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CHAOS AND ORDER OF INVERSE LIMIT SPACE FOR A GRAPH MAP**

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Abstract

With the method of inverse limit, the author obtains several criteria of chaos of piecewise monotone continuous maps on finite graphs.

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§1. Introduction

In the recent years there is a growing interest in investigating the connection between the dynamics of a continuous map of a finite graph and the topological structure of the inverse limit space using the map as a sole bonding map, since some attractor of a dynamical system can be shown to be the inverse limit space of a continuous map of a finite graph[11] (see for instance [1,4,7,8,12]). In [1] Barge and Diamond proved that a piecewise monotone continuous map of a finite graph has topological entropy zero if and only if the inverse limit space using this map as a sole bonding map contains no indecomposable subcontinuum. In this paper, we prove that a piecewise monotone continuous map of a finite graph is non-chaotic in the sense of Li-Yorke if and only if the order of the inverse limit space using the map as a sole bonding map (see §2 for the definition) is at most ω_0. In roughly speaking, if the map is non-chaotic, then the topological structure of its inverse limit space is relatively simple, and vice versa. In addition, we will also prove that a piecewise monotone continuous map of a finite graph is non-chaotic in the sense of Li–Yorke if and only if every point in the intersection of the set of recurrent points of the map and the closure of the set of periodic points of the map is a regularly recurrent point (definition follow). To be more precise we introduce some notions.

By a finite graph we mean a one-dimensional compact connected polyhedron, and by a graph map a continuous map from a finite graph into itself. Let $f : G \to G$ be a graph map. $x \in G$ is called a turning point of f, if for each neighborhood U of x there are $y \neq z \in U$ such that $f(y) = f(z)$. By $T(f)$ denote the set of all turning points of f. f is said to be piecewise monotone if $T(f)$ is finite. Let $C_{\rm PM}(G, G)$ be the set of all continuous piecewise monotone map of finite graph G.

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By $C(X,X)$ we denote the set of all continuous maps of a metric space X. $f \in C(X,X)$ is said to be chaotic (in the sense of Li-Yorke) if there is an uncountable subset $S \subset X$ (the chaotic set of f) such that for any $x \neq y \in S$, $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$ and $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$. $x \in X$ is called a regularly recurrent point of f, if for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $d(f^{km}(x), x) < \varepsilon$ for all $k \in \mathbb{N}$. By $P(f), R(f), RR(f)$ and $\omega(x,f)$ denote the set of periodic points, recurrent points, regularly recurrent points of f and ω limit point of x respectively (see, for instance, [2] for their definitions). Let ω_0 be first limit ordinal number. See §2 for the definitions of inverse limit space (G,f) and its order $\mathrm{Order}(G,f)$. Our main result is

Theorem 1.1. *If $f \in C_{\mathrm{PM}}(G,G)$ with topological entropy zero, then the following statements are equivalent*:

(1) $R(f) \cap \overline{P(f)} = RR(f)$ (*resp.* $R(f) \cap \overline{P(f)} \neq RR(f)$).

(2) $\mathrm{Order}(G,f) \leq \omega_0$ (*resp.* $\mathrm{Order}(G,f) = \omega_0 + 1$).

(3) f *is non-chaotic* (*resp. chaotic*).

Remark that, by the work of Lliber and Misiurewicz in [5], for any graph map (need not be piecewise monotone) positive topological entropy implies chaos. So we shall restrict our attentions on the graph maps which have topological entropy zero.

§2. Preliminaries

By a continuum we mean a nonempty connected compact metric space. A subcontinuum is a subset of a continuum and it is a continuum itself. A continuum is decomposable (resp. indecomposable) if it can (resp. can not) be written as the union of its two proper subcontinua. A continuum is hereditarily decomposable if each of its nondegenerate subcontinua is decomposable (refer to [9] for basic properties of continua).

Given a continuum X with metric d and a map $f \in C(X,X)$, the associated inverse limit space (X,f) is defined by

$$(X,f) = \Big\{ \underline{x} = (x_1 x_2 \cdots) \in \prod_{i=1}^{\infty} X \,\Big|\, f(x_{i+1}) = x_i, \quad i \in \mathbb{N} \Big\}$$

with metric $\underline{d}$ given by $\underline{d}(\underline{x}, \underline{y}) = \sum_{i=1}^{\infty} 2^{-i} \dfrac{d(x_i, y_i)}{1 + d(x_i, y_i)}$. The space (X,f) is a continuum. The map $\hat{f} : (X,f) \to (X,f)$ defined by $\hat{f}(x_1 x_2 x_3 \cdots) = (f(x_1) x_1 x_2 \cdots)$ is called the induced homeomorphism. For every $k \in \mathbb{N}$, the projection map $\pi_k : (X,f) \to X$ given by $\pi_k(\underline{x}) = x_k$ is continuous; if f is onto, so is π_k. Obviously $\pi_k = \pi_{k+1} \circ \hat{f}$ and $\pi_k = f \circ \pi_{k+1}$ for any $k \in \mathbb{N}$. Throughout this paper, we assume that every graph map $f : G \to G$ is onto; since if not, then $(G,f) = (G', f')$, where $G' = \bigcap_{n \geq 0} f^n(G)$ is a finite graph and $f' = f|G' : G' \to G'$ is onto. To introduce the definition of the order of an inverse limit space, we need the following two theorems.

