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NOTES ON GLAISHER'S CONGRUENCES**

HONG Shaofang*

Abstract

Let p be an odd prime and let $n \geq 1, k \geq 0$ and r be integers. Denote by B_k the k-th Bernoulli number. It is proved that (i) If $r \geq 1$ is odd and suppose $p \geq r+4$, then $\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv -\frac{(2n+1)r(r+1)}{2(r+2)} B_{p-r-2} p^2 \pmod{p^3}$. (ii) If $r \geq 2$ is even and suppose $p \geq r+3$, then $\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv \frac{r}{r+1} B_{p-r-1} p \pmod{p^2}$. (iii) $\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}} \equiv -(2n+1)p \pmod{p^2}$. This result generalizes the Glaisher's congruence. As a corollary, a generalization of the Wolstenholme's theorem is obtained.

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§1. Introduction

Several authors (see [2,pp.95-103]) have studied the sums

$$\sum_{\substack{j=1 \\ (j,p)=1}}^{np} \frac{1}{j^r} \tag{1.1}$$

modulo powers of the prime p, especially in the cases where $r=1$ or $n=1$. The well-known Wolstenholme's theorem (see [5]) asserts that if $p \geq 5$ is prime, then

$$\sum_{j=1}^{p-1} \frac{1}{j} \equiv 0 \pmod{p^2}.$$

Define the Bernoulli numbers $B_k (k=0,1,2,\cdots)$ by the series

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{1.2}$$

Glaisher in 1900 found the following strengthened congruences.

Theorem A(see [3],[4], or [7]). *Let r be an integer and let p be an odd prime.*

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*Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.
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(i) *If* $r \geq 1$ *is odd and suppose* $p \geq r+4$, *then*

$$\sum_{j=1}^{p-1} \frac{1}{j^r} \equiv -\frac{r(r+1)}{2(r+2)} B_{p-r-2} p^2 \pmod{p^3}.$$

(ii) *If* $r \geq 2$ *is even and suppose* $p \geq r+3$, *then*

$$\sum_{j=1}^{p-1} \frac{1}{j^r} \equiv \frac{r}{r+1} B_{p-r-1} p \pmod{p^2}.$$

(iii)

$$\sum_{j=1}^{p-1} \frac{1}{j^{p-2}} \equiv -p \pmod{p^2}.$$

Boyd[1] gave an explicit p-adic expansion of the sum (1.1) in the case $r = 1$. Recently, Washington[8] obtained an explicit p-adic expansion of the sum (1.1) as a power series in n and the coefficients are values of p-adic L functions (see Theorem B).

In the present paper we will generalize the Glaisher's results by using the Washington's p-adic expansion of the sum (1.1).The main result in this paper is as follows:

Theorem 1.1. *Let* p *be an odd prime and let* $n \geq 0$ *and* r *be integers.*

(i) *If* $r \geq 1$ *is odd and suppose* $p \geq r+4$, *then*

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv -\frac{(2n+1)r(r+1)}{2(r+2)} B_{p-r-2} p^2 \pmod{p^3}.$$

(ii) *If* $r \geq 2$ *is even and suppose* $p \geq r+3$, *then*

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv \frac{r}{r+1} B_{p-r-1} p \pmod{p^2}.$$

(iii)

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}} \equiv -(2n+1)p \pmod{p^2}.$$

If let $n = 0$, then Theorem 1.1 becomes Theorem A.

§2. Preliminaries on *P*-adic *L* Functions

Let p be a prime and let $L_p(s, \chi)$ be the p-adic function attached to a character χ. In this section we introduce some facts about p-adic-valued L functions.

