, G, k be positive constants. Assume that*

$$kG < E. \tag{2.1}$$

Set

$$\gamma_0(v) := E(2\rho)^{-1} \fint v_x^2(x)dx,$$
$$\gamma_2(v) := E\rho^{-1} + \gamma_0(v), \quad \gamma_1(v) := kG\rho^{-1} + \gamma_0(v).$$

Let $a \in (-\infty, +\infty)$, and consider the initial-boundary value problem

$$\begin{cases} (\partial_{tt} - \gamma_2(u(\cdot,t))\partial_{xx})(\partial_{tt} - \gamma_1(u(\cdot,t))\partial_{xx})u \\ \quad +a(\partial_{tt} - \gamma_0(u(\cdot,t))\partial_{xx})u = 0 \quad (0 < x < L, \quad t > 0), \\ u = \partial_{xx}u = 0 \quad at \quad x = 0, \quad x = L \quad for \quad each \quad t \geq 0, \\ \partial_t^j u(x,0) = u_j(x) \quad (0 \leq x \leq L, \quad j = 0, \cdots, 3), \end{cases} \tag{2.2}$$

and, for $\beta \in (-\infty, +\infty)$, denote $q(\beta) :=$ largest integer $\leq (\beta - 1/2)/2$.

Let us assume that, for some $\alpha \geq 2.5$, the initial data u_j satisfy ($j = 0, \cdots, 3$) the conditions$^{(**)}$

$u_j \in H^{\alpha-j}((0, L))$ *(:=Sobolev space of exponent 2 and derivative order $\alpha - j$)*

$$u_j = \partial_x^2 u_j = \partial_x^4 u_j = \cdots = \partial_x^{2q(\alpha-j)} u_j = 0 \quad \text{at} \quad x = 0, \quad x = L.$$

Then for convenient $T > 0$ problem (2.2) admits a unique weak solution

$$u \in \bigcap_{j=0}^{2\alpha} C^j([0, T); H^{\alpha-j}((0, L))) \subset C^{p(\alpha)}([0, L] \times [0, T)), \tag{2.3}$$

($p(\alpha) :=$ largest integer $< \alpha - 1/2$), which satisfies$^{(**)}$ $(0 \leq j \leq \alpha - 1/2)$

$$\partial_t^j u = \partial_t^j \partial_x^2 u = \cdots = \partial_t^j \partial_x^{2q(\alpha-j)} u = 0 \quad \text{at} \quad x = 0, \quad x = L \quad (t \geq 0). \tag{2.4}$$

Remark 2.1. In the case of beams: it is always $k \leq 1$ (see [20]), and on the other hand for the Young modulus E and the shear modulus G we have $G < E$, therefore the inequality in (2.1) is always satisfied.

$^{(**)}$if $\alpha - j - 1/2 \in 2\mathbf{N}$, the function $\partial_x^{2q(\alpha-j)} u_j$ belongs merely to $H^{1/2}((0, L))$, and so it may happen that it is discontinuous. In that case, the condion of vanishing at the ends of the interval must be interpreted in the weaker sense that $d^{-1/2} \cdot \partial_x^{2q(\alpha-j)} u_j$ belongs to $L^2(0, L)$, where $d(\cdot)$ denotes the distance from the set $\{0, L\}$. In particular, in the limit case $\alpha = 2.5$, the boundary condition "$\partial_{xx}u = 0$ at $x = 0, L$" must be interpreted in this weaker sense.

Remark 2.2. (i) In Theorem 1.1 we can allow α to be any number ≥ 1.5, but in that case the solution does not fulfill the boundary condition "$\partial_{xx}u = 0$ at $x = 0, x = L$" in any sense: the solution is a "mild" one.

(ii) If $\alpha > 4.5$, by (2.3) the solution is classical.

Remark 2.3. Let us compare Theorem 2.1 with the result of M. Tucsnak[58]. He treated a boundary-initial value problem for the quasilinear version of the system (0.9) of [9], with ($G^* = G$, formally $k = 1$ and) $E^* = E$, which is equivalent to Equation (A.17) of [50], under the boundary condition (0.14). However the well-posedness for those two problems are not equivalent: actually if the Cauchy problem for the equation of [50] is solved by passing through the system in [58], one derivative is lost from the initial data to the solution (actually, the system is not sensible to the information whether strict hyperbolicity occurs or not).

