EXTENSIONS OF HILBERT MODULES
AND HANKEL OPERATORS**

GUO Kunyu*

Abstract

Extensions of the Hardy and the Bergman modules over the disc algebra are studied. The author relates extensions of these canonical modules to the symbol spaces of corresponding Hankel operators. In the context of function theory, an explicit formula of $\mathrm{Ext}(L_a^2(D), H^2(D))$ is obtained. Finally, it is also proved that $\mathrm{Ext}(L_a^2(D), L_a^2(D)) \neq 0$. This may be the essential difference between the Hardy and the Bergman modules over the disk algebra.

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§1. Introduction

In the present paper, we will continue the study of Hilbert modules[1−5]. Let H be a Hilbert space and $\mathcal{A}$ a function algebra. We say that H is a Hilbert module over $\mathcal{A}$ if there is a multiplication $(a, f) \to af$ from $\mathcal{A} \times H$ to H, making H into an $\mathcal{A}$-module and if, in addition, the action is jointly continuous in the sup-norm on $\mathcal{A}$ and the Hilbert space norm on H. The category $\mathcal{H}$ of all Hilbert $\mathcal{A}$–modules is a natural setting for numerous questions in operator theory. In [9], Douglas and Paulsen initiated a systematic study of Hilbert modules: they were able to translate many concepts and problems from operator theory in the suggestive language of module theory and in addition they began the study of applications of homological algebra to the categories of Hilbert modules. From an algebraist's point of view, $\mathcal{H}$ is of interest as a nonabelian category. For the analyst, we expect $\mathcal{H}$ to be a fruitful object of study and a useful tool in operator theory. In studying Hilbert modules, as in studying any algebraic structure, the standard procedure is to look at submodules and associated quotient modules. The extension problem then appears quite naturally: given two Hilbert modules H_1, H_2, what module H may be constructed with submodule H_2 and associated quotient module H_1? The set of equivalence classes of such modules H, denoted

by $\text{Ext}_{\mathcal{H}}(H_1, H_2)$, may then be given a natural $\mathcal{A}$-module structure in a way described by Carlson and Clark[6].

In this paper, we will concentrate on studing extensions of the Hardy module $H^2(D)$ and the Bergman module $L_a^2(D)$ on the disk algebra $\mathcal{A} = A(D)$, and relating extensions of these canonical modules to the symbol spaces of corresponding Hankel operators. In Section 2, we introduce some basic homological notions and give some immediate results. Section 3 proves that $\text{Ext}_{\mathcal{H}}(L_a^2(D), H) \neq 0$ for any nonunitarily isometric module H. In particular, we obtain an explicit formula for $\text{Ext}_{\mathcal{H}}(L_a^2(D), H^2(D))$. When one studies the Hardy module and Bergman module over the disk algebra, Hankel operators play an important role. In almost all cases, one must investigate the symbol spaces of corresponding Hankel operators. On the one hand, by using such symbol spaces, one can give the explicit expressions of Ext-groups. On the other hand, by the Hom-Ext sequences, one can determine when a Hankel operator is bounded. In Section 4, we show that

$$\text{Ext}_{\mathcal{H}}(L_a^2(D), L_a^2(D)) \neq 0.$$

As an application, we see that there exists a bounded derivation from $A(D)$ to $\mathcal{B}(L_a^2(D))$ which is not inner, while each bounded derivation from $A(D)$ to $\mathcal{B}(H^2(D))$ is inner. This may be the essential difference between the Hardy and the Bergman modules over the disk algebra.

§2. Homological Preliminaries and Some Immediate Results

In this section, we review some necessary homological notions from the papers [6,7], and give some immediate results which are used in sequent sections. Let H_1, H_2 be two Hilbert modules over $A(D)$. A Hilbert module homomorphism between H_1 and H_2 is a bounded linear operator $L : H_1 \to H_2$ which commutes with the action of $A(D)$. Let $\mathcal{H}$ be the category of all Hilbert modules over $A(D)$ with Hilbert module homomorphisms. The subcategory $\mathcal{C}$ of $\mathcal{H}$ consists of those Hilbert modules H, for which the multiplication by z on H is similar to a contraction (i.e., $\|LzL^{-1}h\| \leq \|h\|$ for some bounded invertible linear operator $L : H \to H'$). It is easy to check that the subcategory $\mathcal{C}$ is full in $\mathcal{H}$. This means that for H_1, $H_2 \in \mathcal{C}$, the set of homomorphisms from H_1 to H_2 in $\mathcal{C}$ is the same as in $\mathcal{H}$, i.e.,