Theorem A.[1] *Suppose that $f \in C_{\mathrm{PM}}(G,G)$. Then the inverse limit space (G,f) is hereditarily decomposable if and only if the topological entropy of f is zero.*

Theorem B.[10] *Suppose that $f : G \to G$ is a graph map. If the inverse limit space (G,f) is hereditarily decomposable, then there exists an upper semi-continuous decomposition $\mathcal{G}$ of (G,f) into disjoint subcontinua such that*

(1) *$\mathcal{G}$, with the quotient topology, is a finite graph;*

(2) *let $g : (G,f) \to \mathcal{G}$ be the quotient map, then the subcontinuum $g^{-1}(g(\underline{x}))$ of (G,f) has empty interior for each $\underline{x} \in (G,f)$;*

(3) *the map* $\psi : \mathcal{G} \to \mathcal{G}$ *defined by* $\psi(g(\underline{x})) = g(\hat{f}(\underline{x}))$ *is well defined and is a homeomorphism.*

Based on the theorems above, if $f \in C_{\rm PM}(G,G)$ with topological entropy zero then, for each $\underline{x} \in (G,f)$, $g^{-1}(g(\underline{x}))$ is precisely a maximal nowhere dense subcontinuum of (G,f) containing $\underline{x}$. Therefore, we have

Definition 2.1. *Let* X *be a hereditarily decomposable continuum.* X *is said to be* K*-decomposable if there is a collection* $\mathcal{D}$ *of pairwise disjoint subcontinua of* X *such that* $\cup\mathcal{D} = X$ *and each element of* $\mathcal{D}$ *is precisely a maximal nowhere dense subcontinua of* X*. Each element of* $\mathcal{D}$ *is called a layer of* X*. In addition,* X *is said to be hereditarily* K*-decomposable if each nondegenerate subcontinuum of* X *is* K*-decomposable.*

Definition 2.2. *Suppose that continuum* X *is hereditarily* K*-decomposable. Let* $\mathcal{D}_0 = \{X\}$*. If* $\alpha = \beta + 1$*, let* $\mathcal{D}_\alpha$ *denote the set of all degenerate elements of* $\mathcal{D}_\beta$ *and all layers of nondegenerate elements of* $\mathcal{D}_\beta$*. If* α *is a limit ordinal number, let* $\mathcal{D}_\alpha = \Big\{ \bigcap\limits_{\beta<\alpha} D_\beta : D_\beta \in \mathcal{D}_\beta \Big\}$*. By* $D_\alpha(x)$ *we denote the element of* $\mathcal{D}_\alpha$ *containing* x *for each* $x \in X$*. We say that the order of* X *is* τ*, written as* $\mathrm{Order}(X) = \tau$*, if* τ *is the minimal ordinal number such that* $D_\tau(x) = \{x\}$ *for each* $x \in X$*. By* $\mathcal{D}'_\alpha$ *we denote the set of nondegenerate elements of* $\mathcal{D}_\alpha$*. In order to emphasize the dependence of* $\mathcal{D}_\alpha$ *(resp.* $\mathcal{D}'_\alpha$*) on* X*, we shall also write* $\mathcal{D}_\alpha(X)$ *(resp.* $\mathcal{D}'_\alpha(X)$*) instead of* $\mathcal{D}_\alpha$ *(resp.* $\mathcal{D}'_\alpha$*).*

Theorem C is an elementary property concerning $\mathrm{Order}((G,f))$.

Theorem C.[7] *Suppose* $f \in C_{\rm PM}(G,G)$ *with topological entropy zero. Then the set of periods of* f *is finite if and only if* $\mathrm{Order}(G,f) < \omega_0$*; moreover, if the set of periods of* f *is infinite, then* $\mathrm{Order}(G,f) \in \{\omega_0, \omega_0+1\}$*. Thus* $(G,f) = \Big[\bigcup\limits_{i=0}^{\infty}(\cup\mathcal{D}'_i \setminus \cup\mathcal{D}'_{i+1})\Big] \bigcup \mathcal{D}'_{\omega_0}$*.*

Lemma 2.1.[6] *Suppose* $f \in C_{\rm PM}(G,G)$ *without periodic point. Then the inverse limit space of* f *is homeomorphic to the circle* $\mathbb{S}^1$*.*

Lemma 2.2.[7] *Suppose* $f \in C_{\rm PM}(G,G)$ *with topological entropy zero. Then*

(1) *if* $\mathrm{Order}(G,f) \geq \omega_0$*, then* $\mathcal{D}'_m$ *is finite* $(\forall m \in \mathbb{N})$ *and* $\pi_1(A) \cap \pi_1(B) = \emptyset$ *for any* $A \neq B \in \mathcal{D}'_m$ $(\forall m \in \mathbb{N})$;

