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A NEW PROPERTY OF BINARY UNDIRECTED de BRUIJN GRAPHS***

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Abstract

The authors obtain a new property of the n-dimensional binary undirected de Bruijn graph $UB(n)$ for $n \geq 4$, namely, there is a vertex x such that for any other vertex y there exist at least two internally disjoint paths of length at most $n-1$ between x and y in $UB(n)$. The result means that the $(n-1, 2)$-dominating number of $UB(n)$ is equal to one if $n \geq 4$.

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§1. Introduction

The n-dimensional binary directed de Bruijn graph, denoted by $B(n)$, has the vertex-set $\{x_1x_2\cdots x_n : x_i \in \{0,1\}\}$. There are two arcs from a vertex $x_1x_2\cdots x_n$ to the vertices $x_2x_3\cdots x_{n-1}x_n0$ and $x_2x_3\cdots x_{n-1}x_n1$. The n-dimensional binary undirected de Bruijn graph, denoted by $UB(n)$, is obtained from $B(n)$ by deleting the orientation of the arcs and then omitting multiple edges and loops.

It is well known that $UB(n)$ has diameter n and connectivity two. The de Bruijn graphs have been widely used in coding theory[6]. They have been also proposed as a possible good computer interconnection network for a parallel architecture[1,5] and received much attention. Many good properties have been found by several researchers. Some of them can be found in [1]. In particular, Li, Sotteau and Xu[3] have shown that there exist at least two internally disjoint paths of length at most n between any two vertices in $UB(n)$. In this paper, we obtain the following result.

Theorem. *If $x = 10\cdots01$ and $n \geq 4$, then for any vertex y other than x there exist at least two internally disjoint paths of length at most $n-1$ between x and y in $UB(n)$.*

Motivated by a problem of resoures sharing in a computer interconnection fault tolerant network of parallel architectures, Li and Xu[4] introduced a notion of (d, m)-dominating numbers as follows: For an m-connected graph G and a given integer d, a nonempty and proper subset S of the vertex set of G is called a (d, m)-dominating set of G if for any vertex y of G but not in S there are at least m internally disjoint paths of length at most d between y and some vertex in S. The parameter

$$s_{d,m}(G) = \min\{|S| : S \text{ is a } (d,m)\text{-dominating set of } G\}$$

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is called (d,m)-dominating number of G. This notion not only generalizes that of the classical dominating numbers of a graph but also gives a good measure of the problem of resoures sharing in fault tolerant networks. For instance, if an m-connected graph G is used to model a computer interconnection network of parallel architectures, then for a given integer d, a (d,m)-dominating set S of G can be taken as a set of computers, which collect, and store resoures or some information, and communicate with every other vertex in the network by m internally disjoint paths of length at most d. An important and practical problem is how to choose a (d,m)-dominating set S such that the number of vertices in S is as small as possible. Thus the (d,m)-dominating numbers in conjunction with other well-known parameters can provide a more accurate analysis of fault tolerance for reliability and efficiency of networks of parallel architectures.

In general, to determine the (d,m)-dominating number of a graph is NP-hard since its special case of $m=d=1$, the dominating number of the graph, is NP-hard[2]. Thus it is of interest to determine the (d,m)-dominating number of some well-known networks for some special values of m and d. The above-mentioned result by Li *et al* means that $s_{n,2}(UB(n))=1$. Our theorem means $s_{n-1,2}(UB(n))=1$ if $n\ge 4$.

§2. Two Key Lemmas

For any two nonadjacent vertices x and y in $B(n)$, since the shortest dipath from x to y is unique, we use $P[x,y]$ to denote the shortest dipath from x to y and $|P[x,y]|$ to denote its length, which is the number of arcs on $P[x,y]$. The following Lemma 2.1 can be easily obtained from Lemma 2.1 and Proposition 2.2 in [3].

Lemma 2.1. *Let $x=10\cdots01$. Then for any vertex $y\ne 00\cdots00$ in $B(n)$ we have*

(a) $|P[x,y]|<n$ *and* $|P[y,x]|<n$*;*

(b) *any closed directed walk that contains x is of length at least $n-1$;*

(c) $P[x,y]\cup P[y,x]$ *consists of at most two directed circuits.*

Lemma 2.2. *Let $x=10\cdots01$ and $y\ne 00\cdots00$ be any vertex other than x in $B(n)$, $n\ge 3$. If $P[x,y]$ intersects $P[y,x]$, and z and u, respectively, are the first and the last vertices that $P[x,y]$ has in common with $P[y,x]$ along the direction from x to y, then $|P[x,z]|=|P[u,x]|$.*