$$\text{Hom}_{\mathcal{C}}(H_1, H_2) = \text{Hom}_{\mathcal{H}}(H_1, H_2).$$

In [6], Carlson and Clark investigated the $\text{Ext}_{\mathcal{H}}$-functor in the category $\mathcal{H}$. What seems to make things most difficult is that the category $\mathcal{H}$ lacks enough projective or injective objects. If we replace $\mathcal{H}$ by the category $\mathcal{C}$, Carlson et al.[7] proved that the category $\mathcal{C}$ has enough projective and injective objects, and the Shilov resolution[9] of a contractive module gives a projective resolution. From [7] and [9], it is not difficult to show that a Hilbert module in $\mathcal{C}$ is projective if and only if it is similar to an isometric Hilbert module; a Hilbert module is projective and injective if and only if it is similar to a unitary Hilbert module. The term isometric (resp., unitary) Hilbert module refers to a Hilbert module H such that the operator of multiplication by z on H is an isometry (resp., a unitary operator). From the definition[7] of $\text{Ext}_{\mathcal{C}}$-functor in the category $\mathcal{C}$, it is easy to see that for H_1 and H_2 in $\mathcal{C}$, there is a canonically injective $A(D)$-module homomorphism

$$i : \quad \text{Ext}_{\mathcal{C}}(H_1, H_2) \to \text{Ext}_{\mathcal{H}}(H_1, H_2).$$

One thus often works in the category $\mathcal{C}$ instead of the category $\mathcal{H}$. In this paper, our main tool is the following Hom-Ext sequences in [6,7]. Let $E : 0 \to H_1 \xrightarrow{\alpha} H_2 \xrightarrow{\beta} H_3 \to 0$ be an exact sequence in the category $\mathcal{C}$. Then for any H in $\mathcal{C}$, there are the following Hom-Ext sequences

$$0 \to \operatorname{Hom}(H, H_1) \xrightarrow{\alpha_*} \operatorname{Hom}(H, H_2) \xrightarrow{\beta_*} \operatorname{Hom}(H, H_3)$$

$$\xrightarrow{\delta} \operatorname{Ext}_{\mathcal{H}}(H, H_1) \xrightarrow{\alpha_*} \operatorname{Ext}_{\mathcal{H}}(H, H_2) \xrightarrow{\beta_*} \operatorname{Ext}_{\mathcal{H}}(H, H_3), \tag{2.1}$$

$$0 \to \operatorname{Hom}(H, H_1) \xrightarrow{\alpha_*} \operatorname{Hom}(H, H_2) \xrightarrow{\beta_*} \operatorname{Hom}(H, H_3)$$

$$\xrightarrow{\delta} \operatorname{Ext}_{\mathcal{C}}(H, H_1) \xrightarrow{\alpha_*} \operatorname{Ext}_{\mathcal{C}}(H, H_2) \xrightarrow{\beta_*} \operatorname{Ext}_{\mathcal{C}}(H, H_3). \tag{2.2}$$

From the above Hom-Ext sequences, we immediately obtain the following proposition which generalizes Proposition 3.2.6 in [6] by completely different method. For another different proof of the next proposition, see [15, Theorem 1.3].

Proposition 2.1. *If H_1, H_2 are similar to isometric Hilbert modules, then*

$$\operatorname{Ext}_{\mathcal{C}}(H_1, H_2) = \operatorname{Ext}_{\mathcal{H}}(H_1, H_2) = 0.$$

Proof. By the Wold decomposition, we may suppose $H_2 = H^2(H)$ for some finite or infinite dimensional Hilbert space H. One thus has the exact sequence of Hilbert modules:

$$E : \ 0 \to H^2(H) \xrightarrow{i} L^2(H) \xrightarrow{\pi} H^2(H)^{\perp} \to 0.$$

Since H_1 is projective in $\mathcal{C}$, this forces the sequence

$$0 \to \operatorname{Hom}(H_1, H^2(H)) \xrightarrow{i_*} \operatorname{Hom}(H_1, L^2(H)) \xrightarrow{\pi_*} \operatorname{Hom}(H_1, H^2(H)^{\perp}) \to 0$$

to be exact. From this fact and $L^2(H)$ being projective in $\mathcal{H}$, the Hom-Ext sequence (2.1) thus implies

$$\operatorname{Ext}_{\mathcal{H}}(H_1, H^2(H)) = 0.$$

This gives what is desired, completing the proof.