Let ω be the p-adic-valued Teichmuller character, so $\omega(a) \equiv a \pmod{p}$ and $\omega(a)^p = \omega(a)$ when $p \geq 3$. If $p \dagger a$, let $\langle a \rangle = a/\omega(a)$. If $x \in \mathbf{Z}_p$(= the ring of the p-adic integers), let $\binom{x}{k} = (x)(x-1)\cdots(x-k+1)/k!$. When p is odd, or when $p = 2$ and $\omega^t = 1$, the p-adic L function for the character ω^t satisfies

$$L_p(s, \omega^t) = \frac{1}{s-1} \frac{1}{p} \sum_{a=0}^{p-1} \omega(a)^t \langle a \rangle^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} (B_j) \Big(\frac{p}{a}\Big)^j$$

for $s \in \mathbf{Z}_p$. This is a p-adic analytic function. In order to prove Theorem 1.1, we need the following results.

Lemma 2.1.[9] (i) *If t is odd, then $L_p(s,\omega^t)$ is identically 0;*
(ii) *If $t \not\equiv 0 \pmod{p-1}$, then for all $s \in \mathbf{Z}_p, L_p(s,\omega^t) \in \mathbf{Z}_p$;*
(iii) *If $t \not\equiv 0 \pmod{p-1}$, then for all $s_1, s_2 \in \mathbf{Z}_p$, we have*

$$L_p(s_1,\omega^t) \equiv L_p(s_2,\omega^t) \pmod{p};$$

(iv) *If $1 \le k \equiv t \pmod{p-1}$, then*

$$L_p(1-k,\omega^t) = -\frac{1-p^{k-1}}{k}B_k.$$

Lemma 2.2.[8] *Assume $p \ge 5, p \ge r$, and $k \ge 3$. If either $r \ne p-3$ or $k \ne 3$, then $L_p(r+k,\omega^{1-k-r})p^k \equiv 0 \pmod{p^3}$. In the case $r = p-3$ and $k=3$, we have $L_p(p,1)p^3 \equiv p^2 \pmod{p^3}$.*

§3. Proof of the Main Result

In order to prove our main result, we need the following p-adic expansion of the sum (1.1) as a power series in n.

Theorem B.[8] *Let p be an odd prime and let $n, r \ge 1$ be integers. Then*

$$\sum_{\substack{j=1\\(j,p)=1}}^{np} \frac{1}{j^r} = -\sum_{k=1}^{\infty} \binom{-r}{k} L_p(r+k,\omega^{1-k-r})(pn)^k.$$

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem A, we only need to consider the case $n \ge 1$.In the following let $n \ge 1$. Clearly we have

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} = \sum_{\substack{j=1\\(j,p)=1}}^{(n+1)p} \frac{1}{j^r} - \sum_{\substack{j=1\\(j,p)=1}}^{np} \frac{1}{j^r}.$$

It then follows from Theorem B that

$$\begin{aligned}\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} &= -\sum_{k=1}^{\infty} \binom{-r}{k} L_p(r+k,\omega^{1-k-r})(p(n+1))^k \\ &\quad + \sum_{k=1}^{\infty} \binom{-r}{k} L_p(r+k,\omega^{1-k-r})(pn)^k \\ &= \sum_{k=1}^{\infty} \binom{-r}{k} L_p(r+k,\omega^{1-k-r})p^k(n^k-(n+1)^k). \qquad (3.1)\end{aligned}$$

(i) Let $r \ge 1$ be odd and suppose $p \ge r+4$. Since $r \le p-4$, by Lemma 2.2 we have that for $k \ge 3$, $L_p(r+k,\ \omega^{1-k-r})p^k \equiv 0 \pmod{p^3}$. Note that r is odd. By Lemma 2.1(i) the summand for $k=1$ vanishes in Equation (3.1). Therefore

$$\begin{aligned}\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} &\equiv \binom{-r}{2} L_p(r+2,\omega^{-1-r})p^2(n^2-(n+1)^2)) \\ &\equiv -\frac{(2n+1)r(r+1)}{2} L_p(r+2,\omega^{-1-r})p^2 \pmod{p^3}. \qquad (3.2)\end{aligned}$$

Using Lemma 2.1(iii)(note that $1+r \not\equiv 0 \pmod{p-1}$), we have

$$L_p(r+2,\omega^{-1-r}) \equiv L_p(r+2-p+1,\omega^{-1-r}) \pmod{p}. \qquad (3.3)$$

