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TWO-DIMENSIONAL APPROXIMATION OF EIGENVALUE PROBLEMS IN SHELL THEORY: FLEXURAL SHELLS

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Abstract

The eigenvalue problem for a thin linearly elastic shell, of thickness 2ϵ, clamped along its lateral surface is considered. Under the geometric assumption on the middle surface of the shell that the space of inextensional displacements is non-trivial, the authors obtain, as $\epsilon \to 0$, the eigenvalue problem for the two-dimensional "flexural shell" model if the dimension of the space is infinite. If the space is finite dimensional, the limits of the eigenvalues could belong to the spectra of both flexural and membrane shells. The method consists of rescaling the variables and studying the problem over a fixed domain. The principal difficulty lies in obtaining suitable a priori estimates for the scaled eigenvalues.

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§1. Introduction

In this paper, we study the limiting behaviour of eigenvalues and eigenfunctions describing the vibrations of a thin linearly elastic shell, clamped along its lateral surface, under a geometric assumption on the middle surface of the shell that the space of inextensional displacements (cf. (4.2)) is non zero. In the stationary case, under additional assumptions on the order of magnitude of the body forces, this leads to the two-dimensional model of the "flexural shell" as shown by Ciarlet, Lods and Miara[5].

Examples of clamped shells which obey the above geometric condition, thus leading to the flexural shell model are plates or, more generally, shells which are 'flat' in some region (cf. Remark 4.1 below). Also if the middle surface of the shell is a cylinder and the shell is clamped on a portion of the lateral surface, the middle line of which is contained in a generatrix of the cylinder, the above geometric condition holds. The results of this paper, though proved for shells clamped along the entire lateral surface, hold for the partially clamped case as well.

Our procedure to study the corresponding eigenvalue problem is the standard one. Starting with the three-dimensional eigenvalue problem (corresponding to the one studied by Ciarlet, Lods and Miara[5] in the stationary case), we rescale the variables and obtain a

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problem posed over a fixed domain where the parameter ϵ (corresponding to the thickness of the shell and the dimension of the three-dimensional domain over which the reference configuration of the shell is defined) now appears in the various bilinear forms. We can then pass to the limit after obtaining suitable a priori estimates.

The key to making this procedure work lies in obtaining the suitable a priori estimates. This is where the principal mathematical contribution of this paper lies. It must be observed that in previous works (cf. [3,5]) the membrane and flexural models were obtained based on two assumptions. First, the nature of the space of inextensional displacements and second, the orders of magnitude of the body forces. If the forces were of order $O(1)$ and the middle surface of the shell is "uniformly elliptic" in the sense that the two principal radii of curvature are either both > 0 or both < 0 at all points of the middle surface, then the above-mentioned space reduces to zero and the membrane shell model was obtained in the limit. If the space was non-trivial and the forces were of order $O(\epsilon^2)$, the flexural shell model was obtained in the limit.

In our case, we do not have the body forces and so we cannot make any extra assumption on their sizes. So how does the shell decide on its limiting behaviour vis-a-vis its vibrations, on the basis of the nature of the space of inextensional displacements? We show in this paper that if the space is infinte-dimensional, then the eigenvalues (at each level l, $l = 0, 1, 2, \cdots$) are of the order $O(\epsilon^2)$, by considering suitable test functions to be used in the variational characterization of the eigenvalues, and that the corresponding scaled eigensolutions converge to the eigensolutions of the two-dimensional flexural shell problem. We also show using the techniques of Ciarlet and Kesavan[2], that all the eigensolutions of the two-dimensional problem are obtained this way. If the space is of finite dimension, say N, then we show that the first N eigenvalues are of order $O(\epsilon^2)$ and the corresponding scaled eigensolutions of the three-dimensional problem converge to the N eigensolutions of the flexural shell model and that either the other eigenfunctions converge weakly to zero in $(H^1(\Omega))^2 \times L^2(\Omega)$ or that the eigensolutions converge to those of the two-dimensional eigenvalue problem for membrane shells.

As in the case of the shallow shell, there will be a difference of a factor of 2 (2ϵ after descaling) between the coefficients obtained here and those obtained on passing to the limit in stationary problems. This is natural and has been discussed in [7]. The difference can be reconciled if, in the stationary model, the modified forces are the means of the body forces over the interval $[-1, 1]$ (resp; $[-\epsilon, \epsilon]$ in the descaled version) rather than just the integrals.

This paper is organized as follows. Section 2 describes the pricipal notations and the formulation in curvilinear co-ordinates, of the three-dimensional problem and its scaled version over a fixed domain. In Section 3, we study the rescaled problem and Section 4 is devoted to the derivation of suitable a priori bounds which will be needed to pass to the limit. In Section 5, we study the limit problem and Section 6 is devoted to concluding remarks.

§2. Statement of the Problem

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz continuous boundary γ, such that the domain lies locally on one side of its boundary. Let $y = (y_\alpha)$ denote a generic point in ω. (Greek indices will vary on the set {1,2} and the Latin indices will vary on {1,2,3}. The summation convention will be used for repeated indices in conjunction with the above-mentioned rule.) Let $\partial_\alpha = \frac{\partial}{\partial y_\alpha}$. Let $\phi : \overline{\omega} \to I\!R^3$ be an injective mapping of class $\mathcal{C}^3$ such that the two vectors

$$\boldsymbol{a}_\alpha(y) = \partial_\alpha \phi(y)$$

are linearly independent vectors for all $y \in \overline{\omega}$, thus forming a covariant basis of the tangent plane to the surface

$$S = \phi(\overline{\omega})$$

at the point $\phi(y)$. The dual basis (contravariant basis) is denoted by $\boldsymbol{a}^\alpha(y)$. We define

$$\boldsymbol{a}^3(y) = \boldsymbol{a}_3(y) = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}.$$

Then we can define

$$\begin{cases} a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, & a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta, \\ b_{\alpha\beta} := a^3 \cdot \partial_\beta \boldsymbol{a}_\alpha, & b_\alpha^\beta := a^{\beta\sigma} b_{\sigma\alpha}, \\ \Gamma_{\alpha\beta}^\sigma := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha \end{cases} \tag{2.1}$$

in covariant, contravariant or mixed components as the case may be. These verify the usual symmetry relations. We also define

$$b_\beta^\sigma|_\alpha = \partial_\alpha b_\beta^\sigma - \Gamma_{\alpha\tau}^\sigma b_\beta^\tau - \Gamma_{\beta\alpha}^\tau b_\tau^\sigma, \tag{2.2}$$

$$c_{\alpha\beta} = b_\alpha^\sigma b_{\sigma\beta}. \tag{2.3}$$

The area element along S is $\sqrt{a}dy$, where

$$a := \det(a_{\alpha\beta}). \tag{2.4}$$

By the continuity of the functions defined above, there exists $a_0 > 0$ such that

$$0 < a_0 \le a(y) \quad \text{for all} \quad y \in \overline{\omega}. \tag{2.5}$$

Given $\epsilon > 0$, we define the sets

$$\Omega^\epsilon = \omega \times (-\epsilon, \epsilon), \quad \Gamma_\pm^\epsilon = \omega \times \{\pm\epsilon\}, \quad \Gamma_0^\epsilon = \gamma \times [-\epsilon, \epsilon], \tag{2.6}$$

where $\Gamma_+^\epsilon \cup \Gamma_-^\epsilon \cup \Gamma_0^\epsilon$ defines a partition of the boundary of Ω^ϵ and Γ_0^ϵ is the lateral surface. Let $x^\epsilon = (x_i^\epsilon)$ denote a generic point in $\overline{\Omega}^\epsilon$ and set $\partial_i^\epsilon = \frac{\partial}{\partial x_i^\epsilon}$. Thus $x_\alpha^\epsilon = y_\alpha$ and so $\partial_\alpha^\epsilon = \partial_\alpha$.