(2) *if* $\mathrm{Order}(G,f) = \omega_0 + 1$ *and* $D \in \mathcal{D}'_{\omega_0}$*, then, for each pair of* $i, j \in \mathbb{N} (i \neq j)$*, we have* $\pi_i(D) \cap \pi_j(D) = \emptyset$ *and* $\pi_i(D) \cap P(f) = \emptyset$*;*

(3) *for each* $m \in \mathbb{N}$ *and* $A \in \mathcal{D}'_m$*, there is* $n \in \mathbb{N}$ *such that*

(3.1) $\pi_i(A) = \pi_j(A) \Longleftrightarrow i \equiv j (\mathrm{mod} n)$*;*

(3.2) *if* $i \not\equiv j (\mathrm{mod} n)$*, then* $\pi_i(A) \cap \pi_j(A) = \emptyset$*;*

(3.3) *for each* $i \in \mathbb{N}, \pi_i(A)$ *is a nondegenerate connected closed subset of* G*, and* A *is homeomorphic to* $(\pi_1(A), f^n|(\pi_1(A)))$*.*

Definition 2.3. *A nondegenerate connected closed subset* K *of a finite graph* G *is said to be a periodic subgraph of* f *with period* n *if* $f^n(K) = K$ *and* $K, f(K), \cdots, f^{n-1}(K)$ *are pairwise disjoint.*

Note that, by Lemma 2.2(3), if $A \in \mathcal{D}'_m((G,f))$, then $\pi_i(A)$ is a periodic subgraph of f for any $i \in \mathbb{N}$, provided $f \in C_{\rm PM}(G,G)$ with topological entropy zero and $\mathcal{D}'_m((G,f)) \neq \emptyset$.

Lemma 2.3. *Suppose that* $f \in C(X,X)$ *is a map on compact metric space* X*. If* A *is a compact subset of the inverse limit space* (X,f) *and* $\hat{f}^n(A) = A$ *for some* $n \in \mathbb{N}$*, then* A *is homeomorphic to the inverse limit space* $(\pi_1(A), f^n|(\pi_1(A)))$*.*

Lemma 2.4. *Suppose that* $f \in (X,X)$ *is a map on compact metric space* X*, and* A, A_i *are compact subsets of inverse limit space* (X,f) *such that* $A_{i+1} \subset A_i$ *and* $A = \bigcap\limits_{i=1}^{\infty} A_i$*. Then* $\pi_j(A) = \bigcap\limits_{i=1}^{\infty} \pi_j(A_i)$ $(\forall j \in \mathbb{N})$*.*

Lemmas 2.3–2.4 are the elementary results in continuum theory (see [9]).

§3. The Proof of the Main Theorem

Proposition 3.1. *Suppose that* $f \in C_{\rm PM}(G,G)$ *with topological entropy zero. If* $R(f) \cap \overline{P(f)} = RR(f)$, *then* $\mathrm{Order}(G,f) \le \omega_0$.

To prove Proposition 3.1, we need some notations and Lemmas 3.1–3.3. $x \in G$ is said to be a branched point of finite graph G if for each neighbourhood U of x there is a neighbourhood $V \subset U$ of x which is homeomorphic to $\{z \in \mathbb{C} | z^n \in [0,1)\}$ for some $n > 2$. By $\mathrm{Br}(G)$ denote the set of all branched points of G. The closure of a connected components of $G \setminus \mathrm{Br}(G)$ is said to be an edge of G. $x \in G$ is said to be an end point of G if there is a neighbourhood of x which is homeomorphic to $[0,1)$. We denote the set of all end points of G by $\mathrm{End}(G)$, and $\{f^n(x) | n \ge 0\}$ by $\mathrm{Orb}_f(x)$.

Lemma 3.1. *Suppose* $f \in C_{\rm MP}(G,G)$ *with topological entropy zero. For any* $n \in \mathbb{N}$, *if* $D \in \mathcal{D}'_n$ *then each end point of* $\pi_1(D)$ *either is a periodic point of* f *or belongs to the orbit of some turning point of* f. *Namely* $\mathrm{End}\Big(\bigcup\limits_{D \in \mathcal{D}'_n} \pi_1(D)\Big) \subset P(f) \cup \Big(\bigcup\limits_{c \in T(f)} \mathrm{Orb}_f(c)\Big)$ *for any* $n \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$ and $D \in \mathcal{D}'_n$. Suppose that $a_1 \in \mathrm{End}(\pi_1(D))$. It is sufficient to prove that $a_1 \in P(f) \cup \Big(\bigcup\limits_{c \in T(f)} \mathrm{Orb}_f(c)\Big)$.

Let k be the period of the periodic subgraph $\pi_1(D)$ under f, $B_D = \bigcup\limits_{i=1}^{k-1} \mathrm{End}(\pi_i(D))$, $s = \mathrm{Card}(B_D)$ and

$$A = \Big\{a_j \in \bigcup_{i=1}^{k-1} \pi_i(D) | f(a_{j+1}) = a_j, \quad j = 1,2,\cdots,s+1\Big\}.$$

If $A \subset B_D$, then $\mathrm{Card}(A) \le s$. Hence there are $1 \le j < p \le s+1$ such that $a_j = a_p$. Assume that $p = j + r$, then $f^r(a_j) = f^r(a_{j+r}) = a_j$, and it implies $a_1 \in P(f)$.

