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# SUFFICIENT CONDITIONS FOR OSCILLATION OF THE LIÉNARD EQUATION

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**Abstract**

Some sufficient conditions for oscillation of the generalized Liénard equations

$$\dot{x} = h(y) - F(x),$$
$$\dot{y} = -g(x)$$

are given, which generalize the results of [1–7].

**Keywords** Liénard equation, Oscillation
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## §1. Introduction

In this paper, we give some sufficient conditions for the oscillation of all solutions of the generalized Liénard equation

$$\begin{aligned}\dot{x} &= h(y) - F(x),\\ \dot{y} &= -g(x).\end{aligned} \tag{1.1}$$

Throughout this paper, we assume $h(\cdot), F(\cdot), g(\cdot)$ are continuous functions on $R$ and satisfy locally Lipschitz condition. Moreover we assume $F(0) = 0; yh(y) > 0, y \neq 0; xg(x) > 0, x \neq 0; h(y)$ is strictly increasing and

$$\lim_{y\to\pm\infty} h(y) = \pm\infty, \quad \lim_{|x|\to\infty} G(x) = \infty,$$

where

$$G(x) = \int_0^x g(u)du, \quad H(y) = \int_0^y h(u)du,$$

$H^{-1}(\cdot), G^{-1}(\cdot)$ are inverse functions of $H(\cdot)$ and $G(\cdot)$. A solution of (1.1) is said to be oscillatory if the solution curve $x(y)$ crosses the $y$-axis infinitely many times. Equation (1.1) is called oscillatory if all its nonzero solutions are oscillatory.

For $x \geq 0$, in order to obtain the oscillation for (1.1), it is necessary to guarantee each positive semiorbit $O^+(P)$, where $P = (0, p)$ with $p > 0$, to cross the vertical isocline $h(y) = F(x)$ and then cross the negative $y$-axis; this property of $O^+(P)$ plays an important role in

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the analysis of center condition, oscillation, asymptotic stability and boundness condition of (1.1). There have been many works in this direction, for example, [1–8]; all of them give sufficient conditions to obtain the above mentioned property of $O^+(P)$ for Equation (1.1) with $h(y) = y, f(x) = F'(x)$.

(1)' From [1], there exists $k > 0$ such that

$$0 < f(x) < kg(x), \quad x > 0.$$

(2)' From [2], there exists $\alpha > \frac{1}{4}$ such that

$$f(x) > 0, \alpha f(x)F(x) \le g(x), \quad x > 0.$$

(3)' From [3], there exists $0 \le \beta < 8$, such that

$$F^2(x) \le \beta G(x), \quad x > 0.$$

(4)' From [4], there exists $\alpha > \frac{1}{4}$ such that

$$\alpha|F(x)| \le \int_0^x g(u)/|F(u)|du, \quad x > 0.$$

(5)' From [5] and [6], there exists $\alpha > \frac{1}{4}$, such that

$$|F(x)| > 0, \frac{1}{F(x)}\int_0^x \frac{g(u)}{F(u)}du \ge \alpha, \quad x > 0.$$

(6)' From [7], there exists $k_1 > 0, k_2 < 0$ such that

$$k_2 \le f(x)/g(x) \le k_1, \quad x > 0.$$

Recently, for $h(y) = m|y|^p$ sgn$y, m > 0, p \ge 1$, J. Sugie[8] obtained the following condition: for $x > 0$.

(7)' $F(x) \ge -\lambda G(x)^{p/(1+p)}$, where $0 \le \lambda < \mu = m(1+p)\left(\frac{(1+p)}{mp}\right)^{p/(1+p)}$.

J. Jiang[9] obtained the similar result.

If we consider the above conditions carefully, it is not difficult to prove (see Section 2) that the above conditions could be written as

$$|F(x)| \le M(g(x)) \quad \text{for} \quad x > 0, \tag{1.2}$$

where $M(r) \ge 0$ for $r \ge 0, M(r) = 0$ if and only if $r = 0$.