Proof. By Lemma 2.1(c) suppose that $P[x,y]\cup P[y,x]$ consists of two directed circuits $C_1=P[x,z]\cup P[z,u]\cup P[u,x]$ and $C_2=P[y,z]\cup P[z,u]\cup P[u,y]$. Let $|P[x,z]|=a, |P[z,u]|=r, |P[u,y]|=b, |P[y,z]|=c$ and $|P[u,x]|=d$. Then $a>0, r\ge 0, b>0, c>0, d>0$ and by Lemma 2.1(a)

$$a+r+b=|P[x,y]|\le n-1,\quad c+r+d=|P[y,x]|\le n-1. \tag{2.1}$$

It follows from Lemma 2.1(b) and (2.1) that

$$a+r+d=|C_1|\ge n-1,\quad c+r+b=|C_2|\le n-1. \tag{2.2}$$

Let $y=y_1y_2\cdots y_n$. Since C_2 is of length $c+r+b\le n-1$ by (2.2), $y_i=y_{i+c+r+b}$ for all $1\le i\le n-(c+r+b)$. Thus we can assume

$$y=y_1y_2\cdots y_{c+r+b}y_1y_2\cdots y_{c+r+b}\cdots y_1y_2\cdots y_{c+r+b}y_1y_2\cdots y_k, \tag{2.3}$$

where $n\equiv k\,(\mathrm{mod}\,(c+r+b)), 1\le k\le c+r+b$. For convenience we call $y_{c+1}y_{c+2}\cdots y_{c+r+b}$ $y_1y_2\cdots y_c$ a majorizing circular segment. Let β be the number of 0's that successively occur in the foremost part of the majorizing circular segment. In order to complete the proof of Lemma 2.2, with the notation given above we will prove $a=d$ by showing

$$n-a-1=\beta=n-d-1.$$

First we note that $d\ge b$ and $a\ge c$ by (2.1) and (2.2).

Since $|P[x,y]|=a+r+b$, the last $n-a-r-b$ coordinates of x overlap the first $n-a-r-b$ coordinates of y. Thus y can be written as

$$y=\underbrace{00\cdots01}_{n-a-r-b}\ y_{n-a-r-b+1}\cdots y_{n-1}y_n. \tag{2.4}$$

Also since $|P[y,x]| = c+r+d$, the last $n-c-r-d$ coordinates of y overlap the first $n-c-r-d$ coordinates of x. So y can also be written as

$$y = y_1 y_2 \cdots y_{c+r+d} \underbrace{100\cdots00}_{n-c-r-d}. \tag{2.5}$$

Consider the vertex z. Since z can be reached in a steps from x along the dipath $P[x,y]$, by (2.4) z can be written as

$$z = \underbrace{00\cdots00}_{r+b} \underbrace{0\cdots01}_{n-a-r-b} \underbrace{y_{n-a-r-b+1}\cdots y_{d-b+1}}_{a+r+d-n+1} \underbrace{y_{d-b+2}\cdots y_{n-r-b}}_{n-r-d-1}. \tag{2.6}$$

Also since z can be reached in c steps from y along the dipath $P[y,x]$, by (2.5) z can also be written as

$$z = \underbrace{y_{c+1}y_{c+2}\cdots y_{c+r+b}}_{r+b} \underbrace{y_{c+r+b+1}\cdots y_{n+c-a}}_{n-a-r-b} \underbrace{y_{n+c-a+1}\cdots y_{c+r+d}\ 1}_{a+r+d-n+1} \underbrace{00\cdots00}_{n-r-d-1}. \tag{2.7}$$

Noting $n+c-a-1 \geq c+r+b$ and comparing (2.6) with (2.7), we have

$$y_{c+1} = \cdots = y_{c+r+b} = y_{c+r+b+1} = \cdots = y_{n+c-a-1} = 0, \quad y_{n+c-a} = 1. \tag{2.8}$$

Noting that $(n+c-a-1)-c = n-a-1$ and $r+b \leq n-a-1 < c+r+b$, from (2.8) we have $\beta = n-a-1$ immediately.

In order to show that $\beta = n-d-1$, we consider the vertex u. Noting that u can be reached in $c+r$ steps from y along the dipath $P[y,x]$, by (2.5) we can write u as

$$u = \underbrace{y_{c+r+1}\cdots y_{c+r+d}}_{d} \underbrace{100\cdots00}_{n-c-r-d} \underbrace{00\cdots00}_{c+r}. \tag{2.9}$$

On the other hand, u can reach y in b steps along the dipath $P[x,y]$. Thus u can also be written as

$$u = \underbrace{y_{c+r+1}\cdots y_{c+r+d}}_{d} \underbrace{y_{c+r+d+1}\cdots y_n}_{n-c-r-d} \underbrace{y_{n-b-c-r+1}\cdots y_{n-b}}_{c+r}. \tag{2.10}$$

Comparing (2.9) with (2.10) and (2.6), we have

$$y_{d-b+1} = 1, \quad y_{d-b+2} = \cdots = y_{n-b} = 0. \tag{2.11}$$

Next, we want to prove that $n-d-1$ coordinates $y_{d-b+2}, \cdots, y_{n-b}$ of y are located in the foremost part of the majorizing circular segment and $y_{n-b+1} = 1$, by which we have $\beta = n-d-1$.

