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POSITIVE PERIODIC SOLUTIONS OF PREDATOR-PREY SYSTEMS WITH INFINITE DELAY**

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Abstract

Conditions for existence of positive periodic solutions of nonautonomous, nonconvoluton type predator-prey system with infinite delay are given.

Keywords Positive periodic solutions, Predator-prey system, Infinite delay
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§1. Introduction

Predator-prey systems are one kinds of ecological systems, they have attracted much attention in recent years. But up to now, most of the known results are concerned with systems without time delay or with finite delay[1−6]. Infinite delay will cause much more trouble than finite delay does, and corresponding results are much fewer[7−9].

In this paper, using the persistence conclusions of [9], we introduce a new argument; without utilizing complicated phase space theory, changing the periodic solution problem of infinite delay to the one of finite delay, new and brief results are obtained.

We consider in this paper the non autonomous, non convolution type predator-prey systems with infinite delay of the form

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t) - c(t)y(t)], \\ y'(t) = y(t)\Big[- d(t) + \int_{-\infty}^{t} K(s,t,x(s),x(t))ds\Big]. \end{cases} \tag{1.1}$$

We assume that

(A) the function $K : R \times R \times R^+ \times R^+ \to R^+$ is measurable, and there is a continuous positive function $h : R^- \to R^+$ with $\int_{-\infty}^{0} h(s)ds = l < +\infty$, and

$$|K(s,t,x,y) - K(s,t,\overline{x},\overline{y})| \le h(s-t)\left(|x-\overline{x}| + |y-\overline{y}|\right), \tag{1.2}$$

and

$$K(s,t,0,0) \equiv 0, \quad K(s+\omega,t+\omega,x,y) \equiv K(s,t,x,y), \quad \omega > 0, \tag{1.3}$$

and the function $K(s,t,x,y)$ is increasing on $x$and y, respectively;

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(B) the functions $a,b,c,d: R\to R^+$ are all positive, continuous, ω-periodic.

We let

$$a^l=\inf_{t\in R}a(t),\quad b^l=\inf_{t\in R}b(t),\quad c^l=\inf_{t\in R}c(t),\quad d^l=\inf_{t\in R}d(t),$$
$$a^u=\sup_{t\in R}a(t),\quad b^u=\sup_{t\in R}b(t),\quad c^u=\sup_{t\in R}c(t),\quad d^u=\sup_{t\in R}d(t).$$

(C)

$$\int_{-\infty}^{t}K\Big(s,t,\frac{a^l}{b^u},\frac{a^l}{b^u}\Big)ds>d^u,\quad t\in R.$$

§2. Some Lemmas

Let $BC(I,S)$ denote the set of all bounded continuous functions defined on I and mapping onto $S\subset R^2$, and let

$$\operatorname{int}BC^+=\{\phi\in BC(R^-,R^2):\ \phi_i(s)>0\,,\ s\le 0,\ \ i=1,2\}.$$

Conditions (A) and (B) imply the existence and uniqueness of initial value problem to (1.1) [11].

For $\phi=(\phi_1,\phi_2)\in BC^+$, let $(x(\sigma,\phi)(t),\ y(\sigma,\phi)(t))$ denote the solution of (1.1) satisfying the initial value condition $x_\sigma=\phi_1,\ y_\sigma=\phi_2$.

It is obvious that $BC\subset BC_h$ (for the definition of BC_h , please refer to [9] p.322), so by Lemmas 1-8 of [9], we can easily get the following Lemmas 2.1-2.8.

Lemma 2.1. *If (A), (B) and (C) hold, then for any $(\sigma,\phi)\in R^+\times\operatorname{int}BC^+$ we have*

$$\limsup_{t\to+\infty}x(\sigma,\phi)(t)\le\frac{a^u}{b^l}.\tag{2.1}$$

Lemma 2.2. *If* (A), (B) *and* (C) *hold, then there is a $T>0$ independent of $(\sigma,\phi)\in R^+\times\operatorname{int}BC^+$ such that*

$$\limsup_{t\to+\infty}y(\sigma,\phi)(t)\le T.\tag{2.2}$$

Lemma 2.3. *If* (A), (B) *and* (C) *hold, then there is a $\xi>0$ independent of $(\sigma,\phi)\in R^+\times\operatorname{int}BC^+$ such that*

$$\liminf_{t\to+\infty}x(\sigma,\phi)(t)\ge\xi.\tag{2.3}$$

Lemma 2.4. If (A), (B) and (C) hold, then there is an $\eta>0$ independent of $(\sigma,\phi)\in R^+\times\operatorname{int}BC^+$ such that

$$\liminf_{t\to+\infty}y(\sigma,\phi)(t)\ge\eta.\tag{2.4}$$

Lemma 2.5. *If* (A), (B) *and* (C) *hold, then for any $(\sigma,\phi)\in R^+\times\operatorname{int}BC^+$ we have*

$$\liminf_{t\to+\infty}y(\sigma,\phi)(t)\le\frac{a^u}{c^l}.\tag{2.5}$$

Lemma 2.6. *If* (A), (B) *and* (C) *hold, then for any $(\sigma,\phi)\in R^+\times\operatorname{int}BC^+$ we have*

$$\limsup_{t\to+\infty}x(\sigma,\phi)(t)>\min\Big(\frac{d^l}{4l},\frac{a^l}{2b^u}\Big).\tag{2.6}$$

Define function $f:R^+\to R^+$ as following

$$f(a)=\inf_{t\ge0}\int_{-\infty}^{t}K(s,t,a,a)ds.$$

It is easy to see that the function $f(a)$ is well defined and increasing.