Define $\Phi : \overline{\Omega}^\epsilon \to R^3$ by

$$\Phi(x^\epsilon) = \phi(y) + x_3^\epsilon a^3(y) \quad \text{for all} \quad x^\epsilon = (y, x_3^\epsilon) \in \overline{\Omega}^\epsilon. \tag{2.7}$$

It can be shown that, for sufficiently small ϵ, the vectors

$$g_i^\epsilon(x^\epsilon) = \partial_i^\epsilon \Phi(x^\epsilon)$$

are linearly independent at all points $x^\epsilon \in \overline{\Omega}^\epsilon$ and that the mapping Φ is injective. These vectors form a covariant basis of the tangent space of $\Phi(\Omega^\epsilon)$ (which is $I\!R^3$) at $\Phi(x^\epsilon)$ and one can, as usual, define the contravariant basis $\{g^{i,\epsilon}(x^\epsilon)\}$ by duality. The covariant and contravariant metric tensors are given respectively by

$$g_{ij}^\epsilon = g_i^\epsilon \cdot g_j^\epsilon \quad \text{and} \quad g^{ij,\epsilon} = g^{i,\epsilon} \cdot g^{j,\epsilon}. \tag{2.8}$$

The Christoffel symbols are defined by

$$\Gamma_{ij}^{p,\epsilon} = g^{p,\epsilon} \cdot \partial_i^\epsilon g_j^\epsilon. \tag{2.9}$$

The volume element is now given by $\sqrt{g^\epsilon}dx$ on $\Phi(\Omega^\epsilon)$, where

$$g^\epsilon = \det(g_{ij}^\epsilon). \tag{2.10}$$

It can be shown that, for sufficiently small ϵ,

$$0 < g_0 \leq g^\epsilon \leq g_1, \tag{2.11}$$

where g_0 and g_1 are constants independent of ϵ.

The set $\Phi(\overline{\Omega}^\epsilon)$ is the reference configuration of a shell of thickness 2ϵ with middle surface $\phi(\overline{\omega})$. We asume that the shell is clamped along its lateral surface Γ_0^ϵ.

Assuming that the material of the shell is homogenious and isotropic and that $\Phi(\overline{\Omega}^\epsilon)$ is natural state, the material is characterized by its Lamé constants $\lambda^\epsilon > 0$ and $\mu^\epsilon > 0$. Then the contravariant components of the three-dimensional elasticity tensor are given by

$$A^{ijkl,\epsilon} = \lambda^\epsilon g^{ij,\epsilon} g^{kl,\epsilon} + \mu^\epsilon (g^{ik,\epsilon} g^{jl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon}). \tag{2.12}$$

Expressed in terms of the curvilinear co-ordinates (x^ϵ) of the reference configuration $\Phi(\overline{\Omega}^\epsilon)$ of the shell, we define the space of admissible displacements by

$$\boldsymbol{V}(\Omega^\epsilon) = \{\boldsymbol{v}^\epsilon = (v_i^\epsilon) \in H^1(\Omega^\epsilon) | \boldsymbol{v}^\epsilon = 0 \text{ on } \Gamma_0^\epsilon\}. \tag{2.13}$$

For a displacement vector $\boldsymbol{v}^\epsilon \in \boldsymbol{V}(\Omega^\epsilon)$, we define the covariant components of the linearized strain tensor by

$$e_{i||j}^\epsilon(\boldsymbol{v}^\epsilon) = \frac{1}{2}(\partial_i^\epsilon v_j^\epsilon + \partial_j^\epsilon v_i^\epsilon) - \Gamma_{ij}^{p,\epsilon} v_p^\epsilon. \tag{2.14}$$

Then the eigenvalue problem consists in finding pairs $(\xi^\epsilon, \boldsymbol{u}^\epsilon) \in I\!R \times \boldsymbol{V}(\Omega^\epsilon) \backslash \{0\}$ such that

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e_{k||l}^\epsilon(\boldsymbol{u}^\epsilon) e_{i||j}^\epsilon(\boldsymbol{v}^\epsilon) \sqrt{g^\epsilon} dx^\epsilon = \xi^\epsilon \int_{\Omega^\epsilon} u_i^\epsilon v_i^\epsilon \sqrt{g^\epsilon} dx^\epsilon \tag{2.15}$$

for all $\boldsymbol{v}^\epsilon \in \boldsymbol{V}(\Omega^\epsilon)$. By classical arguments, we can show that there exists a sequence of eigenvalues

$$0 < \xi^{\epsilon,1} \leq \xi^{\epsilon,2} \leq \cdots \leq \xi^{\epsilon,l} \leq \cdots \to \infty \tag{2.16}$$

and we can choose a corresponding family of eigenfunctions $\{\boldsymbol{u}^{\epsilon,l}\}$ such that

$$\int_{\Omega^\epsilon} u_i^{\epsilon,l} u_i^{\epsilon,m} \sqrt{g^\epsilon} dx^\epsilon = \delta_{lm}. \tag{2.17}$$

The sequence $\{\boldsymbol{u}^{\epsilon,l}\}$ forms an orthonormal basis in the weighted space

$$(L^2(g_\epsilon; \Omega^\epsilon))^3 = \{\boldsymbol{u}^\epsilon | \int_{\Omega^\epsilon} u_i^\epsilon u_i^\epsilon \sqrt{g^\epsilon} dx^\epsilon < \infty\} \tag{2.18}$$

with the obvious inner-product. (However, in view of the inequalities (2.11), it follows that $(L^2(g_\epsilon; \Omega^\epsilon))^3 = (L^2(\Omega^\epsilon))^3$ and that the two topologies are equivalent.)

§3. The Rescaled Problem

We now scale this problem to one posed over a domain independent of ϵ. We set

$$\Omega = \omega \times (-1, 1), \quad \Gamma_\pm = \omega \times \{\pm 1\}, \quad \Gamma_0 = \gamma \times [-1, 1]. \tag{3.1}$$

If $x = (x_i) \in \Omega$ is a generic point, we set $\partial_i = \frac{\partial}{\partial x_i}$ and with $x^\epsilon = (x_i^\epsilon) \in \overline{\Omega}^\epsilon$, we associate $x \in \overline{\Omega}$ by

$$x_\alpha = x_\alpha^\epsilon = y_\alpha, \ x_3 = \frac{1}{\epsilon} x_3^\epsilon. \tag{3.2}$$

Thus, $\partial_\alpha^\epsilon = \partial_\alpha$ and $\partial_3^\epsilon = \frac{1}{\epsilon}\partial_3$.

Given a vector $\boldsymbol{v}^\epsilon \in \boldsymbol{V}(\Omega^\epsilon)$, we associate the vector $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ where

$$\boldsymbol{V}(\Omega) = \{\boldsymbol{v} \in (H^1(\omega))^3 | \boldsymbol{v} = 0 \text{ on } \Gamma_0\} \tag{3.3}$$

by

$$v_i(x) = v_i^\epsilon(x^\epsilon), \tag{3.4}$$

where x and x^ϵ have the correspondence mentioned above. Given an eigenvector $\boldsymbol{u}^{\epsilon,l}$, we denote the corresponding vector obtained via (3.4) by $\boldsymbol{u}^l(\epsilon)$. We assume further that the material properties of the shell do not depend on the thickness, and so we set

$$\lambda^\epsilon = \lambda > 0, \quad \mu^\epsilon = \mu > 0 \tag{3.5}$$

where λ and μ are indepedent of ϵ.

Finally, given an eigenvalue $\xi^{\epsilon,l}$, we associate with it the "scaled" eigenvalue $\xi^l(\epsilon)$ by

$$\xi^{\epsilon,l} = \epsilon^2 \xi^l(\epsilon). \tag{3.6}$$

Remark 3.1. In the case of the shallow shell, the horizontal and vertical components of the vectors were scaled differently. In the present case we have uniform treatment of all components. For the Lamé constants and eigenvalues, we have the same scaling as for shallow shells.

Based on these scalings, we can study the asymptotic behaviour of the functions

$$\begin{cases} \Gamma_{ij}^p(\epsilon)(x) = \Gamma_{ij}^{p,\epsilon}(x^\epsilon), \\ g(\epsilon)(x) = g^\epsilon(x^\epsilon), \\ A^{ijkl}(\epsilon)(x) = A^{ijkl,\epsilon}(x^\epsilon). \end{cases} \tag{3.7}$$

These have all been derived by Ciarlet and Lods[3]. We will not list these results here but we will use them as and when needed.