Let dA denote the usual normalized area measure on the unit disk D, and let $L^2(D)$ be the space of measurable functions f on D such that $\int |f|^2 dA < \infty$. The Bergman module $L_a^2(D)$ over $A(D)$ is the set of analytic functions in $L^2(D)$. In studying extensions of the Hardy and the Bergman modules, we will be concerned with the following four kinds of Hankel operators: from $H^2(D)$ to $H^2(D)^{\perp}$; from $H^2(D)$ to $L_a^2(D)^{\perp}$; from $L_a^2(D)$ to $H^2(D)^{\perp}$; from $L_a^2(D)$ to $L_a^2(D)^{\perp}$. For their definitions, we only see the case from $H^2(D)$ to $L_a^2(D)^{\perp}$. Let ϕ be in $L^2(D)$, a densely defined Hankel operator $H_{\phi} : H^2(D) \to L_a^2(D)^{\perp}$ is defined by $H_{\phi}h = (I - P)\phi h$, $h \in A(D)$, where P is the orthogonal projection from $L^2(D)$ to $L_a^2(D)$. Therefore, the Hankel operator H_{ϕ} is bounded if and only if the H_{ϕ} in $H^2(D)$ can be continuously extended onto $H^2(D)$.

Let μ be a positive finite measure on D. μ is called an i-th ($i = 1, 2$) Carleson measure if there is a constant c such that $\mu(S) \le ch^i$ for each Carleson square

$$S = \{z = re^{i\theta} | \ 1 - h \le r \le 1; \ \theta_0 \le \theta \le \theta_0 + h\}.$$

For $\phi \in L^2(D)$, we say that ϕ is an i-th Carleson function if $|\phi|^2 dA$ is an i-th Carleson measure ($i = 1, 2$). Write $S_i(D)$ for the set of all i-th Carleson functions ($i = 1, 2$). Then it is easily seen that $S_i(D)$ are $A(D)$-modules. It is well-known that a Hankel operator

H_ϕ from $H^2(D)$ to $H^2(D)^\perp$ is bounded if and only if $\phi \in L^\infty(T) + H^2(D)$, where T is the unit circle with arc–length measure. This fact, translated into homological language, is equivalent to the exactness of the following sequence:

$$0 \to \operatorname{Hom}(H^2(D), H^2(D)) \overset{i_*}{\to} \operatorname{Hom}(H^2(D), L^2(T)) \overset{\pi_*}{\to} \operatorname{Hom}(H^2(D), H^2(D)^\perp) \to 0.$$

It is easily verified that each bounded Hankel operator H_ϕ from $H^2(D)$ to $L_a^2(D)^\perp$ is a Hilbert module homomorphism, and every Hilbert module homomorphism from $H^2(D)$ to $L_a^2(D)^\perp$ is a bounded Hankel operator. Since $H^2(D)$ is projective in the category $\mathcal{C}$, one immediately obtains

Proposition 2.2. *A Hankel operator H_ϕ from $H^2(D)$ to $L_a^2(D)^\perp$ is bounded if and only if $\phi \in S_1(D) + L_a^2(D)$.*

Proof. Since $H^2(D)$ is projective in the category $\mathcal{C}$ (see [7]), we have the following exact Hom sequence

$$0 \to \operatorname{Hom}(H^2(D), L_a^2(D)) \overset{i_*}{\to} \operatorname{Hom}(H^2(D), L^2(D)) \overset{\pi_*}{\to} \operatorname{Hom}(H^2(D), L_a^2(D)^\perp) \to 0.$$

Assume that H_ϕ is bounded. Hence, the above exact sequence insures that there is an $\alpha \in \operatorname{Hom}(H^2(D), L^2(D))$ such that $H_\phi = \pi_*(\alpha)$. It is easily seen that there exists a $\phi_0 \in L^2(D)$ such that for any $f \in H^2(D)$, $\alpha(f) = \phi_0 f$. This induces that ϕ_0 is a 1-th Carleson function by Theorem 9.3 in [10]. We conclude thus that $\phi - \phi_0 \in L_a^2(D)$. The other direction is achieved by considering the above exact sequence and Theorem 9.3 in [10], completing the proof.