By Lemma 2.1(iv) we have

$$L_p(r+2-p+1,\omega^{-1-r}) = -\frac{1-p^{p-r-3}}{p-r-2}B_{p-r-2}. \tag{3.4}$$

It can be deduced from Equations (3.2)–(3.4) that

$$\sum_{j=1}^{p-1}\frac{1}{(np+j)^r} \equiv \frac{(2n+1)r(r+1)}{2(p-r-2)}(1-p^{p-r-3})B_{p-r-2}p^2 \pmod{p^3}.$$

Since $\frac{1}{p-r-2} \equiv -\frac{1}{r+2} \pmod p$, we have that

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{1}{(np+j)^r} &\equiv -\frac{(2n+1)r(r+1)}{2(r+2)}(1-p^{p-r-3})B_{p-r-2}p^2\\ &\equiv -\frac{(2n+1)r(r+1)}{2(r+2)}B_{p-r-2}p^2 \pmod{p^3}\end{aligned}$$

as desired.

(ii) Let $r \ge 2$ be even and suppose $p \ge r+3$. By Lemma 2.2 we have that for $k \ge 3$, $L_p(r+k,\omega^{1-k-r})p^k \equiv 0 \pmod{p^2}$. Since r is even, by Lemma 2.1(i) the summand for $k=2$ vanishes in Equation (3.1). By Lemma 2.1(iii) and (iv), it follows from Equation (3.1) that

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{1}{(np+j)^r} &\equiv \binom{-r}{1}L_p(r+1,\omega^{-r})p(n-(n+1))\\ &\equiv rL_p(r+1,\omega^{-r})p\\ &\equiv rL_p(r+1-p+1,\omega^{-r})p\\ &\equiv -r(1-p^{p-r-2})\frac{B_{p-r-1}}{p-r-1}p\\ &\equiv \frac{r}{r+1}B_{p-r-1}p \pmod{p^2}.\end{aligned}$$

(iii) Let $r = p-2$. Then for $k = 1$, $1-k-r$ is odd. Thus $L_p(r+k,\omega^{1-k-r}) = 0$. For $k \ge 3$, by Lemma 2.2 we have that $L_p(r+k,\omega^{1-k-r})p^k \equiv 0 \pmod{p^2}$. For $k=2$, from Lemma 2.2 we deduce that $L_p(p,1)p^2 \equiv p \pmod{p^2}$. Then it follows from Equation (3.1) that

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{1}{(np+j)^{p-2}} &\equiv \binom{-p+2}{2}L_p(p,1)p^2(-2n-1)\\ &\equiv -\frac{(2n+1)(p-1)(p-2)}{2}p\\ &\equiv -(2n+1)p \pmod{p^2}.\end{aligned}$$

The proof is complete.

§4. Corollaries

In the present section, we give some corollaries of the main result.

Corollary 4.1. *Let p be an odd prime and let $n \ge 0$ and $r \ge 1$ be integers. Suppose that r is odd and $p \ge r+4$. Then the following congruences hold:*

(i)

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv 0 \pmod{p^2}.$$

(ii)

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^{r+1}} \equiv 0 \pmod{p}.$$

(iii)

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}} \equiv 0 \pmod{p}.$$

Proof. By the von Staudt-Clausen Theorem (see [6,9]), we have

$$B_{p-r-2} + \sum_{\substack{(l-1)|(p-r-2) \\ l \text{ prime}}} \frac{1}{l} \in \mathbf{Z}. \tag{4.1}$$

Since $p \geq r+4$, we have $\frac{1}{l} \in \mathbf{Z}_p$ for all $1 \leq l \leq p-r-2$. Then it follows from Equation (4.1) that $B_{p-r-2} \in \mathbf{Z}_p$. Note that $p \geq r+4$ implies $\frac{1}{r+2} \in \mathbf{Z}_p$. Thus the result follows from Theorem 1.1. This completes the proof.

Remark 4.1. If let $n = 0$ and $r = 1$, then Corollary 4.1(i) reduces to the Wolstenholme's theorem (see [5]).