But it seems that there is no evident relations between the above conditions and the constants $\alpha, \beta, \mu$. Furthermove, most of the above papers did not give a method in obtaining these constants; hence, the above results are not easy to be extended to more general cases.

In this paper, by using a new method, we obtain more general sufficient conditions to guarantee the oscillation of Equation (1.1); our method has an evident geometric meaning.

## §2. Notations and Lemmas

First, we prove that all the conditions (1)'–(7)' could be rewritten as (1.2). We need only to prove (2)' and (4)'; the other cases are evident.

**Lemma 2.1(Bihari Theorem).**[10] *If $k, m \ge 0$ and $l(s)$ is positive for $s > 0$, then the inequality*

$$u(t) \le k + m\int_a^t v(s)l(u(s))ds, \qquad a \le t \le b, \tag{2.1}$$

*implies that*

$$u(t) \leq L^{-1}\Big[L(k) + m\int_a^t v(s)ds\Big],$$

*where*

$$L(u) = \int_{u_0}^{u} \frac{dt}{l(t)}, \qquad 0 \leq u_0 < u. \tag{2.2}$$

**Proof of (2)'.** Integrating both sides of $\alpha f(x)F(x) \leq g(x)$ for $x > 0$, we have $\alpha F^2(x) \leq 2G(x)$, that is,

$$|F(x)| \leq \sqrt{\frac{2}{\alpha}G(x)} = \sqrt{\frac{2}{\alpha}\int_0^x g(u)du}.$$

**Proof of (4)'.** By Lemma 2.1, let $k = 0, m = \frac{1}{\alpha}, v(t) = g(x), u(t) = |F(x)|, l(u) = \frac{1}{u}$; then

$$L(u) = \int_0^u tdt = \frac{u^2}{2}, \quad L^{-1}(u) = \sqrt{2u},$$

$$|F(x)| \leq \sqrt{\frac{2}{\alpha}\int_0^x g(u)du} = \sqrt{\frac{2}{\alpha}G(x)} < 2\sqrt{2G(x)}$$

$$= 2\sqrt{2\int_0^x g(u)du}, \quad x > 0.$$

Second, we introduce the generalized polar coordinates, which is a generalization of the method used in [11].

Let $x(\theta) = C(\theta), y(\theta) = S(\theta)$ be the solution of the following equations

$$\dot{x} = h(y), \qquad \dot{y} = -g(x) \tag{2.3}$$

satisfying the initial conditions $G(C(0)) = 1, S(0) = 0$.

Then $C(\theta), S(\theta)$ are periodic solutions with period $T > 0$ $(0 \leq \theta \leq T)$ and satisfy the following equations

$$\begin{aligned} &\text{(a)} \quad G(C(\theta)) + H(S(\theta)) = 1, \\ &\text{(b)} \quad dC(\theta)/d\theta = h(S(\theta)), \\ &\text{(c)} \quad dS(\theta)/d\theta = -g(C(\theta)). \end{aligned} \tag{2.4}$$

It is easy to see that the solution curves of (2.3) are simple closed curves surronding the origin:

$$G(x) + H(y) = \text{Const.} \geq 0. \tag{2.5}$$

If we introduce the generalized polar coordinates $(r, \theta), r \geq 0, 0 \leq \theta \leq T$,

$$x = f_1(r)C(\theta), \quad y = f_2(r)S(\theta), \tag{2.6}$$

where $f_1(r) = \sqrt{H(r)/H(1)}, f_2(r) = \sqrt{G(r)/G(1)}$, then (2.5) is changed into

$$G(f_1(r)C(\theta)) + H(f_2(r)S(\theta)) = f_3(r)$$

with

$$f_3(r) = G(f_1(r)C(0)), \qquad r \geq 0.$$

Then it is easy to see that $f_i(r)$ are monotone increasing $C^1$ functions for $r \geq 0$ and $f_i(0) = 0, f_i(1) = 1, f_i(\infty) = \infty, i = 1, 2, 3$.