Given $(\boldsymbol{v}) = (v_i) \in (H^1(\Omega))^3$, we associate the symmetric tensor $(e_{i||j}(\epsilon)(\boldsymbol{v}))$ by

$$\begin{cases} e_{\alpha||\beta}(\epsilon)(\boldsymbol{v}) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma_{\alpha\beta}^\sigma(\epsilon) v_\sigma, \\ e_{\alpha||3}(\epsilon)(\boldsymbol{v}) = \frac{1}{2}(\partial_\alpha v_3 + \frac{1}{\epsilon}\partial_3 v_\alpha) - \Gamma_{\alpha 3}^\sigma(\epsilon) v_\sigma, \\ e_{3||3}(\epsilon)(\boldsymbol{v}) = \frac{1}{\epsilon}\partial_3 v_3. \end{cases} \tag{3.8}$$

Then if $(\xi^\epsilon, \boldsymbol{u}^\epsilon) \in I\!R \times \boldsymbol{V}(\Omega^\epsilon)\backslash\{0\}$ is a solution of (2.15), the scaled variables $(\xi(\epsilon), \boldsymbol{u}(\epsilon)) \in I\!R \times \boldsymbol{V}(\Omega)\backslash\{0\}$ is a solution of the problem

$$\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(\boldsymbol{u}(\epsilon)) e_{i||j}(\epsilon)(\boldsymbol{v})\sqrt{g(\epsilon)}dx = \epsilon^2 \xi(\epsilon) \int_\Omega u_i(\epsilon) v_i \sqrt{g(\epsilon)}dx \tag{3.9}$$

for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$. Once again, it is clear that $\{\xi^l(\epsilon)\}$ corresponding to $\xi^{l,\epsilon}$ via (3.6) are the only eigenvalues of (3.9) and that the corresponding eigenvectors $\{\boldsymbol{u}^l(\epsilon)\}$ are complete in $(L^2(\Omega))^3$ and satisfy the orthogonality conditions

$$\int_\Omega u_i^l(\epsilon) u_i^m(\epsilon)\sqrt{g(\epsilon)}dx = \delta_{lm}. \tag{3.10}$$

Further, we have the following variational characterization of the eigenvalues.

Define the Rayleigh quotient $R(\epsilon)(\boldsymbol{v})$ for $\boldsymbol{v} \in \boldsymbol{V}(\Omega)\backslash\{0\}$ by

$$R(\epsilon)(\boldsymbol{v}) = \frac{\int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\epsilon)(\boldsymbol{v}) e_{i||j}(\epsilon)(\boldsymbol{v})\sqrt{g(\epsilon)}dx}{\epsilon^2 \int_\Omega v_i v_i \sqrt{g(\epsilon)}dx}. \tag{3.11}$$

Then

$$\xi^l(\epsilon) = \min_{\boldsymbol{W} \in \mathcal{V}_l} \max_{\boldsymbol{v} \in \boldsymbol{W}\backslash\{0\}} R(\epsilon)(\boldsymbol{v}), \tag{3.12}$$

where $\mathcal{V}_l$ is the collection of all l-dimensional subspaces of $\boldsymbol{V}(\Omega)$.

§4. A Priori Estimates

In this section, we will show that if the space $V_F(\omega)$ is infinite dimensional, then the scaled eigenvalues $\xi^l(\epsilon)$ are bounded uniformly with respect to ϵ for each fixed positive integer l. If the dimension of $V_F(\omega)$ is finite, say N, then we show that for $1 \leq l \leq N$, $\xi^l(\epsilon)$ is uniformly bounded with respect to ϵ, and for $l > N$, we will show that $\{\epsilon^2 \xi^l(\epsilon)\}$ is bounded uniformly with respect to ϵ and that all the limits for $l > N$ lie in a bounded subset of $\mathbb{R}$.

We henceforth denote by C, a generic constant which is independent of both ϵ and l but whose value differs from place to place.

First of all, we need to define the space of inextensional displacements. Following earlier works (cf. [3, 5]), we define for $\boldsymbol{v} \in (H^1(\Omega))^3$

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta} v_\sigma - b_{\alpha\beta} v_3. \tag{4.1}$$

Then the space of inextensional displacements $V_F(\omega)$ is given by

$$V_F(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in (H^1_0(\omega))^2 \times H^2_0(\omega) | \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}. \tag{4.2}$$

As observed by Ciarlet and Lods[3], Ciarlet, Miara and Lods[5] and Sanchez-Palencia[8], this space may or may not be trivial.

Assumption. We assume henceforth that $V_F(\omega) \neq 0$.

Remark 4.1. It is not clear whether the space $V_F(\omega)$ is always infinite dimensional if it is non-zero. For example, in the case of plates, $b_{\alpha\beta} = 0$ and $V_F(\omega) = \{0\} \times \{0\} \times H^2_0(\omega)$. More generally if $\omega' \subset \omega$ is a subdomain on which the shell is "flat", i.e., $b_{\alpha\beta}|_{\omega'} = 0$, then the space $0 \times 0 \times H^2_0(\omega') \subset V_F(\omega)$, and so $V_F(\omega)$ is infinite dimensional.

We now introduce another important tensor which will play a central role in all that follows.

For $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, we define

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{v}) = {} & \partial_{\alpha\beta} v_3 - \Gamma^\sigma_{\alpha\beta} \partial_\sigma v_3 + b^\sigma_\beta(\partial_\alpha v_\sigma - \Gamma^\tau_{\alpha\sigma} v_\tau) \\ & + b^\sigma_\alpha(\partial_\beta v_\sigma - \Gamma^\tau_{\beta\sigma} v_\tau) + b^\sigma_\alpha|_\beta v_\sigma - c_{\alpha\beta} v_3.\end{aligned} \tag{4.3}$$

A result due to Bernadou and Ciarlet[1] states that there exists a constant $C > 0$ such that

$$\Big(\sum_{\alpha,\beta} ||\gamma_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega} + \sum_{\alpha,\beta} ||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega}\Big)^{\frac{1}{2}} \geq C\Big(\sum_\alpha ||\eta_\alpha||^2_{1,\omega} + ||\eta_3||^2_{2,\omega}\Big)^{\frac{1}{2}} \tag{4.4}$$

for all $\boldsymbol{\eta} = (\eta_i) \in (H^1_0(\omega))^2 \times H^2_0(\omega)$. In particular,

$$\Big(\sum_{\alpha,\beta} ||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega}\Big)^{\frac{1}{2}} \tag{4.5}$$

will be a norm on the space $V_F(\omega)$ equivalent to the $(H^1_0(\omega))^2 \times H^2_0(\omega)$ norm.

If $w \in \mathcal{C}^0(\overline{\Omega})$, we let

$$||w||_{0,\infty,\overline{\Omega}} = \sup\{|w(x)| : x \in \overline{\Omega}\}.$$

Let $\boldsymbol{\eta} = (\eta_i) \in (H^1_0(\omega))^2 \times H^2_0(\omega)$. Then we define, following an idea of Miara and Sanchez-Palencia, $v_\epsilon(\boldsymbol{\eta}) \in V(\Omega)$ by

$$(v_\epsilon(\boldsymbol{\eta}))_\alpha = \eta_\alpha - \epsilon x_3(\partial_\alpha \eta_3 + 2 b^\sigma_\alpha \eta_\sigma), \tag{4.6}$$

$$(v_\epsilon(\boldsymbol{\eta}))_3 = \eta_3. \tag{4.7}$$

For brevity, we will set

$$\theta_\alpha = \partial_\alpha \eta_3 + 2b_\alpha^\sigma \eta_\sigma. \tag{4.8}$$

With these notations we have the following result.

Lemma 4.1. *Let $\boldsymbol{\eta} \in V_F(\omega)$. Then*

$$\epsilon^{-1} e_{\alpha||\beta}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) \to -x_3\rho_{\alpha\beta}(\boldsymbol{\eta}) \text{ in } L^2(\Omega) \text{ as } \epsilon \to 0, \tag{4.9}$$

$$\epsilon^{-1} e_{\alpha||3}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) \text{ is bounded in } L^2(\Omega), \tag{4.10}$$

$$e_{3||3}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) = 0 \text{ for all } \epsilon > 0, \tag{4.11}$$

$$v_\epsilon(\boldsymbol{\eta}) \to \boldsymbol{\eta} \text{ in } V(\Omega) \text{ as } \epsilon \to 0. \tag{4.12}$$

Proof. Relations (4.11) and (4.12) are obvious.

A simple computation shows that

$$\epsilon^{-1} e_{\alpha||3}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) = -\epsilon^{-1}(\Gamma_{\alpha 3}^\sigma(\epsilon) + b_\alpha^\sigma)\eta_\sigma + x_3\Gamma_{\alpha 3}^\sigma(\epsilon)\theta_\sigma. \tag{4.13}$$

Combining with the relation (cf. [5])

$$||\Gamma_{\alpha 3}^\sigma(\epsilon) + b_\alpha^\sigma||_{0,\infty,\overline{\Omega}} \leq C\epsilon \tag{4.14}$$

proves (4.10).