An important question is whether the category $\mathcal{H}$ has enough projectives, in the sense that every object in $\mathcal{H}$ is a quotient of some projective module. Pisier's recent example (see [8]) shows that $\mathcal{C}$ is a proper subcategory of $\mathcal{H}$. Though

$$\operatorname{Ext}_{\mathcal{C}}(H^2(D), L_a^2(D)) = 0,$$

it remains unknown whether

$$\operatorname{Ext}_{\mathcal{H}}(H^2(D), L_a^2(D)) = 0.$$

§3. Nonvanishing of $\operatorname{Ext}_{\mathcal{H}}(L_a^2(D), H)$

In this section, we shall prove that

$$\operatorname{Ext}_{\mathcal{H}}(L_a^2(D), H)) \neq 0$$

for each nonunitarily isometric module H. In particular, an explicit formula is obtained for $\operatorname{Ext}_{\mathcal{H}}(L_a^2(D), H^2(D))$.

Theorem 3.1. *Let H be similar to a nonunitarily isometric module. Then*

$$\operatorname{Ext}_{\mathcal{C}}(L_a^2(D), H) = \operatorname{Ext}_{\mathcal{H}}(L_a^2(D), H) \neq 0.$$

Proof. If H is similar to a unitary module, then H is injective in $\mathcal{H}$ by [1 or 7], and therefore

$$\operatorname{Ext}_{\mathcal{C}}(L_a^2(D), H) = \operatorname{Ext}_{\mathcal{H}}(L_a^2(D), H) = 0.$$

By the Wold decomposition of an isometry, we only need to prove

$$\operatorname{Ext}_{\mathcal{C}}(L_a^2(D), H^2(H)) = \operatorname{Ext}_{\mathcal{H}}(L_a^2(D), H^2(H)) \neq 0$$

for some finite or infinite Hilbert space H. Consider the exact sequence

$$E: \ 0 \to H^2(H) \xrightarrow{i} L^2(H) \xrightarrow{\pi} H^2(H)^\perp \to 0.$$

Since $L^2(H)$ is projective in $\mathcal{H}$ (also in $\mathcal{C}$), the Hom-Ext sequences (2.1), (2.2) induce the following

$$0 \to \mathrm{Hom}(L_a^2(D), H^2(H)) \xrightarrow{i_*} \mathrm{Hom}(L_a^2(D), L^2(H)) \xrightarrow{\pi_*} \mathrm{Hom}(L_a^2(D), H^2(H)^\perp)$$

$$\xrightarrow{\delta} \mathrm{Ext}_{\mathcal{H}}(L_a^2(D), H^2(H)) \to 0;$$

$$0 \to \mathrm{Hom}(L_a^2(D), H^2(H)) \xrightarrow{i_*} \mathrm{Hom}(L_a^2(D), L^2(H)) \xrightarrow{\pi_*} \mathrm{Hom}(L_a^2(D), H^2(H)^\perp)$$

$$\xrightarrow{\delta} \mathrm{Ext}_{\mathcal{C}}(L_a^2(D), H^2(H)) \to 0.$$

We claim that $\mathrm{Hom}(L_a^2(D), L^2(H)) = 0$. In fact, for any $\alpha \in \mathrm{Hom}(L_a^2(D), L^2(H))$, since

$$\|\alpha(z^n)\| = \|z^n \alpha(1)\| = \|\alpha(1)\| \leq \|\alpha\| \Big[\int |z^n|^2 dA\Big]^{\frac{1}{2}} \to 0$$

as $n \to \infty$, it follows that $\alpha(1) = 0$. This means that

$$\alpha(f) = f\alpha(1) = 0, \ \forall f \in A(D).$$

The claim follows. Combining the above exact sequences with the claim immediately shows that

$$\mathrm{Ext}_{\mathcal{C}}(L_a^2(D), H^2(H)) = \mathrm{Ext}_{\mathcal{H}}(L_a^2(D), H^2(H)) = \mathrm{Hom}(L_a^2(D), H^2(H)^\perp).$$