Lemma 4.1.[6] *Let m be even and p a prime such that $(p-1) \dagger m$. Let $S_m(p) = 1^m + 2^m + \cdots + (p-1)^m$. Then $S_m(p) \equiv pB_m \pmod{p^2}$.*

Corollary 4.2. *Let p be an odd prime and let $n \geq 0$ and $r \geq 1$ be integers. Suppose that r is odd and $p \geq r+4$. Then each of the following is true:*

(i) $\sum\limits_{j=1}^{p-1} \frac{1}{(np+j)^{r+1}} \not\equiv 0 \pmod{p^2}$;

(ii) *If* $2n \equiv -1 \pmod{p}$, *then*

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv 0 \pmod{p^3}$$

and

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}} \equiv 0 \pmod{p^2};$$

(iii) *If* $2n \not\equiv -1 \pmod{p}$, *then*

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \not\equiv 0 \pmod{p^3}$$

and

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}} \not\equiv 0 \pmod{p^2}.$$

Proof. We claim that $B_{p-r-2} \not\equiv 0 \pmod{p}$. Otherwise we have $B_{p-r-2} \equiv 0 \pmod{p}$. Since $r \geq 1$ and $p \geq r+4$, we have $(p-1) \dagger (p-r-2)$. By Lemma 4.1 we have

$pB_{p-r-2} \equiv S_{p-r-2}(p) \pmod{p^2}$. Thus one deduces that

$$S_{p-r-2}(p) \equiv 0 \pmod{p^2}. \tag{4.2}$$

On the other hand, we have

$$\begin{aligned} 2^{p-r-2}S_{p-r-2}(p) &= 2^{p-r-2} + 4^{p-r-2} + \cdots + (p-1)^{p-r-2} \\ &\quad + (p+1)^{p-r-2} + (p+3)^{p-r-2} + \cdots + (p+(p-2))^{p-r-2} \\ &\equiv S_{p-r-2}(p) + p(p-r-2)(1+3+\cdots+(p-2)) \\ &\equiv S_{p-r-2}(p) + p(p-r-2)\Big(\frac{p-1}{2}\Big)^2 \pmod{p^2}. \end{aligned} \tag{4.3}$$

Thus Equations (4.2) and (4.3) imply that

$$p-r-2 \equiv 0 \pmod{p}. \tag{4.4}$$

Since $2 \le p-r-2 \le p-3$, Equation (4.4) does not hold and the assertion is true. Note that $r, r+1, \frac{1}{2}$ and $\frac{1}{r+2} \not\equiv 0 \pmod{p}$. Then the result follows from Theorem 1.1. The proof is complete.

For a ρ-adic integer $n \in \mathbf{Z}_p$, let $\mathrm{ord}_p n$ denote the integer m such that $p^m | n$ and $p^{m+1} \nmid n$. Combining Corollaries 4.1 and 4.2, we then have the following theorem.

Theorem 4.1. *Let p be an odd prime and let $n \ge 0$ and $r \ge 1$ be integers. Suppose that r is odd and $p \ge r+4$. Then each of the following is true:*

(i) $\mathrm{ord}_p\Big(\sum_{j=1}^{p-1} \frac{1}{(np+j)^{r+1}}\Big) = 1$;

(ii) *If* $2n \not\equiv -1 \pmod{p}$, *then* $\mathrm{ord}_p\Big(\sum_{j=1}^{p-1} \frac{1}{(np+j)^{r}}\Big) = 2$ *and* $\mathrm{ord}_p\Big(\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}}\Big) = 1$;

(iii) *If* $2n \equiv -1 \pmod{p}$, *then* $\mathrm{ord}_p\Big(\sum_{j=1}^{p-1} \frac{1}{(np+j)^{r}}\Big) \ge 3$ *and* $\mathrm{ord}_p\Big(\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}}\Big) \ge 2$.

Remark 4.2. By Theorem 4.1, one can see that the Wolstenholme's theorem is the best possible in the sense of power divisibled by p.

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