We finally prove (4.9). To start with, by Lemma 3.1 of [5], we have that for $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$

$$||\epsilon^{-1} e_{\alpha||\beta}(\epsilon)(\boldsymbol{v}) - e_{\alpha||\beta}^1(\epsilon)(\boldsymbol{v})||_{0,\Omega} \leq C\epsilon \sum_\alpha ||v_\alpha||_{0,\Omega}, \tag{4.15}$$

where

$$e_{\alpha||\beta}^1(\epsilon)(\boldsymbol{v}) = \epsilon^{-1}\gamma_{\alpha\beta}(\boldsymbol{v}) + x_3 b_\beta^\sigma|_\alpha v_\sigma + x_3 c_{\alpha\beta} v_3. \tag{4.16}$$

Observing that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$, we find (after a tedious computation) that

$$e_{\alpha||\beta}^1(\epsilon)(v_\epsilon(\boldsymbol{\eta})) = -x_3\rho_{\alpha\beta}(\boldsymbol{\eta}) - \epsilon x_3^2 b_\beta^\sigma|_\alpha \theta_\sigma. \tag{4.17}$$

Thus from (4.15) and (4.17) and the definition of $v_\epsilon(\boldsymbol{\eta})$ given by (4.6)–(4.7), we get

$$||\epsilon^{-1} e_{\alpha||\beta}(\epsilon)(v_\epsilon(\boldsymbol{\eta})) + x_3\rho_{\alpha\beta}(\boldsymbol{\eta})||_{0,\Omega} \leq C\epsilon(||\eta_\alpha||_{0,\omega} + ||\eta_3||_{1,\omega}), \tag{4.18}$$

which proves (4.9).

Theorem 4.1. *Assume that $V_F(\omega)$ is an infinite dimensional subspace of $V(\Omega)$. Then for each $l \geq 1$, the sequence $\xi^l(\epsilon)$ is bounded uniformly with respect to ϵ.*

Proof. Let $\mathcal{W}_l$ denote the collection of all l-dimensional subspaces of $V_F(\omega)$.

Consider the map

$T_\epsilon : V_F(\omega) \to \boldsymbol{V}(\Omega)$ defined by

$$T_\epsilon(\boldsymbol{\eta}) = v_\epsilon(\boldsymbol{\eta}). \tag{4.19}$$

For sufficiently small ϵ, T_ϵ is one-one. Thus if $W \in \mathcal{W}_l$, then $T_\epsilon(W) \in \mathcal{V}_l$. Consequently, we have

$$\xi^l(\epsilon) \leq \min_{\boldsymbol{W}\in\mathcal{W}_l} \max_{\boldsymbol{\eta}\in\boldsymbol{W}\setminus\{0\}} R_\epsilon(v_\epsilon(\boldsymbol{\eta})). \tag{4.20}$$

We now proceed to estimate $R_\epsilon(v_\epsilon(\boldsymbol{\eta}))$ for $\boldsymbol{\eta} \in V_F(\omega)$. On one hand

$$\int_\Omega (v_\epsilon(\boldsymbol{\eta}))_i (v_\epsilon(\boldsymbol{\eta}))_i \sqrt{g(\epsilon)} dx \geq g_0 \int_\Omega (v_\epsilon(\boldsymbol{\eta}))_i (v_\epsilon(\boldsymbol{\eta}))_i dx \tag{4.21}$$

$$= 2g_0 \int_\omega \eta_3^2 d\omega + g_0 \sum_\alpha \int_\Omega (\eta_\alpha - \epsilon x_3\theta_\alpha)^2 dx. \tag{4.22}$$

Since $\int_\Omega x_3\eta_\alpha\theta_\alpha dx = 0$, we get

$$\int_\Omega (v_\epsilon(\boldsymbol{\eta}))_i(v_\epsilon(\boldsymbol{\eta}))_i\sqrt{g(\epsilon)}dx \geq 2g_0\int_\omega \eta_i\eta_i d\omega. \tag{4.23}$$

On the other hand, we have

$$\begin{aligned}&\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))e_{i||j}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\sqrt{g(\epsilon)}dx\\ &\leq g_1^{\frac12}\Big\{\int_\Omega A^{\alpha\beta\sigma\tau}(\epsilon)\Big[\frac{1}{\epsilon}e_{\sigma\tau}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\Big]\Big[\frac{1}{\epsilon}e_{\alpha||\beta}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\Big]dx\\ &\quad+4\int_\Omega A^{\alpha3\sigma3}(\epsilon)\Big[\frac{1}{\epsilon}e_{\sigma||3}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\Big]\Big[\frac{1}{\epsilon}e_{\alpha||3}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\Big]dx\Big\},\end{aligned} \tag{4.24}$$

using the symmetries of $A^{ijkl}(\epsilon)$, the fact that $A^{\alpha\beta\sigma3}(\epsilon) = A^{\alpha333}(\epsilon) = 0$ and the relations (4.11) and (2.11). By virtue of the relation (cf. [5])

$$||A^{ijkl}(\epsilon)||_{0,\infty,\overline{\Omega}} \leq C, \tag{4.25}$$

relations (4.9)–(4.12) of Lemma 4.1 above and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))e_{i||j}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\sqrt{g(\epsilon)}dx\\ &\leq C\Big[\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\boldsymbol{\eta})||_{0,\omega}+\epsilon\Big(\sum_\alpha||\eta_\alpha||_{0,\omega}+||\eta_3||_{1,\omega}\Big)\Big]^2+\Big(\sum_\alpha||\eta_\alpha||_{0,\omega}+||\eta_3||_{1,\omega}\Big)^2\\ &\leq C\Big[\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega}+\sum_\alpha||\eta_\alpha||^2_{0,\omega}+||\eta_3||^2_{1,\omega}\Big]\end{aligned} \tag{4.26}$$

for $\epsilon \leq 1$. But from (4.4) it follows that, since $\boldsymbol{\eta} \in V_F(\omega)$,

$$\Big(\sum_\alpha||\eta_\alpha||^2_{0,\omega}+||\eta_3||^2_{1,\omega}\Big) \leq C\Big(\sum_\alpha||\eta_\alpha||^2_{1,\omega}+||\eta_3||^2_{2,\omega}\Big) \leq C\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega}. \tag{4.27}$$

Thus

$$\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))e_{i||j}(\epsilon)(v_\epsilon(\boldsymbol{\eta}))\sqrt{g(\epsilon)}dx \leq C\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega}. \tag{4.28}$$

It follows from (4.23) and (4.28) that

$$R_\epsilon(v_\epsilon(\boldsymbol{\eta})) \leq C\frac{\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{0,\omega}}{\sum_i||\eta_i||^2_{0,\omega}}. \tag{4.29}$$

Let us define the two-dimensional elasticity tensor $a^{\alpha\beta\sigma\tau}$ by

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau}+2\mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}). \tag{4.30}$$

Then it is known that (cf. [1]) there exists $C > 0$ such that

$$\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})\rho_{\sigma\tau}(\boldsymbol{\eta})\sqrt{a}dy \geq C\sum_\alpha||\rho_{\alpha\beta}(\boldsymbol{\eta})||^2_{o,\omega} \tag{4.31}$$

for all $\boldsymbol{\eta} \in V_F(\omega)$. Thus, we have

$$R_\epsilon(v_\epsilon(\boldsymbol{\eta})) \leq C\frac{\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})\rho_{\sigma\tau}(\boldsymbol{\eta})\sqrt{a}d\omega}{\int_\omega \eta_i\eta_i\sqrt{a}d\omega} \tag{4.32}$$

and hence, from (4.20) and (4.32), it follows that

$$\xi^l(\epsilon) \leq C\Lambda^l, \tag{4.33}$$

where Λ^l is the l-th eigenvalue of the two-dimensional problem:

Find $(\Lambda, \boldsymbol{\zeta}) \in I\!R \times V_F(\omega)\backslash\{0\}$ such that

$$\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}d\omega = \Lambda\int_\omega \eta_i\zeta_i\sqrt{a}d\omega \tag{4.34}$$

for all $\boldsymbol{\eta} \in V_F(\omega)$. This completes the proof.

Theorem 4.2. *Assume that* $\dim(V_F(\omega)) = N$. *Then for* $1 \leq l \leq N, \xi^l(\epsilon)$ *is uniformly bounded with respect to* ϵ *and for each positive integer* $l > N$, *there exists constants* C *(independent of* l *and* ϵ*) and* k^l *(independent of* ϵ*) such that* $\epsilon^2\xi^l(\epsilon) \leq C(1+\epsilon^2 k^l)$.