For $\phi \in L^2(H)$, a densely defined Hankel operator $H_\phi : L_a^2(D) \to H^2(H)^\perp$ is defined by $H_\phi h = (I - P)h\phi$; $h \in A(D)$, where P is the orthogonal projection from $L^2(H)$ onto $H^2(H)^\perp$. If the densely defined operator H_ϕ in $L_a^2(D)$ can be continuously extended onto $L_a^2(D)$, then it is easy to check that H_ϕ is a Hilbert module homomorphism from $L_a^2(D)$ to $H^2(H)^\perp$, and each Hilbert module homomorphism from $L_a^2(D)$ to $H^2(H)^\perp$ has such a form. Writing $\mathcal{S}(L_a^2(D), H^2(H))$ for the set of all such ϕ, we have

$$\mathrm{Ext}_{\mathcal{C}}(L_a^2(D), H^2(H)) = \mathrm{Ext}_{\mathcal{H}}(L_a^2(D), H^2(H)) = \mathcal{S}(L_a^2(D), H^2(H))/H^2(H).$$

Taking any $h \in H$ with $\|h\| = 1$, we can easily prove that $\phi = \bar{z}h$ is in $\mathcal{S}(L_a^2(D), H^2(H))$, while ϕ not in $H^2(H)$). This is just what we expect, completing the proof.

Remark 3.1. From the proof of Theorem 3.1, one can deduce the following explicit formula for $\mathrm{Ext}_{\mathcal{C}}(L_a^2(D), H^2(H)) = \mathrm{Ext}_{\mathcal{H}}(L_a^2(D), H^2(H))$, this is,

$$\mathrm{Ext}_{\mathcal{C}}(L_a^2(D), H^2(H)) = \mathrm{Ext}_{\mathcal{H}}(L_a^2(D), H^2(H)) = \mathcal{S}(L_a^2(D), H^2(H))/H^2(H).$$

Let $C_H(T)$ be the set of all continuous functions ϕ on the unit circle T such that Hankel operator $H_\phi : L_a^2(D) \to H^2(D)^\perp$ is bounded. Obviously, $C_H(T)$ is an $A(D)$-module.

Corollary 3.1. $\mathrm{Ext}_{\mathcal{C}}(L_a^2(D), H^2(D)) = \mathrm{Ext}_{\mathcal{H}}(L_a^2(D), H^2(D)) = C_H(T)/A(D)$.

Proof. Let ϕ be in $L^2(T)$ and $H_\phi : L_a^2(D) \to H^2(D)^\perp$ be a bounded Hankel operator. Since $H^2(D)$ is contractively contained in $L_a^2(D)$, one concludes that $H_\phi : H^2(D) \to H^2(D)^\perp$ is a bounded Hankel operator. This shows that there is a $\phi_0 \in L^\infty(T)$ such that $\phi - \phi_0 \in H^2(D)$. Suppose that $\{h_k\} \subset H^2(D)$, and $\{h_k\}$ weakly converges to 0 in $H^2(D)$. Let

$$h_k = \sum_{n \geq 0} a_n^{(k)} z^n$$

be the power series expansion of h_k. Since $\{\|h_k\|_{H^2(D)}\}$ is bounded, there exists a constant c_1 such that $\sum_{n \geq 0} |a_n^{(k)}|^2 < c_1$ for all k. On the other hand, since $H_\phi : L_a^2(D) \to H^2(D)^\perp$ is bounded, there exists a constant c_2 such that

$$\|H_\phi h_k\|_{L^2(T)}^2 = \|H_{\phi_0} h_k\|_{L^2(T)}^2 \quad \leq c_2 \int |h_k|^2 dA = c_2 \Big[\sum_{n \geq 0} \frac{|a_n^{(k)}|^2}{n+1} \Big].$$

For each fixed n, since $a_n^{(k)} \to 0$ as $k \to \infty$, the above discussion implies that

$$\|H_{\phi_0} h_k\|_{L^2(T)}^2 \to 0 \quad \text{as} \quad k \to \infty,$$

and therefore H_{ϕ_0} is a compact Hankel operator from $H^2(D)$ to $H^2(D)^\perp$. So ϕ_0 is in $H^\infty(D) + C(T)$, where $C(T)$ is the set of all continuous functions on T. It now follows that $H_\phi : L_a^2(D) \to H^2(D)^\perp$ is bounded if and only if there is a $\psi \in C_H(T)$ such that $\phi - \psi \in H^2(D)$. Hence from Remark 3.1, we have

$$\begin{aligned}
\mathrm{Ext}_\mathcal{C}(L_a^2(D), H^2(D)) &= \mathrm{Ext}_\mathcal{H}(L_a^2(D), H^2(D)) \\
&\cong [C_H(T) + H^2(D)]/H^2(D) \\
&\cong C_H(T)/A(D).
\end{aligned}$$

This comlpes the proof of Corollary 3.3.