If $a = 1$ (since $a \geq 1$), then $r+d \geq n-2$ by (2.2) and so $c = 1$ by (2.1) (since $c \geq 1$). This implies that x and y are two distinct inneighbors of z. So $y_1 = 0$ since the first coordinate of x is equal to 1. It follows from (2.8) that the first $c+r+b$ coordinates of y are equal to 0 and so $y = 00\cdots00$ by (2.3), which contradicts our assumption of y. Therefore, $a > 1$. Let z_1 and z_2 be two inneighbors of z, respectively, on $P[x,y]$ and $P[y,x]$. Then z_1 and z_2 are different from each other. The first coordinate of z_1 is equal to 0 since $a > 1$. And so the first coordinate y_c of z_2 is equal to 1, i.e., $y_c = 1$.

Similarly, we can prove $d > 1$ and $y_{n-b+1} = 1$.

Since $y_c = 1 = y_{d-b+1}$ and $c \leq n-r-b-1$ by (2.1), we have $c \leq d-b+1$ from (2.6) and (2.7). We claim that $d-b+1 \equiv c \,(\mathrm{mod}\, c+r+b)$. Indeed, let $d-b+1 = c+q+p(c+r+b)$, where q and p are two nonnegative integers, and $q \leq r+b$. Then $y_{c+q} = y_{d-b+1} = 1 = y_c$. But $y_{c+q} = 0$ from (2.8) if $0 < q \leq r+b$. This is a contradiction. Thus $d-b+1 = c+p(c+r+b)$. From (2.11) we have $n-d \leq c+r+b$; otherwise $y = 00\cdots00$. It follows that $d-b < n-b \leq c+r+d$. Note that $c+r+d-(d-b) = c+r+b,\ d-b+1 = c+p(c+r+b), y_{n-b+1} = 1$. $y_{d-b+2}, \cdots, y_{n-b}$ are located in the foremost part of the majorizing circular segment. Thus $\beta = n-d-1$. The proof of Lemma 2.2 is complete.

§3. Proof of the Theorem

Let $x = 100\cdots001$. We prove the theorem by exhibiting two undirected paths P_1 and P_2 between x and any vertex y other than x in $UB(n)$ which are internally disjoint and of length at most $n-1$. Let y be any vertex other than x in $B(n)$.

We first suppose $P[x,y]$ and $P[y,x]$ are internally joint in $B(n)$. Let z and u, respectively, be the first and the last vertices that $P[x,y]$ has in common with $P[y,x]$ along the direction from x to y.

If $y = 00\cdots00$, then it can be directly vertified that $z = 00\cdots010$ and $u = 010\cdots00$, which implies

$$|P[x,z]| = |P[u,x]| = 1,\ |P[u,y]| = |P[y,z]| = 2,\ |P[z,u]| = n-3. \tag{3.1}$$

Let $P_1 = P[x,z] \cup P[y,z]$ and $P_2 = P[u,x] \cup P[u,y]$. Then P_1 and P_2 are internally disjoint in $UB(n)$ since $n \geq 4$, and both are of length three by (3.1).

We suppose $y \neq 00\cdots00$ below. Then by Lemma 2.1(c) $P[x,y] \cup P[y,x]$ consists of two directed circuits, and by Lemma 2.2

$$|P[x,z]| = |P[u,x]|. \tag{3.2}$$

If $z \neq u$, then let $P_1 = P[x,z] \cup P[y,z]$ and $P_2 = P[u,x] \cup P[u,y]$. P_1 and P_2 are internally disjoint in $UB(n)$. By (3.2) and Lemma 2.1(a) P_1 and P_2 are of length

$$|P_1| = |P[y,z]| + |P[x,z]| = |P[y,z]| + |P[u,x]| = |P[y,x]| - |P[z,u| < n-1,$$
$$|P_2| = |P[u,x]| + |P[u,y]| = |P[x,z]| + |P[u,y]| = |P[x,y]| - |P[z,u| < n-1.$$

If $z = u$, then let u_1 and u_2 be the two outneighbors of u on $P[x,y]$ and $P[y,x]$, respectively. Then there is a vertex v other than u in $B(n)$ such that u_1 and u_2 are its two outneighbors. It can be easily vertified that v is not on $P[x,y] \cup P[y,x]$ by the shortness of $P[x,y]$ and $P[y,x]$. Let

$$P_1 = P[x,z] \cup P[y,z] \text{ and } P_2 = P[u_1,y] + (v,u_1) + (v,u_2) + P[u_2,x].$$

P_1 and P_2 are internally disjoint in $UB(n)$ and of length

$$|P_1| = |P[y,u]| + |P[x,u]| = |P[y,u]| + |P[u,x]| = |P[y,x]| \leq n-1,$$
$$|P_2| = |P[u,x]| + |P[u,y]| = |P[x,u]| + |P[u,y]| = |P[x,y]| \leq n-1.$$

Next we suppose that $P[x,y]$ and $P[y,x]$ are internally disjoint in $B(n)$, then $y \neq 00\cdots00$. Let $P_1 = P[x,y]$ and $P_2 = P[y,x]$. Then P_1 and P_2 are of length at most $n-1$ by Lemma 2.1(a). The closed directed walk $P[x,y] \cup P[y,x]$ in $B(n)$ is of length at least $n-1 \geq 3$ by Lemma 2.1(b) since $n \geq 4$. This means that P_1 and P_2 are internally disjoint in $UB(n)$. The proof of the theorem is complete.

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