Proof. The proof that for $1 \leq l \leq N, \xi^l(\epsilon)$ is bounded uniformly with respect to ϵ follows from Theorem 4.1.

Let W_l denote the collection of all l-dimensional subspaces of $H_0^2(\omega)$.

For $\eta \in W_l$, define $w_\epsilon(\eta) \in \boldsymbol{V}(\Omega)$ by

$$(w_\epsilon(\eta))_\alpha = -\epsilon x_3\partial_\alpha\eta, \tag{4.35}$$

$$(w_\epsilon(\eta))_3 = \eta. \tag{4.36}$$

Then a simple computation shows that

$$e_{\alpha||\beta}(\epsilon)(w_\epsilon(\eta)) = -\epsilon x_3(\partial_{\alpha\beta}\eta + \Gamma^\sigma_{\alpha\beta}(\epsilon)\partial_\sigma\eta) - \Gamma^3_{\alpha\beta}(\epsilon)\eta, \tag{4.37}$$

$$e_{\alpha||3}(\epsilon)(w_\epsilon(\eta)) = -\epsilon x_3\Gamma^\sigma_{\alpha 3}\partial_\alpha\eta, \tag{4.38}$$

$$e_{3||3}(w_\epsilon(\eta)) = 0. \tag{4.39}$$

For $W \in W_l$, define

$$\boldsymbol{W} = \{w_\epsilon(\eta) : \eta \in W\}. \tag{4.40}$$

Then $\boldsymbol{W} \in \mathcal{V}_l$ and hence it follows from (3.12) that

$$\xi^l(\epsilon) \leq \min_{W\in W_l}\max_{\eta\in W\backslash\{0\}} R_\epsilon(w_\epsilon(\eta)). \tag{4.41}$$

We now proceed to calculate $R_\epsilon(w_\epsilon(\eta))$. On one hand

$$\int_\Omega (w_\epsilon(\eta))_i(w_\epsilon(\eta))_i\sqrt{g(\epsilon)}dx \geq g_0\int_\Omega (w_\epsilon(\eta))_i(w_\epsilon(\eta))_i dx \geq 2g_0\int_\omega \eta^2 dx. \tag{4.42}$$

On the other hand, we have

$$\begin{aligned}&\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(w_\epsilon(\eta))e_{i||j}(\epsilon)(w_\epsilon(\eta))\sqrt{g(\epsilon)}dx\\ &\leq g_1^{\frac{1}{2}}\Big\{\int_\Omega A^{\alpha\beta\sigma\tau}(\epsilon)[e_{\sigma||\tau}(\epsilon)(w_\epsilon(\eta))][e_{\alpha||\beta}(\epsilon)(w_\epsilon(\eta))]dx\\ &\quad + 4\int_\Omega A^{\alpha 3\sigma 3}[e_{\sigma||3}(\epsilon)(w_\epsilon(\eta))][e_{\alpha||3}(\epsilon)(w_\epsilon(\eta))]dx\Big\},\end{aligned} \tag{4.43}$$

using the symmetries of $A^{ijkl}(\epsilon)$, the fact that $A^{\alpha 3\sigma 3}(\epsilon) = A^{\alpha 333}(\epsilon) = 0$, relations (4.39) and (2.11). By virtue of relations (4.25),(4.37)–(4.38), and the Cauchy-Schwarz inequality, we get

$$\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(w_\epsilon(\eta))e_{i||j}(\epsilon)(w_\epsilon(\eta))\sqrt{g(\epsilon)}dx$$
$$\leq C\Big[\epsilon\Big(\sum_{\alpha,\beta}||\partial_{\alpha\beta}\eta||_{0,\omega}+\sum_{\alpha}||\partial_\alpha\eta||_{0,\omega}\Big)+||\eta||_{0,\omega}\Big]^2+C\epsilon^2\sum_\alpha||\partial_\alpha\eta||^2_{0,\omega}$$
$$\leq C\Big[\epsilon^2\sum_{\alpha,\beta}||\partial_{\alpha\beta}\eta||^2_{0,\omega}+||\eta||^2_{0,\omega}\Big]. \tag{4.44}$$

It follows from (4.42) and (4.44) that

$$R_\epsilon(w_\epsilon(\eta))\leq C\frac{\Big(\epsilon^2\sum\limits_{\alpha,\beta}||\partial_{\alpha\beta}\eta||^2_{0,\omega}+||\eta||^2_{0,\omega}\Big)}{\epsilon^2||\eta||^2_{0,\omega}}. \tag{4.45}$$

Hence from (4.41) and (4.45), it follows that

$$\epsilon^2\xi^l(\epsilon)\leq C(\epsilon^2k^l+1), \tag{4.46}$$

where k^l is the l-th eigenvalue of the two-dimensional problem:

Find $(k,\zeta)\in I\!R\times H_0^2(\omega)$ such that

$$\begin{cases}\Delta^2\zeta=k\zeta \quad \text{in} \quad \omega,\\ \zeta=0 \quad \text{on} \quad \partial\omega.\end{cases} \tag{4.47}$$

This completes the proof.

§5. Limit Problem

In this section we show that if the space $V_F(\omega)$ is infinite dimensional, then for each fixed integer $l\geq 1$, the scaled eigensolution $(\xi^l(\epsilon),\boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ converges towards a limit $(\xi^l,\boldsymbol{u}^l)$ which can be identified with the eigensolution of the two-dimensional "flexural shell" problem posed over the set ω. If the dimension of the space $V_F(\omega)$ is finite, say N, then we will show that the first N scaled eigensolutions converge to the N eigensolutions of the two-dimensional "flexural shell" problem and the other eigensolutions either converge to the solution of the two-dimensional "membrane shell" problem or the eigenvectors converge weakly to zero in $(H^1(\Omega))^2\times L^2(\Omega)$.

The next three lemmas are crucial; they play an important role in the proof of the convergence of the scaled unknowns as $\epsilon\to 0$.

Lemma 5.1. *Let $\boldsymbol{V}(\Omega)$ be the space defined in* (3.3) *and the functions $e_{i||j}(\epsilon)(\boldsymbol{v})\in L^2(\Omega)$, $\gamma_{\alpha\beta}(\boldsymbol{v})\in L^2(\Omega),\rho_{\alpha\beta}(\boldsymbol{v})\in H^{-1}(\Omega)$ be defined for any function $\boldsymbol{v}\in\boldsymbol{V}(\Omega)$ as in* (3.8), (4.1) *and* (4.3). *Let $(\boldsymbol{v}(\epsilon))_{\epsilon>0}$ be a sequence of functions in $\boldsymbol{V}(\Omega)$ such that*

$$\boldsymbol{v}(\epsilon)\rightharpoonup\boldsymbol{v} \quad \textit{weakly in} \quad H^1(\Omega), \tag{5.1}$$
$$\frac{1}{\epsilon}e_{i||j}(\epsilon)(\boldsymbol{v}(\epsilon))\rightharpoonup e^1_{i||j} \quad \textit{weakly in} \quad L^2(\Omega) \tag{5.2}$$

as $\epsilon\to 0$. Then

$$\boldsymbol{v}=(v_i) \textit{ is independent of the transverse variable } x_3\ , \tag{5.3}$$
$$\overline{\boldsymbol{v}}=(\overline{v_i})=\frac{1}{2}\int_{-1}^1 vdx_3\in(H_0^1(\omega))^2\times H_0^2(\omega) \textit{ and} \tag{5.4}$$
$$\gamma_{\alpha\beta}(\boldsymbol{v})=0. \tag{5.5}$$

Proof. See the proof of Lemma 3.3 of [5].

The key to the convergence theorem (Theorem 5.4) is the generalized Korn's inequality (5.6), which involves the functions $e_{i||j}(\epsilon)(\boldsymbol{v})$ defined in (3.8) instead of the traditional function $e_{ij}(\boldsymbol{v})$. This generalized Korn's inequality is valid for an arbitrary surface $S=\phi(\omega)$ (the only requirements are that the set ω and the mapping ϕ satisfy the assumptions of Section 2), irrespective of whether the space $V_F(\omega)$ defined in (4.2) reduces to zero or not.

Lemma 5.2. *Let the space $\boldsymbol{V}(\Omega)$ be defined as in* (3.3). *Then There exists* $0<\epsilon\le\epsilon_0$ *and* $C>0$ *such that for all* $0<\epsilon\le\epsilon_0$

$$||\boldsymbol{v}||_{1,\Omega}\le\frac{C}{\epsilon}\Big(\sum_{i,j}||e_{i||j}(\epsilon)(\boldsymbol{v})||^2_{0,\Omega}\Big)^{\frac{1}{2}}\quad \textit{for all}\quad \boldsymbol{v}\in\boldsymbol{V}(\Omega), \tag{5.6}$$

where the tensor $(e_{i||j}(\epsilon)(\boldsymbol{v}))$ *is defined as in* (3.8).