Remark 3.2. From the proof of Corollary 3.1, we observe that a Hankel operator $H_\phi : L_a^2(D) \to H^2(D)^\perp$ is bounded if and only if $\phi \in C_H(T) + H^2(D)$. However it remains unknown whether $C_H(T)$ is equal to $C(T)$.

§4. Nonvanishing of $\mathrm{Ext}_\mathcal{H}(L_a^2(D), L_a^2(D))$

From the Hom-Ext sequences, it is easy to see that a Hilbert module H in $\mathcal{H}$ (resp., in $\mathcal{C}$) is injective if and only if $\mathrm{Ext}_\mathcal{H}(\widetilde{H}, H) = 0$ (resp., $\mathrm{Ext}_\mathcal{C}(\widetilde{H}, H) = 0$) for each Hilbert module $\widetilde{H}$ in $\mathcal{H}$ (resp., in $\mathcal{C}$). The question naturally arises as to whether it is necessary to use all Hilbert modules $\widetilde{H}$ in $\mathrm{Ext}(\widetilde{H}, H)$ to test whether H is injective; might it not happen that there exists a small family of Hilbert modules H_λ such that if $\mathrm{Ext}(H_\lambda, H) = 0$ for each H_λ in the family, then H is injective? By Theorem 3.1, when H is similar to an isometric module, H is injective in $\mathcal{H}$ if and only if $\mathrm{Ext}_\mathcal{H}(L_a^2(D), H) = 0$. Hence the Bergman module $L_a^2(D)$ seems to play a role to test injective modules. In the following we shall show that

$$\mathrm{Ext}_\mathcal{C}(L_a^2(D), L_a^2(D)) \neq 0,$$

and therefore

$$\mathrm{Ext}_\mathcal{H}(L_a^2(D), L_a^2(D)) \neq 0.$$

Consider the exact sequence

$$E: \ 0 \to L_a^2(D) \xrightarrow{i} L^2(D) \xrightarrow{\pi} L_a^2(D)^\perp \to 0.$$

Hence it induces the following Hom-Ext sequence

$$0 \to \mathrm{Hom}(L_a^2(D), L_a^2(D)) \xrightarrow{i_*} \mathrm{Hom}(L_a^2(D), L^2(D)) \xrightarrow{\pi_*} \mathrm{Hom}(L_a^2(D), L_a^2(D)^\perp)$$

$$\xrightarrow{\delta} \mathrm{Ext}_\mathcal{C}(L_a^2(D), L_a^2(D)) \xrightarrow{i_*} \mathrm{Ext}_\mathcal{C}(L_a^2(D), L^2(D)) \xrightarrow{\pi_*} \mathrm{Ext}_\mathcal{C}(L_a^2(D), L_a^2(D)^\perp).$$

To prove that $\mathrm{Ext}_\mathcal{C}(L_a^2(D), L_a^2(D)) \neq 0$, one only needs to show that

$$\pi_* : \mathrm{Hom}(L_a^2(D), L^2(D)) \to \mathrm{Hom}(L_a^2(D), L_a^2(D)^\perp)$$

is not surjective. For this aim, we first give the descriptions of $\mathrm{Hom}(L_a^2(D), L^2(D))$ and $\mathrm{Hom}(L_a^2(D), L_a^2(D)^\perp)$. Let $\alpha \in \mathrm{Hom}(L_a^2(D), L^2(D))$. It is easy to check that α is a multiplier from $L_a^2(D)$ to $L^2(D)$, that is, there is a $\phi \in L^2(D)$ such that $\alpha = M_\phi$. According to [11 or 12], M_ϕ is a multiplier from $L_a^2(D)$ to $L^2(D)$ if and only if ϕ is a 2-nd Carleson function. We thus have

Lemma 4.1. $\mathrm{Hom}(L_a^2(D), L^2(D)) \cong S_2(D)$.

Write $B_H(D)$ for all those $\phi \in L^2(D)$ such that $H_\phi : L_a^2(D) \to L_a^2(D)^\perp$ is bounded. Clearly, $B_H(D)$ is an $A(D)$-module.

Lemma 4.2. $\mathrm{Hom}(L_a^2(D), L_a^2(D)^\perp) \cong B_H(D)/L_a^2(D)$.