**Lemma 2.2.** *Consider the following equations*

$$\dot{x} = h(y) + \lambda R(x),$$
$$\dot{y} = -g(x), \tag{2.7$_\lambda$}$$

*where $R(x) \neq 0$ for $x \neq 0, \lambda \geq 0$ is a constant. Then there exists a $\lambda_0 > 0$ such that all solutions of* $(2.7)_\lambda$ *are oscillatory for* $0 \leq \lambda < \lambda_0$; $(2.7)_\lambda$ *is not oscillatory for* $\lambda \geq \lambda_0$.

**Proof.** By using the generalized polar coordinates introduced above, $(2.7)_\lambda$ is changed into the following form

$$\overline{R}(r,\theta)\frac{dr}{dt} = S(r,\theta,\lambda),$$
$$\overline{R}(r,\theta)\frac{d\theta}{dt} = T(r,\theta,\lambda), \tag{2.8$_\lambda$}$$

where

$$\overline{R}(r,\theta) = f_1'(r)f_2(r)\cdot C(\theta)g(C(\theta)) + f_1(r)f_2'(r)h(S(\theta))S(\theta),$$
$$S(r,\theta,\lambda) = f_2(r)g(C(\theta))h(y) - g(x)f_1(r)h(S(\theta)) + \lambda f_2(r)g(C(\theta))R(x),$$
$$T(r,\theta,\lambda) = -\left[\frac{g(r)}{2G(r)}yh(y) + \frac{h(r)}{2H(r)}xg(x) + \frac{\lambda g(r)S(\theta)R(x)}{\sqrt{2G(r)G(1)}}\right],$$

and $x, y$ are given by (2.6), $r = 0$ if and only if $x = y = 0$.

It is obvious that $\overline{R}(r,\theta) \geq 0$ and $\overline{R}(r,\theta) = 0$ if and only $r = 0$. When $\overline{R}(r,\theta) > 0$, $(2.8)_\lambda$ is equivalent to the following equation

$$\frac{dr}{d\theta} = \frac{S(r,\theta,\lambda)}{T(r,\theta,\lambda)}. \tag{2.9$_\lambda$}$$

Write $T(r,\theta,\lambda) = T(r,\theta,0) + \lambda T_1(r,\theta)$. Then by the assumptions of $h$ and $g$,

$$T(r,\theta,0) = -\left[\frac{g(r)}{2G(r)}yh(y) + \frac{h(r)}{2H(r)}xg(x)\right] < 0 \quad \text{for } r > 0,$$
$$T_1(r,\theta) = -g(r)S(\theta)R(x)/\sqrt{2G(r)G(1)}.$$

If for some $\theta$, $T_1(r,\theta) > 0$, then for fixed $r, \theta$, letting $\lambda > 0$ be large enough, we have $T(r,\theta,\lambda) > 0$. Hence $T(r,\theta,\lambda)$ changes sign.

But if $\lambda > 0$ is small enough, $T(r,\theta,\lambda)$ has the same sign as $T(r,\theta,0)$.

If $T(r,\theta,\lambda) \geq T_0(r) > 0$ for $r > 0$ and $0 \leq \lambda < \lambda_0$ for some $\lambda_0 > 0$ and if $y$-axis is not an exceptional direction for $(2.7)_\lambda$, then all solutions of $(2.7)_\lambda$ are oscillatory.

If for some $\lambda_1 > 0$ there exists at least one $\overline{\theta}$ such that $T(r,\overline{\theta},\lambda)$ changes sign for $\lambda \geq \lambda_1$, for all $r > 0$ , then $\overline{\theta}$ is an exceptional direction for $(2.7)_\lambda$. Hence $(2.7)_\lambda$ is not oscillatory.