If $A \not\subset B_D$, then there is $a_j \in B_D$ such that $a_i \in B_D$ for all $1 \le i \le j$ and $a_{j+1} \in \mathrm{int}\Big(\bigcup\limits_{i=1}^{k-1} \pi_i(D)\Big)\Big(= \Big(\bigcup\limits_{i=1}^{k-1} \pi_i(D)\Big) \setminus B_D\Big)$. Suppose a_j and a_{j+1} belong to $\pi_{i_j}(D)$ and $\pi_{i_j+1}(D)$ respectively. Note that $\pi_{i_j}(D) = f(\pi_{i_j+1}(D))$. Thus, by the definition of turning point, a_{j+1} is a turning point of f, and a_1 belongs to its orbit.

Lemma 3.2. *Suppose* $f \in C_{\rm MP}(G,G)$ *with topological entropy zero. If* $\mathrm{Order}(G,f) = \omega_0 + 1$, *then for each* $D \in \mathcal{D}'_{\omega_0}$, *there is an* $l \in \mathbb{N}$ *such that* $\pi_i(D) \cap \Big(\bigcup\limits_{c \in T(f)} \mathrm{Orb}_f(c)\Big) = \emptyset$ *when* $i > l$.

Proof. Fix $D \in \mathcal{D}'_{\omega_O}$ and $c \in T(f)$. As $T(f)$ is finite, it is sufficient to prove that there is $l \in \mathbb{N}$ such that $\pi_i(D) \cap \mathrm{Orb}_f(c) = \emptyset$ when $i > l$.

We first prove that, for any $n \in \mathbb{N}$, $\pi_1(\hat{f}^n(D)) \bigcap \Big[\bigcup\limits_{i \ge 1} \pi_i(D)\Big] = \emptyset$. In fact, as $\hat{f} : (G,f) \to (G,f)$ is a homeomorphism, then $\hat{f}^n(D) \in \mathcal{D}'_{\omega_0}$ for any $n \in \mathbb{N}$. Note that $\pi_{n+k}(\hat{f}^n(D)) = \pi_k(D)$ for any $k \in \mathbb{N}$, then, by Lemma 2.2(2),

$$\{\pi_i(\hat{f}^n(D)) | 1 \le i \le n\} \bigcup \{\pi_i(D) | i \ge 1\} = \{\pi_i(\hat{f}^n(D)) | i \ge 1\}$$

are pairwise disjoint. Thus $\pi_1(\hat{f}^n(D)) \bigcap \Big[\bigcup\limits_{i \ge 1} \pi_i(D)\Big] = \emptyset$.

Suppose $\Big[\bigcup\limits_{i\geq 1}\pi_i(D)\Big]\bigcap \mathrm{Orb}_f(c)\neq\emptyset$ (otherwise, we have done). Let

$$m=\min\Big\{m'\in\mathbb{N}\cup\{0\}\Big|f^{m'}(c)\in\bigcup_{i\geq 1}\pi_i(D)\Big\}.$$

Then there is $l\in\mathbb{N}$ such that $f^m(c)\in\pi_l(D)$. Note that

$$f^{m+p}(c)\in f^p(\pi_l(D))=\pi_l(\hat{f}^p(D))=\pi_1(\hat{f}^{p+l-1}(D))$$

for any $p\in\mathbb{N}$. Then, by the above, $f^{m+p}(c)\notin\bigcup\limits_{i>l}\pi_i(D)$ for any $p\in\mathbb{N}\cup\{0\}$. Note also that $\{f^i(c)|0\leq i\leq m\}\cap\Big(\bigcup\limits_{i>l}\pi_i(D)\Big)=\emptyset$, then $\Big(\bigcup\limits_{i>l}\pi_i(D)\Big)\cap\mathrm{Orb}_f(c)=\emptyset$.

Lemma 3.3.[2,p.131] *Suppose that f is a continuous map on the unit interval $[0,1]$, $y,z\in\overline{P}(f)$ with $y<z$ and $[y,z]\cap P(f)=\emptyset$. If one of the points y,z is regularly recurrent, then the other is not recurrent.*

Proof of Proposition 3.1. Suppose it is not true, then $\mathrm{Order}(G,f)=\omega_0+1$. We claim that there is $D\in\mathcal{D}'_{\omega_0}$ such that

(i) $\pi_1(D)\bigcap\Big[\mathrm{Br}(f)\bigcup\Big[\bigcup\limits_{c\in T(f)}\mathrm{Orb}_f(c)\Big]\Big]=\emptyset$, and $\pi_1(D)$ is homeomorphic to $[0,1]$;

(ii) $\mathrm{End}(\pi_1(D))\subset R(f)\cap\overline{P(f)}$.