Proof. See the proof of Theorem 4.1 of [5].

Lemma 5.3. *There exists a constant* ϵ_1 *such that for all* $0<\epsilon\le\epsilon_1$,

$$\Big\{\sum_{\alpha}||v_\alpha||^2_{1,\Omega}+||\epsilon v_3||^2_{1,\Omega}\Big\}\le C\Big\{\sum_{i,j}||e_{i||j}(\epsilon)(\boldsymbol{v})||^2_{0,\Omega}+\sum_i||v_i||^2_{0,\Omega}\Big\} \tag{5.7}$$

for all $\boldsymbol{v}=(v_i)\in(H^1(\Omega))^3$.

Proof. Given $\boldsymbol{v}=(v_i)\in(H^1(\Omega))^3$, let $\boldsymbol{v}(\epsilon)=(v_1,v_2,\epsilon v_3)\in(H^1(\Omega))^3$. Then

$$e_{\alpha\beta}(\boldsymbol{v}(\epsilon))=e_{\alpha||\beta}(\epsilon)(\boldsymbol{v})+\Gamma^p_{\alpha\beta}(\epsilon)v_p, \tag{5.8}$$

$$e_{\alpha3}(\boldsymbol{v}(\epsilon))=\epsilon e_{\alpha||3}(\epsilon)(\boldsymbol{v})+\epsilon\Gamma^\sigma_{\alpha3}(\epsilon)v_\sigma, \tag{5.9}$$

$$e_{33}(\boldsymbol{v}(\epsilon))=\epsilon^2e_{3||3}(\epsilon)(\boldsymbol{v}), \tag{5.10}$$

where $e_{ij}(\boldsymbol{v})=\frac{1}{2}(\partial_jv_i+\partial_iv_j)$, and consequently by virtue of the relation (cf. [3])

$$||\Gamma^\sigma_{\alpha\beta}(\epsilon)-\Gamma^\sigma_{\alpha\beta}||_{0,\infty,\overline{\Omega}}+||\Gamma^3_{\alpha\beta}(\epsilon)-b_{\alpha\beta}||_{0,\infty,\overline{\Omega}}+||\Gamma^\sigma_{\alpha3}(\epsilon)+b^\sigma_\alpha||_{0,\infty,\overline{\Omega}}\le C\epsilon, \tag{5.11}$$

we get

$$\Big\{\sum_{i,j}||e_{ij}(\boldsymbol{v}(\epsilon))||^2_{0,\Omega}\Big\}\le C\Big\{\sum_{i,j}||e_{i||j}(\epsilon)(\boldsymbol{v})||^2_{0,\Omega}+\sum_i||v_i||^2_{0,\Omega}\Big\} \tag{5.12}$$

for $\epsilon\le1$. By the classical Korn's inequality,

$$||\boldsymbol{v}(\epsilon)||^2_{1,\Omega}=\sum_\alpha||v_\alpha||^2_{1,\Omega}+||\epsilon v_3||^2_{1,\Omega}\le C\Big\{\sum_{i,j}||e_{ij}(\boldsymbol{v}(\epsilon))||^2_{0,\Omega}+||\boldsymbol{v}(\epsilon)||^2_{0,\Omega}\Big\} \tag{5.13}$$

and the lemma follows from inequalities (5.12) and (5.13).

Theorem 5.1. *Assume that the space $V_F(\omega)$ is infinite dimensional. Then*

(a) *For each integer* $l\ge1$, *there exists a subsequence* (*still indexed by* ϵ *for convenience*) *such that* $(\xi^l(\epsilon),\boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ *converges strongly in* $I\!R\times H^1(\Omega)$ *to* $(\xi^l,\boldsymbol{u}^l)$; *further* $\boldsymbol{u}^l$ *is independent of the transverse variable* x_3 *and* $\overline{\boldsymbol{u}}^l\in V_F(\omega)$.

(b) *The pair* $(\xi^l,\overline{\boldsymbol{u}}^l)$ *solves the two-dimensional eigenvalue problem for the flexural shell, viz, find* $(\xi,\boldsymbol{\zeta})\in I\!R\times V_F(\omega)\backslash\{0\}$ *such that*

$$\frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy=\xi\int_\omega\zeta_i\eta_i\sqrt{a}dy\quad \textit{for all}\quad \boldsymbol{\eta}=\eta_i\in V_F(\omega), \tag{5.14}$$

where $a^{\alpha\beta\sigma\tau}$ *and* $\rho_{\alpha\beta}(\boldsymbol{v})$ *are defined as in* (4.30) *and* (4.3).

Proof. The proof is divided into several steps.

Step 1. Boundedness of the eigenvectors in $H^1(\Omega)$:

From the variational equation (3.9), relation (3.10), inequality (2.11), the boundedness of the eigenvalues $\xi^l(\epsilon)$, the generalized Korn's inequality (5.6) and by virtue of relation (cf. [5])

$$A^{ijkl}(\epsilon)t_{kl}t_{ij} \geq Ct_{ij}t_{ij} \tag{5.15}$$

for all symmetric tensors (t_{ij}), we infer that

$$\begin{aligned}\epsilon^2 C^{-2}||\boldsymbol{u}^l(\epsilon)||^2_{1,\Omega} &\leq \sum_{i,j} ||e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon))||^2_{0,\Omega}\\ &\leq Cg_0^{-\frac{1}{2}} \int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\boldsymbol{u}^l(\epsilon))e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon))\sqrt{g(\epsilon)}dx\\ &= \epsilon^2 Cg_0^{-\frac{1}{2}}\xi^l(\epsilon)\int_\Omega |\boldsymbol{u}^l(\epsilon)|^2\sqrt{g(\epsilon)}dx\\ &= \epsilon^2 Cg_0^{-\frac{1}{2}}\Lambda^l. \end{aligned} \tag{5.16}$$

Hence the assertion follows.

Step 2. It follows from Step 1 that $\boldsymbol{u}^l(\epsilon) \rightharpoonup \boldsymbol{u}^l$ weakly in $H^1(\Omega)$ (hence strongly in $L^2(\Omega)$) and $\frac{1}{\epsilon}e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon)) \rightharpoonup e^{1,l}_{i||j}$ weakly in $L^2(\Omega)$. Hence it follows from Lemma (4.3) that $\boldsymbol{u}^l$ is independent of x_3 and $\gamma_{\alpha\beta}(\boldsymbol{u}^l) = 0$, *ie*, $\overline{\boldsymbol{u}}^l \in V_F(\omega)$.

Step 3. The limit functions $e^{1,l}_{i||j}$ are related to the limit function $\boldsymbol{u}^l$ by

$$-\partial_3 e^{1,l}_{\alpha||\beta} = \rho_{\alpha\beta}(\boldsymbol{u}^l), \tag{5.17}$$

$$e^{1,l}_{\alpha||3} = 0, \tag{5.18}$$

$$e^{1,l}_{3||3} = \frac{-\lambda}{\lambda+2\mu}a^{\alpha\beta}e^{1,l}_{\alpha||\beta}. \tag{5.19}$$

(The argument is as in [5]. On the right-hand side of the relevant equation, we have $\xi^l(\epsilon)\boldsymbol{u}^l(\epsilon)$ which replaces the forces $f_i(\epsilon)$. All that is needed to pass to the limit is the boundedness of these functions in $H^1(\Omega)$ which we have.)

Step 4. Taking $\boldsymbol{v}$ in Equation (3.9) of the form $(\eta_\alpha - \epsilon x_3\theta_\alpha, \eta_3)$, where $\theta_\alpha = \partial_\alpha\eta_3 + 2b^\sigma_\alpha\eta_\sigma$ with $\boldsymbol{\eta} = (\eta_i) \in V_F(\omega)$ and passing to the limit in Equation (3.9) and taking into account of the relation (5.17)–(5.19), it follows that $(\xi^l, \overline{\boldsymbol{u}}^l)$ satisfies Equation (5.14).

Step 5. The strong convergence of $(\boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ to $\boldsymbol{u}^l$ in $H^1(\Omega)$ follows once again as in [5].