Therefore Theorem 2.1 above is in any case independent from the result of [58].

Proof of Theorem 2.1 and Remark 2.2. Let us make the positions

$$H := L^2(0,L), \quad D(A) := \{u \in H^2((0,L)) : u(0) = u(L) = 0\}, \quad A := -L^{-1}\partial_{xx},$$
$$\gamma^{-1} = a, \quad m_0(r) := Er(2\rho)^{-1}, \quad c_2 := E\rho^{-1}, \quad c_1 := kG\rho^{-1}.$$

Then condition (2.1) implies (1.3). Moreover, the initial data belong to the phase space (1.5). So from Theorem 1.1 we get a unique weak solution $u \in \bigcap_{j=0}^{3} C^j([0,T); H^{\alpha-j}((0,L)))$ which satisfies condition (2.4) for $j = 0, \cdots, 3$. In particular, since $\alpha \geq 2.5$, for $j = 0$, we get $u = \partial_{xx}u = 0$ at $x = 0, \quad x = L \quad (t \geq 0)^{(**)}$, which is the boundary condition in (2.2).

The further regularity stated in (2.3) and (2.4) follows directly by differentiating the equation in (2.2) (note that in the present case $m_0 \in C^\infty$), thanks to the Sobolev imbedding theorem.

§3. The Kirchhoff Correction to the Mindlin-Timoshenko Plate

The analogous of Equation (0.3) for a plate, which takes into account both the effects of rotary inertia and transverse shear, and which is currently used in applications in Control Theory (cf. [32, 35]), is the so-called Mildlin-Timoshenko Equation[59,37]

$$d_1^{-1}(\partial_{tt} - d_2\triangle_x) \cdot (\partial_{tt} - d_1\triangle_x)u + K^{-2}\partial_{tt}u = 0, \tag{3.1}$$

where $K^2 := h^2/12, \quad d_1 := \tau G\rho^{-1}$, and $d_2 := (1-\sigma^2)^{-1}E\rho^{-1}$. The letters σ, τ (resp. h) denote the physical (resp. geometrical) parameters of the plate. More precisely, σ :=Poisson's ratio, τ is Mindlin's constant which depends in a nonlinear fashion upon σ (however, $\tau \approx 0.76 + 0.3\sigma$ for $0 \leq \sigma \leq 0.5$), h := thickness of the plate. For simplicity's sake, we assume that for $u \equiv 0$ the plate is tension-free.

The analogous of Equation (0.10) for the small transverse vibrations of the simply supported plate (the extensible plate) was proposed by W. A. Nash & J. R. Modeer[41], and by T. Wah[61] as a dynamic analogue of H. M. Berger's Equation [11]

$$\begin{cases} d_2\triangle_x^2 u + K^{-2}(\partial_{tt} - \chi_0(u(\cdot,t))\triangle_x)u = 0, \\ u = \triangle_x u = 0 \text{ at the boundary of the plate } (t \geq 0), \end{cases} \tag{3.2}$$

where

$$\chi_0(v) := E(2\rho)^{-1}(1-\sigma^2)^{-1} \fint |\nabla_x v|^2 dx.$$

See [43] and the references quoted there, and also [2]. Let us set

$$\chi_1(v) := d_1 + \chi_0(v), \quad \chi_2(v) := d_2 + \chi_0(v).$$

The <u>formal</u> analogue of Equation (0.12) for the plate reads as follows:

$$(\partial_{tt} - \chi_2(u(\cdot,t))\triangle_x)(\partial_{tt} - \chi_1(u(\cdot,t))\triangle_x)u + a(\partial_{tt} - \chi_0(u(\cdot,t))\triangle_x)u = 0, \tag{3.3}$$

where $a := d_1 K^{-2} = 12\tau G h^{-2}\rho^{-1}$.

We may call Equation (3.3) the extensible (3.1)-plate (or Mildlin- Timoshenko-Kirchhoff Equation). Equation (3.3) must be compared with [65, 53, 27, 33, 51, 66] (which anyway are interested in particular solutions). Now, Equation (3.3) may still be written in the abstract form (0.12), and so Theorem 1.1 may be applied as well, yielding a result analogous to Theorem 2.1.

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