Fix $D_0\in\mathcal{D}'_{\omega_0}$. Since $\mathrm{Br}(f)$ is finite and $\{\pi_i(D_0)|i\geq 1\}$ is pairwise disjoint, there is n_1 such that $\pi_n(D_0)\cap\mathrm{Br}(f)=\emptyset$ when $n\geq n_1$. By Lemma 3.2, there is $n_2\geq n_1$ such that $\pi_n(D_0)\bigcap\Big[\bigcup\limits_{c\in T(f)}\mathrm{Orb}_f(c)\Big]=\emptyset$ when $n\geq n_2$. Let $D=\hat{f}^{-n_2}(D_0)$. Then $\pi_1(D)$ satisfies (i) in the above.

By the definition of $\mathcal{D}'_{\omega_0}$, we have $A_i\in\mathcal{D}'_i(i>0)$ such that $A_{i+1}\subset A_i$ and $\bigcap\limits_{i\geq 1}A_i=D$. By d_H we denote the Hausdorff metric on 2^G. By Lemma 2.3, we have $\lim\limits_{i\to\infty}d_H(\pi_1(A_i),\pi_1(D))=0$. Note that $\pi_1(D)$ is homeomorphic to the unit interval $[0,1]$ and $d(\pi_1(D),\mathrm{Br}(f))=\delta>0$, then there is $N\in\mathbb{N}$ such that $\pi_1(A_i)$ is also homeomorphic to $[0,1]$ when $i\geq N$. By Lemma 2.2(3), there is $m\in\mathbb{N}$ such that $f^m(\pi_1(A_N))=\pi_1(A_N)$. Hence $f^m|\pi_1(A_N)$ is a map on closed interval $\pi_1(A_N)$.

Let $\pi_1(D)=[a',b']$ and $\pi_1(A_{N+i})=[a_i,b_i]$. By Lemma 3.1,

$$\{x|x=a_i\ \text{ or }\ b_i,i\geq 1\}\subset P(f)\bigcup\Big[\bigcup_{c\in T(f)}\mathrm{Orb}_f(c)\Big].$$

Note that $\lim\limits_{i\to\infty}a_i=a'$, $\lim\limits_{i\to\infty}b_i=b'$ and $\pi_1(D)\cap\Big[\bigcup\limits_{c\in T(f)}\mathrm{Orb}_f(c)\Big]=\emptyset$. Then $\{a',b'\}\subset\overline{P(f)}\cup\Big[\bigcup\limits_{c\in T(f)}\omega(c,f))\Big]\subset R(f)$. Let $F=f^m|\pi_1(A_N)$, then F is a piecewise monotone map with topological entropy zero, and $R(F)=\overline{R(F)}=\overline{P(F)}$. Then

$$\{a',b'\}\subset R(f)\cap\pi_1(A_N)=R(f^m)\cap\pi_1(A_N)=R(F)=\overline{P(F)}\subset\overline{P(f)}.$$

Hence, D satisfies (ii) above. This ends the proof of the claim.

Since $R(f)\cap\overline{P(f)}=RR(f)$, we have

$$\{a',b'\}\subset RR(f)\cap\pi_1(A_N)\subset RR(f^m)\cap\pi_1(A_N)=RR(F). \qquad (*)$$

But, note the facts that $[a',b']\cap P(F)=\emptyset$ and $\{a',b'\}\subset\overline{P(F)}$, at most one of the points a',b' is regularly recurrent point of F. This contradicts $(*)$. Hence $\mathrm{Order}(G,f)\leq\omega_0$.

Proposition 3.2. *Suppose $f \in C_{\rm PM}(G,G)$ with topological entropy zero. If* Order$(G,f) \le \omega_0$ *then f is non-chaotic.*

To prove Proposition 3.2, we need Lemmas 3.4–3.7. In the following, by $\mathcal{K}(=\mathcal{K}(f,G))$ we denote the set of all periodic subgraphs of G which contain no periodic points of the graph map $f: G \to G$.

Lemma 3.4. *Suppose that $f: G \to G$ is a graph map.*

(1) *if $P(f) = \emptyset$, then f is non-chaotic;*

(2) *for each $K \in \mathcal{K}$, $f\Big|\Big(\bigcup\limits_{i=0}^{m-1} f^i(K)\Big)$ is non-chaotic, where m is the period of K.*

Proof. (1) By [3], we know that f is semi-conjugate to an irrational rotation r on the circle $\mathbb{S}^1$, and the semi-conjugate is monotone. That is, there is a continuous surjection $\eta: G \to \mathbb{S}^1$ such that $\eta \circ f = r \circ \eta$ and $\eta^{-1}(x')$ is connected for each $x' \in \mathbb{S}^1$.

Assume that f is chaotic and E is its chaotic set. Fix $x_1 \ne x_2 \in E$, and set $\eta(x_i) = z_i \in \mathbb{S}^1$. We first prove that $z_1 \ne z_2$. In fact, by the compactness of G, $\lim\limits_{n\to\infty} \mathrm{diam}[f^n(\eta^{-1}(z))] = 0$ for any $z \in \mathbb{S}^1$, then $z_1 = z_2$ implies $\lim\limits_{n\to\infty} d(f^n(x_1), f^n(x_2)) = 0$. This contradicts $x_1 \ne x_2 \in E$. Thus $z_1 \ne z_2$.