Though we have proved that each subsequence $(\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon))_{\epsilon>0}, l \geq 1$, strongly converges in $\mathbb{R}\times H^1(\Omega)$ to a solution $(\xi^l, \overline{\boldsymbol{u}}^l)$ of the two-dimensional eigenvalue problem for the flexural shells, nothing tells us so far whether ξ^l is precisely the l-th eigenvalue (counting multiplicities) of (5.14), nor whether the set $(\overline{\boldsymbol{u}}^l)^\infty_{l=1}$ forms a complete set in the space $V_F(\omega)$. We shall answer these questions in the affirmative in the next lemma using the ideas developed by Kesavan[6] and Ciarlet and Kesavan[2].

Lemma 5.4. *Let* $(\xi^l, \overline{\boldsymbol{u}}^l), l \geq 1$, *be the eigensolutions of Problem* (5.14) *found as limits of the subsequence* $(\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon))_{\epsilon>0}, l \geq 1$ *of eigensolutions, orthonormalized as in* (3.10) *of Problem* (3.9). *Then the sequence* $(\xi^l)^\infty_{l=1}$ *comprises all the eigenvalues, counting multiplicities, of Problem* (5.14) *and the associated sequence* $(\overline{\boldsymbol{u}}^l)^\infty_{l=1}$ *of eigenfunctions forms a complete orthonormal set in the space* $V_F(\omega)$.

Proof. Passing to the limit in the orthogonality relation (3.10), we get

$$\int_\omega \overline{u}^l_i\overline{u}^m_i\sqrt{a}d\omega = \frac{1}{2}\delta_{lm}. \tag{5.20}$$

We first show that

$$0 < \xi^1 \leq \xi^2 \leq \cdots \leq \xi^l \leq \cdots \to \infty. \tag{5.21}$$

Since $0 < \xi^1(\epsilon) \leq \xi^2(\epsilon) \leq \cdots \leq \xi^l(\epsilon) \leq \cdots \to \infty$, it follows that $0 \leq \xi^1 \leq \xi^2 \leq \cdots$; since the bilinear form associated with the left-hand side of Equation (5.14) is coercive over $V_F(\omega)$, it follows that $\xi^1 > 0$. Since the operator associated to the limit problem is compact, the eigevalues are all of finite multiplicity and cannot have a finite accumulation point. Hence the relation (5.21) holds.

We next show that if ξ is any eigenvalue of the Problem (5.14), there exists an integer $l \geq 1$ such that $\xi = \xi^l$.

Suppose the contrary holds, i.e., $\xi \neq \xi^l$ for all $l \geq 1$, and let $\boldsymbol{\zeta}$ denote an associated eigenfunction, which satisfies

$$\frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}d\omega = \xi\int_\omega \zeta_i\eta_i\sqrt{a}d\omega \quad \text{for all} \quad \boldsymbol{\eta} \in V_F(\omega), \tag{5.22}$$

$$\int_\omega \zeta_i\zeta_i\sqrt{a}d\omega = \frac{1}{2}, \ \int_\omega \zeta_i\overline{u}_i^l\sqrt{a}d\omega = 0 \quad \text{for all} \ \ l. \tag{5.23}$$

For each $\epsilon > 0$, let $\boldsymbol{w}(\epsilon)$ be the unique solution of

$$\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\boldsymbol{w}(\epsilon))e_{i||j}(\epsilon)(v)\sqrt{g(\epsilon)}dx = \epsilon^2\xi\int_\Omega \zeta_i v_i\sqrt{g(\epsilon)}dx \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}(\Omega). \tag{5.24}$$

Then proceeding as in Theorem 5.1, we can show that $\boldsymbol{w}(\epsilon) \to \boldsymbol{w}$ in $\boldsymbol{V}(\Omega)$ and $\overline{\boldsymbol{w}} \in V_F(\omega)$. Further $\overline{\boldsymbol{w}}$ satisfies

$$\frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\overline{\boldsymbol{w}})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}d\omega = \xi\int_\omega \overline{w}_i\eta_i\sqrt{a}d\omega \quad \text{for all} \quad \boldsymbol{\eta} \in V_F(\omega). \tag{5.25}$$

By the uniqueness of the solution, it follows that $\overline{\boldsymbol{w}} = \zeta$. Since the sequence ξ^l is unbounded, we can choose an l such that

$$\xi < \xi^l. \tag{5.26}$$

For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(\Omega)$, define

$$D(\epsilon)(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega u_i v_i\sqrt{g(\epsilon)}dx. \tag{5.27}$$

Consider the vector

$$\boldsymbol{v}(\epsilon) = \boldsymbol{w}(\epsilon) - \sum_{k=1}^{l} D(\epsilon)(\boldsymbol{w}(\epsilon), \boldsymbol{u}^k(\epsilon))\boldsymbol{u}^k(\epsilon).$$

Then

$$D(\epsilon)(\boldsymbol{v}(\epsilon), \boldsymbol{u}^k(\epsilon)) = 0 \quad \text{for all} \ \ 1 \leq k \leq l. \tag{5.28}$$

Therefore it follows from the variational characterization of the eigenvalues that

$$\xi^{l+1}(\epsilon) \leq \frac{\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\boldsymbol{v}(\epsilon))e_{i||j}(\epsilon)(\boldsymbol{v}(\epsilon))\sqrt{g(\epsilon)}dx}{\epsilon^2 D(\epsilon)(\boldsymbol{v}(\epsilon), \boldsymbol{v}(\epsilon))}. \tag{5.29}$$

Passing to the limit in the above inequality, it can be shown that

$$\xi^{l+1} \leq \xi \tag{5.30}$$

which contradicts (5.26) and the proof is complete.

Theorem 5.2. *Assume that* $\dim(V_F(\omega)) = N$ *and let the space* $V_m(\omega)$ *be defined by*

$$V_m(\omega) = \{\boldsymbol{\eta} = (\eta_i) : \eta_\alpha \in H_0^1(\omega), \eta_3 \in L^2(\omega)\}. \tag{5.31}$$

Then

(a) *For* $1 \le l \le N, (\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ *converges strongly in* $I\!R \times H^1(\Omega)$ *to the* N *eigensolutions of the two-dimensional "flexural shell" problem, viz, find* $(\xi, \boldsymbol{\zeta}) \in I\!R \times V_F(\omega)$ *such that*

$$\frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy = \xi\int_\omega \zeta_i\eta_i\sqrt{a}dy \quad \text{for all} \quad \boldsymbol{\eta} \in V_F(\omega). \tag{5.32}$$

(b) *For each integer* $l > N$, *there exists a subsequence* (*still denoted by* ϵ) *such that*

$$u_\alpha^l(\epsilon) \rightharpoonup u_\alpha^l \quad \text{weakly in} \quad H^1(\Omega), \tag{5.33}$$

$$u_3^l(\epsilon) \rightharpoonup u_3^l \quad \text{weakly in} \quad L^2(\Omega), \tag{5.34}$$

$$\epsilon^2\xi^l(\epsilon) \to \xi^l, \tag{5.35}$$

$$\boldsymbol{u}^l = (u_i^l) \quad \text{is independent of the transverse variable} \quad x_3. \tag{5.36}$$

(c) *The pair* $(\xi^l, \overline{\boldsymbol{u}}^l)$ *solves the two-dimensional eigenvalue problem for the "membrane shell", viz, find* $(\xi, \boldsymbol{\zeta}) \in I\!R \times V_m(\omega)$ *such that*

$$\frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy = \xi\int_\omega \zeta_i\eta_i\sqrt{a}dy \quad \text{for all} \quad \boldsymbol{\eta} \in V_m(\omega). \tag{5.37}$$

Thus, either $(\boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ *converges weakly to zero in* $(H^1(\Omega))^2 \times L^2(\Omega)$ *or* $(\xi^l, \overline{\boldsymbol{u}}^l)$ *is an eigensolution of the "membrane shell" problem.*

Proof. For clarity, it is divided into several steps.

Step 1. The proof that for $1 \le l \le N, (\xi^l(\epsilon), \boldsymbol{u}^l(\epsilon))_{\epsilon>0}$ converges strongly in $I\!R\times H^1(\Omega)$ to the solution of (5.32) follows from Theorem 5.1.