Lemma 2.7. *If* (C) *holds, that is, if* $f(\frac{a^l}{b^u}) > d^u$, *then there is a* $\delta > 0$ *such that*

$$f\Big(\frac{a^l - c^u\delta}{b^u}\Big) > d^u. \tag{2.7}$$

Lemma 2.8. *If* (A), (B) *and* (C) *hold, then for any* $(\sigma, \phi) \in R^+ \times \mathrm{int}\, BC^+$ *we have*

$$\limsup_{t\to+\infty} y(\sigma,\phi)(t) \geq \frac{\delta}{2}. \tag{2.8}$$

Define function $f_n : R^+ \to R^+$ as follows:

$$f_n(a) = \inf_{t\geq 0} \int_{t-n\omega}^{t} K(s,t,a,a)ds.$$

Lemma 2.9. *If* (C) *holds, then there is an* $n_0 > 0$ *such that*

$$f_n\Big(\frac{a^l - c^u\delta}{b^u}\Big) > d^u$$

provided $n > n_0$.

Proof. From Lemma 2.7, we can set

$$f\Big(\frac{a^l - c^u\delta}{b^u}\Big) - d^u = \epsilon > 0.$$

We have

$$\begin{aligned}&\int_{-\infty}^{t} K\Big(s,t,\frac{a^l - c^u\delta}{b^u},\frac{a^l - c^u\delta}{b^u}\Big)ds - \int_{t-n\omega}^{t} K\Big(s,t,\frac{a^l - c^u\delta}{b^u},\frac{a^l - c^u\delta}{b^u}\Big)ds\\ &= \int_{-\infty}^{t-n\omega} K\Big(s,t,\frac{a^l - c^u\delta}{b^u},\frac{a^l - c^u\delta}{b^u}\Big)ds\\ &\leq 2\frac{a^l - c^u\delta}{b^u}\int_{-\infty}^{-n\omega} h(s)ds.\end{aligned}$$

There is an $n_0 > 0$ such that

$$\int_{-\infty}^{-n\omega} h(s)ds \leq \frac{b^u\epsilon}{4(a^l - c^u\delta)}$$

provided $n > n_0$. It follows that

$$\int_{-\infty}^{t} K\Big(s,t,\frac{a^l - c^u\delta}{b^u},\frac{a^l - c^u\delta}{b^u}\Big)ds \leq \frac{\epsilon}{2} + \int_{t-n\omega}^{t} K\Big(s,t,\frac{a^l - c^u\delta}{b^u},\frac{a^l - c^u\delta}{b^u}\Big)ds.$$

Then we have

$$f\Big(\frac{a^l - c^u\delta}{b^u}\Big) \leq \frac{\epsilon}{2} + f_n\Big(\frac{a^l - c^u\delta}{b^u}\Big).$$

Therefore

$$f_n\Big(\frac{a^l - c^u\delta}{b^u}\Big) - d^u \geq f\Big(\frac{a^l - c^u\delta}{b^u}\Big) - \frac{\epsilon}{2} - d^u = \frac{\epsilon}{2} > 0. \tag{2.9}$$

The lemma is proved.

Now we consider the associate equation

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t) - c(t)y(t)],\\ y'(t) = y(t)\big[-d(t) + \int_{t-n\omega}^{t} K(s,t,x(s),x(t))ds\big].\end{cases} \tag{2.10}$$

Let

$$K_1(s,t,x,y)=\begin{cases}K(s,t,x,y), & s\ge t-n\omega,\\ 0, & s<t-n\omega.\end{cases}$$

Then Equation (2.10) can be rewritten as

$$\begin{cases}x'(t)=x(t)[a(t)-b(t)x(t)-c(t)y(t)],\\ y'(t)=y(t)\big[-d(t)+\int_{-\infty}^{t}K_1(s,t,x(s),x(t)ds\big].\end{cases}\tag{2.11}$$

When $n\ge n_0$, it is easy to check that the conditions (A) and (B) are also hold for the functions K_1,a,b,c,d. By Lemma 2.9, when $n\ge n_0$, condition (C) also holds for Equation (2.10). Therefore, when $n\ge n_0$, Lemmas 2.1–2.8 keep true for Equation (2.10). Let

$$BC_n=BC([-n\omega,0]\to R^2),$$

and

$$\text{int } BC_n^+=\{\phi\in BC_n:\ \phi_1(s)>0\,,\ \phi_2(s)>0\,,\ s\in[-n\omega,0]\}\ .$$

For $(\sigma_n,\phi^n)\in R^+\times BC_n$, let $(x(\sigma_n,\phi^n)(t),y(\sigma_n,\phi^n)(t))$ denote the solution of Equation (2.10) satisfying the initial condition

$$x_{\sigma_n}=\phi_1^n,\quad y_{\sigma_n}=\phi_2^n.$$

Lemma 2.10. *If (A), (B) and* (C) *hold, then there is a* $T^*>0$ *independent of* n *such that for any* $(\sigma_n,\phi^n)\in R^+\times\text{int}\,BC_n^+$, $n=1,2,\cdots$, $n\ge n_0$, *we have*

$$\limsup_{t\to+\infty}y(\sigma_n,\phi^n)(t)\le T^*.\tag{2.12}$$

Proof. Since equation (2.10) is dependent on n, this lemma can not be obtained by Lemma 2.2 directly. But by Lemma 2.6 we have

$$\liminf_{t\to+\infty}y(\sigma,\phi)(t)\le\frac{a^u}{c^l}.$$

If (2.12) fails, then there exist $(\sigma_n,\phi^n)\in R^+\times\text{int }BC_n^+$, $n\ge n_0$, and $\tau_k^{(n)},t_k^{(n)}$, $k=1,2,\cdots$, with $\sigma_n\le\tau_k^{(n)}<t_k^{(n)}$, and $\tau_k^{(n)}\to+\infty$ as $k\to+\infty$ such that

$$y(\sigma_n,\phi^n)(\tau_k^{(n)})=\frac{2a^u}{c^l},\quad y(\sigma_n,\phi^n)(t_k^{(n)})=n+\frac{2a^u}{c^l}$$

and

$$\frac{2a^u}{c^l}<y(\sigma_n,\phi^n)(\tau_k^{(n)})<n+\frac{2a^u}{c^l}\ \text{ for }\ t\in(\tau_k^{(n)},t_k^{(n)}).\tag{2.13}$$