Since r is an irrational rotation, $d(r^n(z_1), r^n(z_2)) = \delta > 0$ for any $n \in \mathbb{N}$. However, by the definition of chaotic set E, $\liminf\limits_{n\to\infty} d(f^n(x_1), f^n(x_2)) = 0$. It implies $\liminf\limits_{n\to\infty} d(r^n(z_1), r^n(z_2)) = 0$ by the continuity of η. This is a contradiction.

(2) By (1), we have that $f^m|[f^i(K)]$ is non-chaotic. By the definition of chaos, it is easy to check that $f\Big|\Big(\bigcup\limits_{i=0}^{m-1} f^i(K)\Big)$ is non-chaotic.

Lemma 3.5. *Suppose $f \in C_{\rm PM}(G,G)$ with topological entropy zero.*

(1) *If $P(f) \ne \emptyset$ and $\omega(x,f)$ is infinite, then $\omega(x,f) \subset \pi_1(\cup \mathcal{D}'_1)$.*

(2) *If $\omega(x,f)$ is infinite and* $\mathrm{Orb}_f(x) \cap (\cup\mathcal{K}) = \emptyset$, *then $\omega(x,f) \subset \pi_1(\cup\mathcal{D}'_i)$ for any $i \in \mathbb{N}$.*

Proof. (1) Since the homeomorphic image of a layer of (G,f) is still a layer, we see that $\cup\mathcal{D}'_1$ and $(G,f) \setminus \cup\mathcal{D}'_1$ are both complete invariable under $\hat{f}$. Hence $f(\pi_1(\cup\mathcal{D}'_1)) = \pi_1 \circ \hat{f}(\cup\mathcal{D}'_1) = \pi_1(\cup\mathcal{D}'_1)$. Then we need only to prove that, for any $y \in G \setminus \pi_1(\cup\mathcal{D}'_1)$, $\omega(y,f)$ is finite.

By Theorem B there are a finite graph $\mathcal{G}$, a map $g: (G,f) \to \mathcal{G}$ and a homeomorphism $\varphi: \mathcal{G} \to \mathcal{G}$, such that $\varphi \circ g = g \circ \hat{f}$ and $g^{-1}(t)$ is a maximal nowhere dense subcontinuum (layer) of (G,f) for each $t \in \mathcal{G}$.

Fix $\underline{y} \in (G,f)$ such that $\pi_1(\underline{y}) = y \in G \setminus \pi_1(\cup\mathcal{D}'_1)$. Then $\underline{y}$ belongs to $(G,f) \setminus (\cup\mathcal{D}'_1)$ (otherwise, $y \in \cup\pi_1(\mathcal{D}'_1)$, a contradiction). Since $\omega(\underline{y}, \hat{f})$ is precisely the inverse limit space $(\omega(y,f), f)$ (see [4]), we have $\omega(y,f) = \pi_1(\omega(y,f),f) = \pi_1(\omega(\underline{y},\hat{f}))$. Hence, it is sufficient to prove that $\omega(\underline{y},\hat{f})$ is finite. Since φ is a homeomorphism and $P(\varphi) \ne \emptyset$, we see that $\omega(g(\underline{y}),\varphi) \subset \Omega(\varphi) = P(\varphi)$ is finite. Note that $g|[(G,f)\setminus\cup\mathcal{D}'_1]$ is a homeomorphism, then $\omega(\underline{y},\hat{f}) = g^{-1}[\omega(g(\underline{y}),\varphi)]$ is finite. This ends the proof of (1).

(2) Suppose that $\omega(x,f)$ is infinite and $\mathrm{Orb}_f(x) \cap (\cup\mathcal{K}) = \emptyset$. By (1), we assume that for $1 \le i \le k$, we have $\omega(x,f) \subset \pi_1(\cup\mathcal{D}'_i)$. We will show that $\omega(x,f) \subset \pi_1(\cup\mathcal{D}'_{k+1})$.

Since $\omega(x,f) \subset \pi_1(\cup\mathcal{D}'_k)$, there are $D \in \mathcal{D}'_k$ and $m = m(D) \in \mathbb{N}$ such that $f^m(\pi_i(D)) = \pi_i(D)$ $(\forall i \in \mathbb{N})$ and $\omega(x,f) \subset \bigcup\limits_{i=1}^{m-1} \pi_i(D)$. Note that $\omega(x,f) = \bigcup\limits_{i=0}^{m-1} \omega(f^i(x), f^m)$. Without loss of generality, we assume that $\omega(x,f^m) \subset \pi_1(D)$. Let $F = f^m$. Then $F \in C_{\rm MP}(\pi_1(D), \pi_1(D))$ with topological entropy zero. Since $\mathrm{Orb}_f(x) \cap (\cup\mathcal{K}) = \emptyset$, we have $P(F|(\pi_1(D)) \ne \emptyset$. Replacing f and G by F and $\pi_1(D)$ respectively in (1), we have that

$\omega(x,F) \subset \pi_1[\cup\mathcal{D}'_1(\pi_1(D),F)]$. Note that the inverse limit space $(\pi_1(D),F)$ is homeomorphic to D, then

$$\omega(x,f^m) \subset \pi_1[\cup\mathcal{D}'_1((\pi_1(D),F))] \subset \pi_1[\cup\mathcal{D}'_1(D)] \subset \pi_1(\cup\mathcal{D}'_{k+1}).$$

Similarly, we have $\omega(f^i(x),f^m) \subset \pi_1(\cup\mathcal{D}'_{k+1})(1 \le i \le m-1)$. Hence $\omega(x,f) \subset \pi_1(\cup\mathcal{D}'_{k+1})$.