Step 2. From the variational equation (3.9), relation (3.10), inequalities (2.11), (4.46), (5.7) and (5.15), it follows that

$$\begin{aligned}\sum_\alpha ||u_\alpha^l(\epsilon)||_{1,\Omega}^2 + ||\epsilon u_3^l(\epsilon)||_{1,\Omega}^2 &\le C\Big\{\sum_{ij}||e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon))||_{0,\Omega}^2 + ||\boldsymbol{u}^l(\epsilon)||_{0,\Omega}^2\Big\}\\ &\le Cg_0^{-\frac{1}{2}}\Big\{\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\boldsymbol{u}^l(\epsilon))e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon))\sqrt{g(\epsilon)}dx + 1\Big\}\\ &\le Cg_0^{-\frac{1}{2}}\Big\{\epsilon^2\xi^l(\epsilon)\int_\Omega|\boldsymbol{u}^l(\epsilon)|^2\sqrt{g(\epsilon)}dx + 1\Big\}\\ &\le Cg_0^{-\frac{1}{2}}(\epsilon^2k^l + 1).\end{aligned} \tag{5.38}$$

Hence the norms $||e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon))||_{0,\Omega}, ||u_\alpha^l||_{1,\Omega}, ||u_3^l(\epsilon)||_{0,\Omega}$ are bounded independent of ϵ. Consequently, there exists a subsequence (still indexed by ϵ for convenience) and there exist functions $e_{i||j}^l \in L^2(\Omega), u_\alpha^l \in H^1(\Omega)$, satisfying $u_\alpha^l = 0$ on Γ_0 and $u_3^l \in L^2(\Omega)$ such that

$$e_{i||j}(\epsilon)(\boldsymbol{u}^l(\epsilon)) \rightharpoonup e_{i||j}^l \quad \text{weakly in} \quad L^2(\Omega), \tag{5.39}$$

$$u_\alpha^l(\epsilon) \rightharpoonup u_\alpha^l \quad \text{weakly in} \quad H^1(\Omega), \tag{5.40}$$

$$u_3^l(\epsilon) \rightharpoonup u_3^l \quad \text{weakly in} \quad L^2(\Omega). \tag{5.41}$$

Step 3. The limit functions u_i^l found in (5.40)–(5.41) are independent of x_3.

By (5.11) and Step 2,

$$\partial_3u_\alpha^l(\epsilon) + \epsilon\partial_\alpha u_3^l(\epsilon) = 2\epsilon\left\{e_{\alpha||3}(\epsilon)(\boldsymbol{u}^l(\epsilon)) + \Gamma_{\alpha3}^\sigma(\epsilon)u_3^l(\epsilon)\right\} \to 0 \quad \text{strongly in} \quad L^2(\Omega). \tag{5.42}$$

Let $\phi \in \mathcal{D}(\Omega)$; since $u^l_\alpha(\epsilon) \rightharpoonup u^l_\alpha$ weakly in $H^1(\Omega)$ and since $(u^l_3(\epsilon))_{\epsilon>0}$ is bounded in $L^2(\Omega)$ by Step 2,

$$\int_\omega \partial_3 u^l_\alpha \phi dx = \lim_{\epsilon\to 0}\int_\Omega \partial_3 u^l_\alpha(\epsilon)\phi dx, \tag{5.43}$$

$$\lim_{\epsilon\to 0}\int_\Omega \epsilon\partial_3 u^l_3\phi dx = -\lim_{\epsilon\to 0}\int_\Omega \epsilon u^l_3(\epsilon)\partial_\alpha\phi dx = 0, \tag{5.44}$$

whence $\int_\Omega \partial_3 u^l_\alpha \phi dx = 0$. Therefore $\partial_3 u^l_\alpha = 0$ in $L^2(\Omega)$.

Also by Step 2,

$$\partial_3 u^l_3(\epsilon) = \epsilon e_{3||3}(\epsilon)(\boldsymbol{u}^l(\epsilon)) \to 0 \quad \text{strongly in} \quad L^2(\Omega). \tag{5.45}$$

Let $\phi \in \mathcal{D}(\Omega)$; since $u^l_3(\epsilon) \rightharpoonup u^l_3$ weakly in $L^2(\Omega)$ by Step 2,

$$\int_\Omega u^l_3\partial_3\phi dx = \lim_{\epsilon\to 0}\int_\Omega u^l_3(\epsilon)\partial_3\phi dx = -\lim_{\epsilon\to 0}\int_\Omega \partial_3 u^l_3(\epsilon)\phi dx = 0, \tag{5.46}$$

whence $\partial_3 u_3 = 0$ in the sence of distributions. Hence it follows that u_3 is independent of x_3.

Step 4. The limit functions $e^l_{i||j}$ found in (5.39) are independent of x_3, moreover they are related to the limit function (u^l_i) by

$$e^l_{\alpha||\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}^l), \tag{5.47}$$

$$e^l_{\alpha||3} = 0, \tag{5.48}$$

$$e^l_{3||3} = \frac{-\lambda}{\lambda+2\mu}a^{\alpha\beta}e^l_{\alpha||\beta}. \tag{5.49}$$

(The argument is as in [3]. On the right hand side of the relevant equation, we have $\epsilon^2\xi^l(\epsilon)\boldsymbol{u}^l(\epsilon)$ which replaces the forces $f_i(\epsilon)$. All that is needed to pass to the limit is the boundedness of these functions in $L^2(\Omega)$ which we have.)

Step 5. Taking $\boldsymbol{v}$ in Equation (3.9) of the form $\boldsymbol{v} = (\eta_i)$ with $\eta_i \in H^1_0(\omega)$ and passing to the limit as $\epsilon \to 0$, taking into account of the relation (5.47)-(5.49), we get

$$\frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\overline{\boldsymbol{u}}^l)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy = \xi^l\int_\omega \overline{u}^l_i\eta_i\sqrt{a}dy \quad \text{for all} \quad \boldsymbol{\eta} \in (H^1_0(\omega))^3. \tag{5.50}$$

Since both sides of Equation (5.50) are continuous, linear forms with respect to $\eta_3 \in L^2(\omega)$ and $H^1_0(\omega)$ is dense in $L^2(\omega)$, these equations are valid for all $\boldsymbol{\eta} \in V_m(\omega)$.

Remark 5.1. Note that if $V_F(\omega)$ is finite dimensional of dimension, say N, then $\{\xi^l(\epsilon)\}$ for $l > N$ cannot be of order ϵ^2. For, if this were the case, we can get convergence of $\{\boldsymbol{u}^l(\epsilon)\}$ in $\boldsymbol{V}(\Omega)$ to $\overline{\boldsymbol{u}}^l$, an eigenvector of the flexural shell problem. This $\overline{\boldsymbol{u}}^l, l > N$ will be orthogonal to $\overline{\boldsymbol{u}}^i, 1 \le i \le N$ and will contradict the fact that $\dim(V_F(\omega)) = N$.

§6. Conclusions

As mentioned in the introduction, we have investigated the behaviour of eigensolutions of a thin shell based uniquely on the non-trivial nature of the space of inextensional displacements $V_F(\omega)$.

In the stationary case, if $V_F(\omega)$ were nontrivial and the body forces were of order $O(\epsilon^2)$, one got the flexural shell model. Here we have no supplimentary assumption. If $V_F(\omega)$ were infinite dimensional, all the eigenvalues were shown to be of order $O(\epsilon^2)$ and they converge, for each fixed level l, to those of the flexural shell model. Further, all the eigenvalues of the flexural shell are obtained this way. The eigenvectors converge strongly.

If the dimension of $V_F(\omega)$ were finite, the above results hold only upto the level equal to that dimension. Higher eigenvalues are bounded but are not of order $O(\epsilon^2)$. These higher eigenvalues converge to eigenvalues of the membrane shell model, unless the corresponding eigenvectors converge weakly to zero.

Sanchez-Palencia[8], when discussing the eigenvalues of the shells via the Koiter's model, says that when $V_F(\omega) \neq 0$, the eigenvalues are "low frequency" type and converge to the flexural eigenvalues while when dim $V_F(\omega) = 0$ one could get (for instance under the additional assumption that the shell is "uniformly elliptic") the eigenvalues of the membrane shell in the limit. Such eigenvalues are said to be of "medium frequency".

Here we observe that if dim $V_F(\omega) = N < \infty$, then both kinds of eigenvalues—low and medium frequency—may be present.

Of course, we do not know if the eigenvectors for $l > N$ converge weakly to zero or not. If they all converge weakly to zero, then no medium frequency eigenvalues exist. It will be nice to know if this is indeed the case. If so, it will also be nice to know how to characterize the limits of $\epsilon^2 \xi^l(\epsilon)$ for $l > N$.

Of course, to the best of our knowledge, we do not know of any examples of shells for which if $V_F(\omega) \neq 0$, then it is finite dimensional. Sanchez-Palencia states that, in general, $V_F(\omega)$ is infinite dimensional. This is yet another open question.

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