From above discussion, we define $\lambda_0 \geq 0$ as follows:

let

$$P(r) = \max_{0\leq\theta\leq T}\left|\frac{g(r)S(\theta)R(x)}{2\sqrt{G(r)G(1)}}\right|_{x=f_1(r)C(\theta)}, \tag{2.10}$$

$$Q(r) = \min_{0\leq\theta\leq T}\left[\frac{g(r)yh(y)}{2G(r)} + \frac{h(r)xg(x)}{2H(r)}\right]_{\substack{x=f_1(r)C(\theta),\\ y=f_2(r)S(\theta)}} \tag{2.11}$$

with $f_1(r) = \sqrt{H(r)/H(1)}$, $\quad f_2(r) = \sqrt{G(r)/G(1)}$, and define

$$\lambda_0 = \inf_{r>0} \frac{Q(r)}{P(r)}. \tag{2.12}$$

Then $\lambda_0$ is the desired value. (If $\lambda_0 = 0$, then for all $\lambda > 0$, $(2.7)_\lambda$ is not oscillatory).

## §3. Main Results

**Theorem 3.1.** *Let all conditions of $h, F, g$ stated in Introduction be satisfied, $R(x) \neq 0$ for $x \neq 0$. Then for $|\lambda| < \lambda_0, |F(x)| \leq \lambda |R(x)|$, Equation* (1.1) *is oscillatory, where $\lambda_0$ is given by* (2.12).

The proof of Theorem 3.1 is a direct consequence of Lemma 2.2 and the comparison theorem of differential equations if we write (1.1) and $(2.7)_\lambda$ as

$$\frac{dy}{dx} = \frac{-g(x)}{h(y) - F(x)} \tag{3.1$'$}$$

and

$$\frac{dy}{dx} = \frac{-g(x)}{h(y) + \lambda R(x)} \tag{3.2$'_\lambda$}$$

respectively.

**Theorem 3.2.** *Let $R(x) = h(H^{-1}(G(x)))\operatorname{sgn} x$ in Theorem* 3.1. *Especially if $h(y) = m|y|^p \operatorname{sgn} y$, with $m > 0, p > 0$, then*

$$R(x) = m \left( \frac{(p+1)G(x)}{m} \right)^{\frac{p}{p+1}} \operatorname{sgn} x,$$

*and*

$$\lambda_0 = (p+1)\Big(\frac{1}{p}\Big)^{\frac{p}{1+p}}. \tag{3.3}$$

**Proof.** We first change variables

$$u = \sqrt{2G(x)} \operatorname{sgn} x, \quad y = y, \quad d\tau = \frac{g(x)\operatorname{sgn} x\, dt}{\sqrt{2G(x)}}. \tag{3.4}$$

Then by denoting $\tau$ by $t, u$ by $x$ again, $(2.7)_\lambda$ is transformed into the following form

$$\begin{aligned} \dot{x} &= h(y) + \lambda h(H^{-1}(\frac{x^2}{2}))\operatorname{sgn} x, \\ \dot{y} &= -x. \end{aligned} \tag{3.5$_\lambda$}$$

Therefore for $h(y) = m|y|^p \operatorname{sgn} y$, the $P(r)$ and $Q(r)$ in (2.10) and (2.11) become

$$\begin{aligned} Q(r) &= \min_{0\leq\theta\leq T} \left( \frac{yh(y)}{2r} + \frac{h(r)}{2H(1)} C^2(\theta) \right) \Big|_{y=rS(\theta)} \\ &= \min_{0\leq\theta\leq T} r^p \left( m|S(\theta)|^{p+1} + (p+1)\frac{C^2(\theta)}{2} \right) \\ &= (p+1)r^p. \end{aligned}$$

Here we used the fact $\frac{C^2(\theta)}{2} + \frac{m|S(\theta)|^{p+1}}{p+1} = 1$, which is the result of (a) of (2.4).

$$P(r) = \max_{0\leq\theta\leq T} m\Big(\frac{p+1}{m}\Big)^{\frac{p}{p+1}} r^p |S(\theta)||C^{\frac{2p}{1+p}}(\theta)|.$$