By Lemma 2.1, there exists an $M_1^{(n)}\ge\sigma_n$ such that

$$x(\sigma_n,\phi^n)(t)\le\frac{2a^u}{b^l}$$

provided $t\ge M_1^{(n)}$. There exists a $K_1^{(n)}>0$ such that when $k\ge K_1^{(n)}$, we have $\tau_k^{(n)}>M_1^{(n)}+n\omega$. When $k\ge K_1^{(n)}$ and $t\ge\tau_k^{(n)}$, we have

$$\begin{aligned}y'(\sigma_n,\phi^n)(t)&\le y(\sigma_n,\phi^n)(t)\int_{t-n\omega}^{t}h(s-t)(x(\sigma_n,\phi^n)(s)+x(\sigma_n,\phi^n)(t))ds\\&\le y(\sigma_n,\phi^n)(t)\int_{\tau_k^{(n)}-n\omega}^{t}h(s-t)\frac{4a^u}{b^l}ds\\&\le\frac{4a^u}{b^l}y(\sigma_n,\phi^n)(t):=\alpha y(\sigma_n,\phi^n)(t).\end{aligned}\tag{2.14}$$

It follows that

$$y(\sigma_n,\phi^n)(t)\le y(\sigma_n,\phi^n)(\tau_k^{(n)})e^{\alpha(t-\tau_k^{(n)})}\le\frac{2a^u}{c^l}e^{\alpha(t-\tau_k^{(n)})},$$

and then

$$t_k^{(n)}-\tau_k^{(n)}\ge\frac{1}{\alpha}\ln\Big(\frac{nc^l}{2a^u}+1\Big). \tag{2.15}$$

On the other hand, when $k\ge K_1^{(n)},\ t\in[\tau_k^{(n)},t_k^{(n)}]$, we have

$$x'(\sigma_n,\phi^n)(t)\le x(\sigma_n,\phi^n)(t)\Big[a^u-c^l\frac{2a^u}{c^l}\Big]=-a^u x(\sigma_n,\phi^n)(t).$$

Therefore

$$x(\sigma_n,\phi^n)(t)\le x(\sigma_n,\phi^n)(\tau_k^{(n)})e^{-a^u(t-\tau_k^{(n)})}\le\frac{2a^u}{b^l}e^{-a^u(t-\tau_k^{(n)})},$$

$$n\ge n_0,k\ge K_1^{(n)},\ t\in t\in[\tau_k^{(n)},\quad t_k^{(n)}]. \tag{2.16}$$

There exists an $L_1>0$ such that

$$\frac{2a^u}{b^l}e^{-a^uL_1}<\frac{d^l}{8l}. \tag{2.17}$$

There exists an $L_2>L_1$ such that

$$\frac{4a^u}{b^l}\int_{-\infty}^{L_1-L_2}h(s)ds<\frac{d^l}{4}. \tag{2.18}$$

From (2.15), there exists an $n_1>0$ such that $t_k^{(n)}-\tau_k^{(n)}\ge 2(L_1+L_2)$ provided $n\ge n_1$. It follows from (2.16) and (2.17) that when $n\ge n_1,\ \ k\ge K_1^{(n)}\ \ t\in(\tau_k^{(n)}+L_2,t_k^{(n)})$, we have

$$\begin{aligned}y'(\sigma_n,\phi^n)(t)&\le y(\sigma_n,\phi^n)(t)\Big[-d^l+\int_{t-n\omega}^{t}h(s-t)(x(\sigma_n,\phi^n)(s)+x(\sigma_n,\phi^n)(t))ds\Big]\\&\le y(\sigma_n,\phi^n)(t)\Big[-d^l+\int_{-\infty}^{\tau_k^{(n)}+L_1}h(s-t)\frac{4a^u}{b^l}ds+\int_{\tau_k^{(n)}+L_1}^{t}h(s-t)\frac{d^l}{4l}ds\Big]\\&\le y(\sigma_n,\phi^n)(t)\Big[-d^l+\int_{-\infty}^{L_1-L_2}h(s-t)\frac{4a^u}{b^l}ds+\frac{d^l}{4}\Big]\\&\le-\frac{d^l}{2}y(\sigma_n,\phi^n)(t).\end{aligned}$$

Therefore, when $n\ge n_1$, $k\ge K_1^{(n)}$, we have

$$n+\frac{2a^u}{c^l}=y(\sigma_n,\phi^n)(t_k^{(n)})<y(\sigma_n,\phi^n)(\tau_k^{(n)}+L_2)<n+\frac{2a^u}{c^l}.$$

This contradiction completes the proof of this lemma.