From Lemmas 4.1 and 4.2, we have

Lemma 4.3. $\mathrm{coker}\pi_* = B_H(D)/[S_2(D) + L_a^2(D)]$.

Theorem 4.1. $B_H(D) \supsetneqq S_2(D) + L_a^2(D)$.

Proof. To prove Theorem 4.1, we must find a function b such that $b \in B_H(D)$, while $b \notin S_2(D) + L_a^2(D)$. Firstly notice that the analytic function $f(z) = \sum\limits_{n=0}^{\infty} z^{2^n}$ is in $L_a^2(D)$. We claim that $f(z)$ is in the Bloch space. This claim is easily from the theorem on Hadamard power series[10]. For reader's convenience, we write the proof's detail. That is, we need to prove the following

$$\sup\{(1 - |z|^2)|f'(z)| : z \in D\} < +\infty.$$

Since $f'(z) = \sum\limits_{n=0}^{\infty} 2^n z^{2^n - 1}$, one needs to show

$$\sum_{n=0}^{\infty} 2^n r^{2^n - 1} = O\Big(\frac{1}{1-r}\Big)$$

for $0 < r < 1$. This is equivalent to estimating the integral

$$\int_0^\infty 2^x r^{2^x - 1} dx = \frac{1}{\ln 2} \int_1^\infty r^{t-1} dt \quad (t = 2^x).$$

Since $\ln r < r - 1$, the second integral is majorized by

$$\int_1^\infty r^{t-1} dt = \int_1^\infty e^{(t-1)\ln r} dt \leq \int_1^\infty e^{(r-1)(t-1)} dt = \frac{1}{1-r},$$

which proves the claim, and hence $f(z)$ is in the Bloch space. Therefore, Theorem 6 of [13] shows that Hankel operator $H_{\bar f}$ is bounded. The rest of the proof is to check $\bar f \notin S_2(D) + L_a^2(D)$.

For any $z \in D$, let k_z be the normalized Bergman reproducing kernel at z (i.e., with $\|k_z\| = 1$). If there is a $\phi \in S_2(D)$ and a $\psi \in L_a^2(D)$ such that $\phi = \bar f + \psi$, then

$$|\bar f(z) + \psi(z)| = \Big| \int (\bar f + \psi)|k_z|^2 dA \Big| = \Big| \int \phi |k_z|^2 dA \Big| \leq \|M_\phi\|.$$

This implies that $\phi(z) = \bar f(z) + \psi(z)$ is a bounded harmonic function. Since $\bar f(re^{i\theta}) - \bar f(0)$ and $\psi(re^{i\theta}) + \bar f(0)$ are orthogonal in $L^2(T)$, it follows that

$$\|\phi(re^{i\theta})\|_{L^2(T)}^2 = \|f(re^{i\theta}) - f(0)\|_{H^2(D)}^2 + \|\psi(re^{i\theta}) + \bar f(0)\|_{H^2(D)}^2.$$

Since ϕ is bounded, the above equality shows that there exists some positive constant c such that

$$\|f(re^{i\theta}) - f(0)\|_{H^2(D)} \leq c \quad \text{for any} \quad r.$$

So f is in $H^2(D)$. This is impossible and hence $S_2(D) + L_a^2(D)$ is a proper subset of $B_H(D)$, completing the proof.

Corollary 4.1. $\mathrm{Ext}_{\mathcal{C}}(L_a^2(D), L_a^2(D)) \neq 0$, and $\mathrm{Ext}_{\mathcal{H}}(L_a^2(D), L_a^2(D)) \neq 0$.

Remark 4.1. Let $\mathcal{B}$ be a Banch $A(D)$-bimodule. A derivation δ from $A(D)$ to $\mathcal{B}$, by definition, is a continuously linear map and such that

$$\delta(ab) = a\delta(b) + \delta(a)b \quad \text{for all} \ \ a, b \in A(D).$$

Let $\mathcal{B}(L_a^2(D))$ (resp., $\mathcal{B}(H^2(D))$) denote the set of all bounded linear operators on $L_a^2(D)$ (resp., $H^2(D)$) with natural $A(D)$-bimodule structure. By Theorem 2.2.2 in [6], we see that each derivation from $A(D)$ to $\mathcal{B}(H^2(D))$ is inner. However, Corollary 4.1 shows that there is some derivation from $A(D)$ to $\mathcal{B}(L_a^2(D))$ which is not inner. This may be the essential defference between the Hardy and the Bergman modules over $A(D)$.

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