Lemma 3.6. *Suppose $f \in C_{\mathrm{PM}}(G,G)$ with topological entropy zero. If* $\mathrm{Order}(G,f) \ge \omega_0$, *then there is $N \in \mathbb{N}$ such that when $n \ge N$ and $A \in D'_n$, $\pi_1(A) \cap P(f) \ne \emptyset$.*

Proof. Let $\mathcal{M} = \{\pi_1(A)|A \in \mathcal{D}'_i,\ i \ge 0\}$. It is sufficient to prove that $\mathcal{M} \cap \mathcal{K}$ is finite. If $\pi_1(A) \in \mathcal{M} \cap \mathcal{K}$, then $\pi_1(A)$ contain at least one simple closed curve by Lemma 2.2(3). Moreover, if $A_i \in \mathcal{D}'_{m_i}$ ($i \in 1,2$ and $m_1 < m_2$) and $\pi_1(A) \in \mathcal{M}\cap\mathcal{K}$, then $\pi_1(A_1)\cap\pi_1(A_2) = \emptyset$ by Lemma 2.1. Thus the conclusion holds the fact that G contains only finite many simple closed curves.

Suppose that $f : X \to X$ is a continuous map on metric space X. The set of approximately periodic point of f, written by $\mathrm{App}(f)$, is definited by: $x \in \mathrm{App}(f)$ if and only if for any $\varepsilon > 0$ given, there are $p \in P(f)$ and $N \in \mathbb{N}$ such that $d(f^n(x),f^n(p)) < \varepsilon$ for every $n > N$. Similarly with the proof of Lemma 28 in [2, p.144], we have

Lemma 3.7.[2] *If f is a continuous map on metric space X, then*

$$\liminf_{n\to\infty} d(f^n(x),f^n(y)) > 0 \ \text{or}\ \lim d(f^n(x),f^n(y)) = 0 \quad (\forall x,y \in \mathrm{App}(f)).$$

Proof of Proposition 3.2. By Lemma 3.4, $f|(\cup\mathcal{K})$ is non-chaotic. Hence, $f|(\mathrm{Orb}^-_f(\mathcal{K}))$ is also non-chaotic, where $\mathrm{Orb}^-_f(\mathcal{K}) = \bigcup_{i\ge0} f^{-i}(\cup\mathcal{K})$.

We will show that $G\backslash\mathrm{Orb}^-_f(\mathcal{K}) \subset \mathrm{App}(f)$. Obviously, if $\omega(x,f)$ is finite, then $x \in \mathrm{App}(f)$. In the following, assume that $x \in G \setminus \mathrm{Orb}^-_f(\mathcal{K})$ and $\omega(x,f)$ is infinite.

Fix $x_1 \in \omega(x,f)$, let $\underline{x} = (x_1x_2\cdots)$, where $x_{i+1} = (f|\omega(x,f))^{-1}(x_i)$ $(i \in \mathbb{N})$. By Lemma 3.5, $\omega(x,f) \subset \bigcap_{i\ge0}(\pi_1(\cup\mathcal{D}'_i))$. Then there is $D_i \in \mathcal{D}'_i$ such that $D_{i+1} \subset D_i$ and $\underline{x} \in \bigcap_{n\ge1} D_i$. For any $n \in \mathbb{N}$, by M_n denote the period of periodic subgraph $\pi_1(D_n)$ under f. Let $\mathcal{T}_n = \{J^i_n \subset G|J^i_n = \pi_i(D_n), 1 \le i \le M_n\}$ and $\alpha_n = \max\{\mathrm{diam}(J)|J \in \mathcal{T}_n\}$. We claim that (1) $\alpha_{n+1} \le \alpha_n(\forall n \in \mathbb{N})$, and (2) $\lim_{n\to\infty} \alpha_n = 0$.