Lemma 2.11. *If* (A), (B) *and* (C) *hold, then there exists a* $\xi^*>0$ *independent of* n *such that for any* $(\sigma_n,\phi^n)\in R^+\times\mathrm{int}\,BC_n^+$, $n=1,2,\cdots,n\ge n_0$ *we have*

$$\liminf_{t\to+\infty}x(\sigma_n,\phi^n)(t)\ge\xi^*. \tag{2.19}$$

Proof. By Lemma 2.6 we know that for any $(\sigma_n,\phi^n)\in R^+\times$ int BC_n^+ , $n=1,2,\cdots,n\ge n_0$ we have

$$\limsup_{t\to+\infty}x(\sigma_n,\phi^n)(t)\ge\min\Big(\frac{d^l}{4l},\frac{a^l}{2b^u}\Big):=\lambda.$$

There is a $\lambda_1 < \lambda$ such that

$$2\lambda_1 l < \frac{d^l}{3}, \quad b^u \lambda_1 < \frac{a^l}{3}. \tag{2.20}$$

Choose two positive number sequences α_n, β_n such that for any n, $1 > \alpha_n > \beta_n$, and $\alpha_n \to 1$, $\beta_n \to 0$, as $n \to \infty$. There is an $n_1 > n_0$ such that $\alpha_n > \frac{1}{2}$, and $\alpha_n \lambda_1 > \beta_n \lambda$, provided $n \geq n_1$.

If (2.19) fails, then there exists $(\sigma_n, \phi^n) \in R^+ \times$ int BC_n^+, $n \geq n_1$, and $\tau_k^{(n)}, t_k^{(n)}$, $k = 1, 2, \cdots$, with $\sigma_n \leq \tau_k^{(n)} < t_k^{(n)}$, and $\tau_k^{(n)} \to +\infty$ as $k \to +\infty$ such that

$$x(\sigma_n, \phi^n)(\tau_k^{(n)}) = \alpha_n \lambda_1, \quad x(\sigma_n, \phi^n)(t_k^{(n)}) = \beta_n \lambda$$

with

$$\beta_n \lambda < x(\sigma_n, \phi^n)(t) < \alpha_n \lambda_1 \ \text{ for } \ t \in (\tau_k^{(n)}, t_k^{(n)}). \tag{2.21}$$

By Lemma 2.1 and Lemma 2.2, we know that there exists an $M_1^{(n)} \geq \sigma_n + n\omega$ such that

$$x(\sigma_n, \phi^n)(t) < \frac{2a^u}{b^l}, \qquad y(\sigma_n, \phi^n)(t) < 2T \tag{2.22}$$

provided $t \geq M_1^{(n)}$. So, there exists a $K_1^{(n)} > 0$ such that $\tau_k^{(n)} \geq M_1^{(n)}$ provided $k \geq K_1^{(n)}$. It follows that

$$x'(\sigma_n, \phi^n)(t) \geq x(\sigma_n, \phi^n)(t)\Big[-b^u \frac{2a^u}{b^l} - c^u \cdot 2T\Big] := -\beta x(\sigma_n, \phi^n)(t)$$

provided $n \geq n_1$, $k \geq K_1^{(n)}$ and $t \geq \tau_k^{(n)}$. Then, we have

$$x(\sigma_n, \phi^n)(t) \geq x(\sigma_n, \phi^n)(\tau_k^{(n)}) e^{-\beta(t - \tau_k^{(n)})} \geq \frac{\lambda}{2} e^{-\beta(t-\tau_k^{(n)})}.$$

Therefore, we get

$$t_k^{(n)} - \tau_k^{(n)} \geq \frac{1}{\beta} \ln \frac{\lambda_1}{2\beta_n \lambda} \quad \text{for} \quad n \geq n_1 \quad \text{and} \quad k \geq K_1^{(n)}. \tag{2.23}$$

So, when $n \geq n_1$, $k \geq K_1^{(n)}$ and $t \geq \tau_k^{(n)}$, we have

$$\begin{aligned} y'(\sigma_n, \phi^n)(t) &\leq y(\sigma_n, \phi^n)(t)\Big[-d^l + \int_{t-n\omega}^{t} h(s-t)(x(\sigma_n, \phi^n)(s) + x(\sigma_n, \phi^n)(t))ds\Big] \\ &\leq y(\sigma_n, \phi^n)(t)\Big[-d^l + \frac{4a^u}{b^l}\int_{-\infty}^{\tau_k^{(n)}} h(s-t)ds + 2\alpha_n\lambda_1 \int_{\tau_k^{(n)}}^{t} h(s-t)ds\Big] \\ &\leq y(\sigma_n, \phi^n)(t)\Big[-\frac{2}{3}d^l + \frac{4a^u}{b^l}\int_{-\infty}^{\tau_k^{(n)}-t} h(s-t)ds\Big]. \end{aligned} \tag{2.24}$$

There exists an $L_1 > 0$ such that

$$\frac{4a^u}{b^l}\int_{-\infty}^{-L_1} h(s-t)ds < \frac{d^l}{3}. \tag{2.25}$$

By (2.25) there is an $n_2 > n_1$ such that

$$t_k^{(n)} - \tau_k^{(n)} \geq 2L_1$$

provided $n \geq n_2$. Therefore, when $n \geq n_2$, $k \geq K_1^{(n)}$ and $t \in [\tau_k^{(n)} + L_1, t_k^{(n)}]$, (2.24) and (2.25) imply that

$$y'(\sigma_n, \phi^n)(t) \leq -\frac{d^l}{3} y(\sigma_n, \phi^n)(t),$$

and then we get

$$y(\sigma_n,\phi^n)(t) \le y(\sigma_n,\phi^n)(\tau_k^{(n)}+L_1)e^{-\frac{d^l}{3}(t-\tau_k^{(n)}-L_1)} \le 2Te^{-\frac{d^l}{3}(t-\tau_k^{(n)}-L_1)}. \tag{2.26}$$

There exists an $L_2 > 0$ such that

$$2Te^{-\frac{d^l}{3}L_2} < \frac{a^l}{3c^u}. \tag{2.27}$$

There exists an $n_3 > n_2$ such that when $n \ge n_3$, we have $t_k^{(n)} - \tau_k^{(n)} \ge 2(L_1+L_2)$.

So, when $n \ge n_3$, $k \ge K_1^{(n)}$ and $t \in [\tau_k^{(n)}+L_1+L_2,\ t_k^{(n)}]$, (2.26) and (2.27) imply that

$$y(\sigma_n,\phi^n)(t) \le \frac{a^l}{3c^u}\ .$$

By (2.20) we get

$$x'(\sigma_n,\phi^n)(t) \ge x(\sigma_n,\phi^n)(t)\Big[a^l - b^u\alpha_n\lambda_1 - c^u\frac{a^l}{3c^u}\Big]$$
$$\ge \frac{a^l}{3}x(\sigma_n,\phi^n)(t) > 0 \text{ for } \quad n \ge n_3, \quad k \ge K_1^{(n)}$$

and $t \in [\tau_k^{(n)}+L_1+L_2,\ t_k^{(n)}]$.

Therefore, when $n \ge n_3,\ k \ge K_1^{(n)}$, we have

$$x(\sigma_n,\phi^n)(t_k^{(n)}) = \beta_n\lambda > x(\sigma_n,\phi^n)(\tau_k^{(n)}+L_1+L_2).$$

This contradicts (2.21), and this lemma is proved.

Lemma 2.12. *If* (A), (B) *and* (C) *hold, then there exists a* $\eta^* > 0$ *independent of* n *such that for any* $(\sigma_n,\phi^n) \in R^+ \times \mathrm{int}\, BC_n^+$, $n = 1,2,\cdots, n \ge n_0$ *we have*

$$\liminf_{t\to+\infty} y(\sigma_n,\phi^n)(t) \ge \eta^*. \tag{2.28}$$

Proof. By Lemma 2.8, we have

$$\limsup_{t\to+\infty} y(\sigma_n,\phi^n)(t) \ge \frac{\delta}{2}.$$

Note that δ is independent of n by Lemma 2.7. By Lemma 2.9, we can choose a δ satisfies

$$f_n\Big(\frac{a^l - c^u\delta}{b^u}\Big) > d^u. \tag{2.29}$$

Choose two positive number sequences $\alpha_n,\ \beta_n$ such that for any n, $1 > \alpha_n > \beta_n$, and $\alpha_n \to 1, \beta_n \to 0$, as $n \to \infty$. There is an $n_1 > n_0$ such that $\alpha_n > \frac{1}{2}$. If (2.28) fails, then there exists $(\sigma_n,\phi^n) \in R^+ \times \mathrm{int}\, BC_n^+$, $n \ge n_1$, and $\tau_k^{(n)}, t_k^{(n)}$, $k = 1,2,\cdots$, with $\sigma_n \le \tau_k^{(n)} < t_k^{(n)}$, and $\tau_k^{(n)} \to +\infty$ as $k \to +\infty$ such that

$$y(\sigma_n,\phi^n)(\tau_k^{(n)}) = \frac{\alpha_n\delta}{4}, \quad y(\sigma_n,\phi^n)(t_k^{(n)}) = \frac{\beta_n\delta}{4}$$

with

$$\frac{\beta_n\delta}{4} < y(\sigma_n,\phi^n)(\tau_k^{(n)}) < \frac{\alpha_n\delta}{4} \quad \text{for } t \in (\tau_k^{(n)}, t_k^{(n)}). \tag{2.30}$$

Since

$$y'(\sigma_n,\phi^n)(t) \ge -d^u y(\sigma_n,\phi^n)(t) \quad \text{for} \quad t \ge \sigma_n,$$

when $n \ge n_1$ we have

$$y(\sigma_n,\phi^n)(t) \ge y(\sigma_n,\phi^n)(\tau_k^{(n)})e^{-d^u(t-\tau_k^{(n)})} \ge \frac{\delta}{8}e^{-d^u(t-\tau_k^{(n)})}.$$

It follows that

$$t_k^{(n)} - \tau_k^{(n)} \ge \frac{1}{d^u} \ln \frac{1}{2\beta_n}. \tag{2.31}$$

Let us consider the equation

$$y'(t) = y(t)\Big(a^l - b^u y(t) - c^u \frac{\delta}{4}\Big), \qquad y(\sigma) = \frac{\alpha_n \delta}{4}.$$

From the definition of δ, we have $a^l - c^u \frac{\delta}{4} > a^l - c^u\delta > 0$. Simple calculation gives

$$y(t) = \Big[\frac{b^u}{a^l - c^u \frac{\delta}{4}} + \Big(\frac{1}{y(\sigma)} - \frac{b^u}{a^l - c^u\frac{\delta}{4}}\Big)e^{-(a^l - c^u\frac{\delta}{4})(t-\sigma)}\Big]^{-1}.$$

It is easy to see that

$$y(t) \to \frac{a^l - c^u\frac{\delta}{4}}{b^u} \quad \text{as} \quad t \to +\infty.$$

So, there is an $L_1 > 0$ such that when $t \ge \sigma + L_1$ we have

$$y(t) \ge \frac{a^l - c^u\frac{\delta}{2}}{b^u}. \tag{2.32}$$

By (2.31), there is an $n_2 > n_1$ such that when $n \ge n_2$

$$t_k^{(n)} - \tau_k^{(n)} \ge 2L_1.$$

Therefore, when $n \ge n_2$ and $t \in [\tau_k^{(n)}, t_k^{(n)}]$, we have

$$x'(\sigma_n, \phi^n)(t) \ge x(\sigma_n, \phi^n)(t)\Big[a^l - b^u x(\sigma_n, \phi^n)(t) - c^u \frac{\delta}{4}\Big].$$