The item (1) can be easily checked by the fact that $\pi_k(D_{n+1}) \subset \pi_k(D_n)$ for every $k \in \mathbb{N}$. To prove the item (2), we assume on the contrary that $\lim_{n\to\infty} \alpha_n = \varepsilon > 0$. Then, for every $n \in \mathbb{N}$, $\mathcal{M}_n = \{J \in \mathcal{T}_n|\mathrm{diam}(J) \ge \varepsilon\} \ne \emptyset$. Furthermore, we have that $\cup\mathcal{M}_{n+1} \subset \cup\mathcal{M}_n$. In fact, for any $J_{n+1} \in \mathcal{M}_{n+1}$, there is $J_n \in \mathcal{T}_n$ such that $J_{n+1} \subset J_n$; moreover, $J_n \in \mathcal{M}_n$ by the fact that $\mathrm{diam}(J_{n+1}) \le \mathrm{diam}(J_n)$. Hence $\bigcap_{n\ge1}(\cup\mathcal{M}_n) \ne \emptyset$. Thus there is $J_n \in \mathcal{M}_n$ for every $n \in \mathbb{N}$ such that $J_{n+1} \subset J_n$ and $\bigcap_{n\ge1} J_n \ne \emptyset$. Since $J_n \in \mathcal{M}_n \subset \mathcal{T}_n$, there is $n' \in \mathbb{N}$ such that $J_n = \pi_{n'+1}(D_n) = \pi_1 \circ \hat{f}^{-n'}(D_n)$. Let $C_n = \hat{f}^{-n'}(D_n)$. Then $C_n \in \mathcal{D}'_n$ by the fact that $\hat{f}$ is a homeomorphism. Let $\delta = [2(1+\mathrm{diam}(G))]^{-1}$. Then

$$\begin{aligned}\mathrm{diam}\bigcap_{n\ge1} C_n &= \mathrm{diam}\Big[\bigcap_{n\ge1}\hat{f}^{-n'}(D_n)\Big] \ge \delta\cdot\mathrm{diam}\Big[\pi_1\Big(\bigcap_{n\ge1}\hat{f}^{-n'}(D_n)\Big)\Big]\\ &= \delta\cdot\mathrm{diam}\Big[\bigcap_{n\ge1}\pi_1(\hat{f}^{-n'}(D_n))\Big] = \delta\cdot\mathrm{diam}\Big[\bigcap_{n\ge1} J_n\Big] \ge \delta\cdot\varepsilon > 0,\end{aligned}$$

which contradicts $\mathrm{Order}(G,f) \le \omega_0$. This ends the proof of the item (2).

By Lemma 3.6 and the claim above, for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\max\{\mathrm{diam}(J)| J \in \mathcal{T}_n\} < \varepsilon$ and $(\cup\mathcal{T}_n) \cap P(f) \ne \emptyset$ when $n \ge n_0$. Since $x_1 \in \omega(x,f) \subset \cup\mathcal{T}_{n_0}$, there is $N \in \mathbb{N}$

such that $\{f^n(x)|n > N\} \subset \cup\mathcal{T}_{n_0}$. Let $p \in [\cup\mathcal{T}_{n_0}] \cap P(f)$, then $d(f^n(x), f^n(p)) < \varepsilon$ when $n > N$. That is, $x \in \mathrm{App}(f)$. By Lemma 3.7, $f|[G \setminus \mathrm{Orb}_f^-(\mathcal{K})]$ is also non-chaotic. Hence, f is non-chaotic.

Lemma 3.8.[2] *A continuous map $f : [0,1] \to [0,1]$ with topological zero is chaotic if and only if $R(f) \neq RR(f)$.*

Proposition 3.3. *Suppose $f \in C_{\mathrm{PM}}(G,G)$ with topological entropy zero. If $R(f)\cap\overline{P(f)} \neq RR(f)$, then f is chaotic.*

Proof. Given $x \in [R(f) \cap \overline{P(f)}] \setminus RR(f)$. Then $\omega(x,f)$ is infinite, otherwise, $x \in \omega(x,f) \subset P(f) \subset RR(f)$, and this is a contradiction. Moreover, $\omega(x,f) \subset G \setminus \mathrm{Orb}_f^-(\mathcal{K})$. Fix $x_1 \in \omega(x,f)$, and let $\underline{x} = (x_1x_2\cdots)$, where $x_{i+1} \in (f|\omega(x,f))^{-1}(x_i)$. By Lemma 3.5, there are $D_{i+1} \subset D_i \in \mathcal{D}'_i$ such that $\underline{x} \in \bigcap\limits_{i\geq 1} D_i$. For each $n \in \mathbb{N}$, let m_n be the period of $\pi_1(D_n)$ under f and

$$\mathcal{T}_n = \{J_n^i \subset G | J_n^i = \pi_i(D_n), 1 \leq i \leq m_n\}.$$

Then there are $n, i \in \mathbb{N}$ such that J_n^i is contained in some edge of G (see [7, Lemma 4.6]). Namely J_n^i is homeomorphic to $[0,1]$. Note that $f^{m_n} \in C_{MP}(J_n^i, J_n^i)$ with topological entropy zero and $R(f^{m_n}|J_n^i) \neq RR(f^{m_n}|J_n^i)$ (since $\omega(x,f) \subset \cup\mathcal{T}_n$ ($\forall n \in \mathbb{N}$) and

$$f(RR(f)) = RR(f) = RR(f^n)).$$

Then, by Lemma 3.8, $f^{m_n}|J_n^i$ is chaotic, and so is f.

Proof of Theorem 1.1. By Propositions 3.1–3.3, (1), (2) and (3) imply respectively (2), (3) and (1).

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