There is a $K_1^{(n)} > 0$, such that when $k \ge K_1^{(n)}$, we have $x(\sigma_n, \phi^n)(t) < \frac{2a^u}{b^l}$. From (2.32) and comparison theorem one can easily proved that

$$x(\sigma_n, \phi^n)(t) \ge \frac{a^l - c^u\frac{\delta}{2}}{b^u} \quad \text{for} \quad t \in [\tau_k^{(n)} + L_1, t_k^{(n)}]$$

provided $n \ge n_2$ *and* $k \ge K_1^{(n)}$. Then, when $n \ge n_2$, $k \ge K_1^{(n)}$ and $t \in [\tau_k^{(n)} + L_1, t_k^{(n)}]$, we have

$$\begin{aligned}
&\int_{t-n\omega}^{t} K(s, t, x(\sigma_n, \phi^n)(s), x(\sigma_n, \phi^n)(t))ds - \int_{t-n\omega}^{t} K\Big(s, t, \frac{a^l - c^u\delta}{b^u}, \frac{a^l - c^u\delta}{b^u}\Big)ds \\
&\ge \int_{\tau_k^{(n)} - n\omega}^{t_k^{(n)} + L_1} h(s-t)\Big(\Big|x(\sigma_n, \phi^n)(s) - \frac{a^l - c^u\delta}{b^u}\Big| + \Big|x(\sigma_n, \phi^n)(t) - \frac{a^l - c^u\delta}{b^u}\Big|\Big)ds \\
&\ge -2\Big(\frac{2a^u}{b^l} + \frac{a^l - c^u\delta}{b^u}\Big)\int_{-\infty}^{\tau_k^{(n)} + L_1 - t} h(s)ds.
\end{aligned} \tag{2.33}$$

There exists an $L_2 > L_1$ such that

$$2\Big(\frac{2a^u}{b^l} + \frac{a^l - c^u\delta}{b^u}\Big)\int_{-\infty}^{L_1 - L_2} h(s)ds < \frac{1}{2}\Big[f_n\Big(\frac{a^l - c^u\delta}{b^u}\Big) - d^u\Big]. \tag{2.34}$$

There exists an $n_3 > n_2$ such that when $n \ge n_3$, we have

$$t_k^{(n)} - \tau_k^{(n)} \ge 2(L_1 + L_2).$$

Then, when $n \geq n_3$, $k \geq K_1^{(n)}$ and $t \in [\tau_k^{(n)} + L_1 + L_2, t_k^{(n)}]$, by (2.33) and (2.34) we get

$$\begin{aligned}&\int_{t-n\omega}^{t} K(s,t,x(\sigma_n,\phi^n)(s),x(\sigma_n,\phi^n)(t))ds\\ &\geq \int_{t-n\omega}^{t} K\Big(s,t,\frac{a^l-c^u\delta}{b^u},\frac{a^l-c^u\delta}{b^u}\Big)ds - \frac{1}{2}\Big[f_n\Big(\frac{a^l-c^u\delta}{b^u}\Big)-d^u\Big]\\ &\geq f_n\Big(\frac{a^l-c^u\delta}{b^u}\Big) - \frac{1}{2}\Big[f\Big(\frac{a^l-c^u\delta}{b^u}\Big)-d^u\Big] = \frac{1}{2}f_n\Big(\frac{a^l-c^u\delta}{b^u}\Big)+\frac{1}{2}d^u.\end{aligned}$$

It follows that

$$-d^u + \int_{t-n\omega}^{t} K(s,t,x(\sigma_n,\phi^n)(s),x(\sigma_n,\phi^n)(t))ds \geq \frac{1}{2}f_n\Big(\frac{a^l-c^u\delta}{b^u}\Big) - \frac{d^u}{2} > 0.$$

This implies that when $n \geq n_3$, $k \geq K_1^{(n)}$ and $t \in [\tau_k^{(n)} + L_1 + L_2, t_k^{(n)}]$, we have, $y'(\sigma_n,\phi^n)(t) > 0$. Thus, we get

$$y(\sigma_n,\phi^n)(t_k^{(n)}) > y(\sigma_n,\phi^n)(\tau_k^{(n)} + L_1 + L_2) > \frac{\beta_n\delta}{4}.$$

This contradicts (2.30), and this lemma is proved.

Lemma 2.13. *If* (A), (B) *and* (C) *hold, and if* S *is a given bounded set in* $\operatorname{int} BC_n^+$, *then there exists a* $P(S) > 0$ *such that for any* $\phi^n \in S$,

$$x(0,\phi^n)(t),\ y(0,\phi^n)(t) \leq P(S), \quad t \geq 0.$$

Proof. If the conclusion of this lemma is failed for $y(0,\phi^n)(t)$, then by Lemma 2.5, there are a $\phi_m^{(n)} \in S$ and τ_m, t_m with $0 \leq \tau_m < t_m$, $\tau_m \to \infty$ as $m \to \infty$ such that

$$\begin{aligned}&y(0,\phi_m^{(n)})(\tau_m) = N_2\,, \quad y(0,\phi_m^{(n)})(t_m) = m + N_2,\\ &N_2 < y(0,\phi_m^{(n)})(t) < m + N \quad \text{for} \quad t \in (\tau_m, t_m),\end{aligned} \tag{2.35}$$

where $N_2 = \sup\limits_{\phi\in S}(\phi_2(0) + \frac{2a^u}{c^l})$. It is easy to check that

$$x(0,\phi_m^{(n)})(t) < N_1 := \sup_{\phi\in S}\Big(\phi_1(0) + \frac{2a^u}{c^l}\Big) \quad \text{for} \quad t \geq 0.$$

Thus, we have

$$y'(0,\phi_m^{(n)})(t) \leq y(0,\phi_m^{(n)})(t) \int_{t-n\omega}^{t} K(s,t,N_1,N_1)ds \leq 2N_1 l\, y(0,\phi_m^{(n)})(t).$$

From the comparison theorem we can easily get

$$y(0,\phi_m^{(n)})(t) \leq N_2 e^{2N_1 l(t-\tau_m)} \quad \text{for} \quad t \geq \tau_m.$$

It follows that

$$t_m - \tau_m \geq \frac{1}{2N_1 l}\ln\Big(\frac{m}{N_2}+1\Big). \tag{2.36}$$

When $t \in [\,\tau_m\,, t_m]$, we have

$$x'(0,\phi_m^{(n)})(t) \leq x(0,\phi_m^{(n)})(t)\Big(a^u - c^l\frac{2a^u}{c^l}\Big) = -a^u x(0,\phi_m^{(n)})(t).$$

This implies that

$$x(0,\phi_m^{(n)})(t) \leq N_1 e^{-a^u(t-\tau_m)} \quad \text{for} \quad t \in [t_m,\ \tau_m]. \tag{2.37}$$

There exists an $L_1 > 0$ such that

$$N_1 e^{-a^u L_1} < \frac{d^l}{4l}. \tag{2.38}$$

From (2.36), there exists an $m_1 > 0$ such that when $m \geq m_1$,

$$t_m - \tau_m \geq 2(L_1 + m\omega).$$

Therefore, when $m \geq m_1$, $t \in [\tau_m + L_1 + n\omega, t_m]$, we have

$$y'(0, \phi_m^{(n)})(t) \leq y(0, \phi_m^{(n)})(t)\Big(-d^l + \frac{d^l}{2}\Big) = -\frac{d^l}{2}\, y(0, \phi_m^{(n)})(t) < 0.$$

It follows that

$$y(0, \phi_m^{(n)})(t_n) < y(0, \phi_m^{(n)})(\tau_n + L_1 + n\omega) < m + N_2.$$

This contradicts (2.35), so this lemma is true for $y(0, \phi^n)(t)$. It is obvious that this lemma is true also for $x(0, \phi^n)(t)$, and this lemma is proved.

Lemma 2.14 (Horn).[10] *Suppose that F is a complete continuous mapping from a Banach space X to itself. If there exists a bounded set E such that for any $x \in E$, there exists an $m = m(x)$ with $F^m(x) \in E$, then F has fixed points on E.*

Let

$$k_n(s,t,x,y) = K(s,t,x,y) + K(s-\omega,t,x,y) + \cdots + K(s-(n-1)\omega,t,x,y).$$

Let us consider the equation

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)\,x(t) - c(t)\,y(t)], \\ y'(t) = y(t)\,[-d(t) + \int_{t-\omega}^{t} k_n(t,s,x(s),x(t))ds]. \end{cases} \tag{2.39}$$

Lemma 2.15. *If $(x = \phi(t),\ y = \psi(t))$ is an ω-periodic solution of* (2.39)*, then it is also an ω-periodic solution of* (2.10).

Proof. We have

$$\begin{aligned}\psi'(t) &= \psi(t)\Big[-d(t) + \int_{t-\omega}^{t} k_n(s,t,\phi(s),\phi(t))ds\Big] \\ &= \psi(t)\Big[-d(t) + \int_{t-\omega}^{t} (K(s,t,\phi(s),\phi(t)) + K(s-\omega,t,\phi(s),\phi(t)) + \cdots \\ &\quad + K(s-(n-1)\omega,t,\phi(s),\phi(t)))ds\Big] \\ &= \psi(t)\Big[-d(t) + \int_{t-\omega}^{t} K(s,t,\phi(s),\phi(t))ds + \int_{t-2\omega}^{t-\omega} K(s,t,\phi(s+\omega),\phi(t))ds + \cdots \\ &\quad + \int_{t-n\omega}^{t-(n-1)\omega} K(s,t,\phi(s+(n-1)\omega),\phi(t))ds\Big] \\ &= \psi(t)\Big[-d(t) + \int_{t-n\omega}^{t} K(s,t,\phi(s),\phi(t))ds\Big].\end{aligned}$$

This lemma is proved.

Lemma 2.16. *If $n \geq n_0$, then*

$$\int_{t-\omega}^{t} k_n\Big(s,t,\frac{a^l}{b^u},\frac{a^l}{b^u}\Big)ds > d^u,, \quad t \in R.$$

Proof. By the similar arguments as above, we can easily prove that

$$\int_{t-\omega}^{t} k_n\Big(s,t,\frac{a^l}{b^u},\frac{a^l}{b^u}\Big)ds = \int_{t-n\omega}^{t} K\Big(s,t,\frac{a^l}{b^u},\frac{a^l}{b^u}\Big)ds.$$

From Lemma 2.9 and the definition of n_0 , we complete the proof immediately.

Therefore, Equation (2.39) can be regarded as a special case of Equation (2.10), and then, as a special case of equation (1.1). Conditions (A), (B) and (C) are all satisfied for (2.39), so Lemmas 2.1-2.4 are all true for it. Corresponding $T,\ \xi,\ \eta$ are denoted as T_1, ξ and η_1, respectively.

Define set

$$S=\Big\{\phi\in BC_1:\frac{\xi_1}{2}\le\phi_1(s)\le\frac{2a^u}{b^l}\ ,\ \ \frac{\eta_1}{2}\le\phi_2(s)\le 2T_1\Big\}.$$

Lemma 2.17. *If* $n\ge n_0$, *Equation* (2.10) *has* $\omega-$*periodic solution* $x=\phi^n(t),\ t\in R$, *with*

$$\frac{\xi^*}{2}\le\phi_1^n(s)\le\frac{2a^u}{b^l},\ \ \frac{\eta^*}{2}\le\phi_2^n(s)\le 2T^*,\quad t\in R. \tag{2.40}$$

Proof. For $\phi\in BC_1$, $\sigma\in R$, let $\bar{x}(\sigma,\phi)(t)$ denote the solution of (2.39) satisfying the initial condition $x_\sigma=\phi$. It is obvious that BC_1 with the sup norm is a Banach space. Define mapping $F:BC_1\to BC_1$ as follows:

$$F(\phi)=\bar{x}_\omega(0,\phi).$$

By Lemma 2.13, it is easy to see that F is complete continuous.

If $\phi\in S$, by Lemmas 2.1-2.4, there exists a $m>0$ such that when $t\ge(m+1)\omega$,

$$\frac{\xi_1}{2}\le\bar{x}_1(0,\phi)(s)\le\frac{2a^u}{b^l},\frac{\eta_1}{2}\le\bar{x}_2(0,\phi)(s)\le 2T_1. \tag{2.41}$$

That is, $\bar{x}_{m\omega}(0,\phi)\in S$ and then $F^m(\phi)\in S$.

From Horn Theorem (Lemma 2.14), there exists a $\phi^*\in S$ such that $F(\phi^*)=\phi^*$, and $\phi^n(t)=\bar{x}(0,\phi^*)(t)$ is an ω-periodic solution of Equations (2.10) and (2.40) holds.

§3. The Main Results

Theorem 3.1. *If conditions* (A), (B) *and* (C) *hold, then system* (1.1) *has* $\omega-$*periodic solution* $\phi(t)=(\phi_1(t),\,\phi_2(t))$ *with*

$$\xi\le\phi_1(t)\le\frac{a^u}{b^l}\ ,\quad \eta\le\phi_2(t)\le T\quad for\quad t\in R. \tag{3.1}$$

Proof. From Lemma 2.17 , when $n\ge n_0$, Equation (2.10) has an ω-periodic solution $\phi^n(t)$ with

$$\frac{\xi^*}{2}\le\phi_1^n(s)\le\frac{2a^u}{b^l},\ \ \frac{\eta^*}{2}\le\phi_2^n(s)\le 2T^*,\quad t\in R,$$

and

$$|\phi^n(s_1)-\phi^n(s_2)|\le L\,|s_1-s_2|\,.$$

Thus, the function sequence $\{\phi^n(s):s\in[0,\omega],\ n=n_0,n_0+1,n_0+2,\cdots\}$ is uniformly bounded and equicontinuous. From Ascoli Lemma, there is a subsequence $\{\phi^{n_k}(s):k=1,2,\cdots\}$ of $\{\phi^n(s):s\in[0,\omega],\ n=n_0,n_0+1,n_0+2,\cdots\}$ which uniformly converges to a continuous function $\bar{\phi}(s),\ s\in[0,\omega]$. Since all $\phi^{n_k}(s)$ are ω-periodic, we have $\bar{\phi}(0)=\bar{\phi}(\omega)$. Let

$$\phi(t)=\bar{\phi}(t-k\omega)\ \text{ for }\ t\in[k\omega,(k+1)\omega),\ k=0,\pm1,\pm2,\cdots.$$

It is easy to see that $\phi(t)$ is a continuous ω-periodic function, and $\phi^{n_k}(t)$ uniformly converges to $\phi(t)$ for $t\in R$.

Define

$$K_n(s,t,x,y)=\begin{cases}K(s,t,x,y), & s\geq t-n\omega,\\ 0, & s<t-n\omega.\end{cases}$$

It is obvious that $K_n(s,t,x,y)$ is measurable. Since

$$\begin{cases}\phi_1^n(t)=\phi_1^n(0)+\int_0^t\phi_1^n(s)[a(s)-b(s)\phi_1^n(s)-c(s)\phi_2^n(s)]ds,\\ \phi_2^n(t)=\phi_2^n(0)+\int_0^t\phi_2^n(s)\Big[-d(s)+\int_{-\infty}^s K_n(\tau,s,\phi_1^n(\tau),\phi_1^n(s))d\tau\Big]ds,\end{cases}\tag{3.2}$$

$$\Big|\int_{-\infty}^s K_n(\tau,s,\phi_1^n(\tau),\phi_1^n(s))d\tau\Big|\leq\frac{4a^u l}{b^l},$$

$$\Big|\int_{-\infty}^s K_n(\tau,s,\phi_1^n(\tau),\phi_1^n(s))d\tau\int_{-\infty}^s K(\tau,s,\phi_1^n(\tau),\phi_1^n(s))d\tau\Big|$$

$$\leq\frac{4a^u l}{b^l}\int_{-\infty}^{-n\omega}h(s)ds\to 0\ \text{ as }\ n\to\infty,$$

letting $n\to\infty$ on the two sides of (3.2) by dominant convergence theorem we get

$$\begin{cases}\phi_1(t)=\phi_1(0)+\int_0^t\phi_1(s)[a(s)-b(s)\phi_1(s)-c(s)\phi_2(s)]ds,\\ \phi_2(t)=\phi_2(0)+\int_0^t\phi_2(s)\Big[-d(s)+\int_{-\infty}^s K(\tau,s,\phi_1(\tau),\phi_1(s))d\tau\Big]ds.\end{cases}$$

Thus, $\phi(t)$ is an ω-periodic solution of (1.1). By Lemmas 2.1-2.4, (3.2) can be easily obtained. This theorem is proved.

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