Now what we need is to calculate

$$\max_{0\le\theta\le T}|S(\theta)||C(\theta)|^{\frac{2p}{1+p}}.$$

From $\frac{C^2(\theta)}{2}+\frac{m|S(\theta)|^{p+1}}{p+1}=1$ and Hölder inequality $A^{\frac{1}{\alpha}}\cdot B^{\frac{1}{\beta}}\le\frac{A}{\alpha}+\frac{B}{\beta}$ (the equality holds iff $B^{\frac{1}{\beta}}=A^{\frac{1}{\alpha}(\alpha-1)}$), where $A\ge 0, B\ge 0, \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1$, by letting

$$A=C^2(\theta)\frac{(p+1)}{2p},\quad B=|S(\theta)m^{\frac{1}{p+1}}|^{p+1},$$

we obtain

$$\begin{aligned}\max_{0\le\theta\le T}|S(\theta)||C(\theta)|^{\frac{2p}{1+p}}&=\max_{0\le\theta\le T}A^{\frac{p}{1+p}}\cdot B^{\frac{1}{p+1}}\Big/\Big[\Big(\frac{p+1}{p}\Big)^{\frac{p}{1+p}}\cdot m^{\frac{1}{p+1}}\Big]\\&=\Big[A\frac{p}{p+1}+B\cdot\frac{1}{p+1}\Big]\Big/\Big[\Big(\frac{p+1}{p}\Big)^{\frac{p}{p+1}}\cdot m^{\frac{1}{p+1}}\Big]\\&=\Big[\frac{C^2(\theta)}{2}+\frac{m|S(\theta)|^{p+1}}{p+1}\Big]\Big/\Big[\Big(\frac{p+1}{p}\Big)^{\frac{p}{p+1}}\cdot m^{\frac{1}{p+1}}\Big]\\&=\frac{p}{p+1}^{\frac{p}{p+1}}\Big(\frac{1}{m}\Big)^{\frac{1}{p+1}}.\end{aligned}$$

The equality holds iff $|S(\theta)|^{p+1}=C^2(\theta)\cdot\frac{p+1}{2pm}$, hence $P(r)=p^{\frac{p}{1+p}}r^p$, and

$$\lambda_0=\frac{Q(r)}{P(r)}=(p+1)\Big(\frac{1}{p}\Big)^{\frac{p}{1+p}},\quad p>0. \tag{3.6}$$

Now let $\overline{f}(p)=\ln\lambda_0$. Then

$$\overline{f}'(p)=\frac{\ln p}{(1+p)^2}\begin{cases}>0, & \text{if } p>1,\\ <0, & \text{if } p<1.\end{cases}$$

We know

$$\max_{p>0}\lambda_0=\lambda_0|_{p=1}=2,\quad \inf_{p>0}\lambda_0=\lim_{p\to+\infty}s\lambda_0=\lim_{p\to 0^+}\lambda_0=1.$$

Since

$$h(H^{-1}(G(x)))=m\Big(\frac{p+1}{m}\Big)^{\frac{p}{1+p}}\cdot G^{\frac{p}{1+p}}(x),$$

we have

$$\mu=\lambda_0 m\Big(\frac{p+1}{m}\Big)^{\frac{p}{1+p}}=m(p+1)\Big(\frac{1+p}{pm}\Big)^{\frac{p}{1+p}}, \tag{3.7}$$

which is the value defined in [8] and [9]. But our result holds also for $0<p<1$, so our result cover the corresponding result of [8] and [9].

**Remark 3.1.** Let $m=1, p=1$. Then

$$\mu=\sqrt{8}, |F(x)|<\sqrt{8G(x)}=\frac{\sqrt{8}}{\sqrt{G(x)}}G(x).$$

Because $G(x)\to\infty$ as $|x|\to\infty$, we obtain the results of (1)'–(3)' and (6)'.

**Theorem 3.3.** *Assume $\lambda\ge 0$ and $l(t)$ is a positive function for $t>0; v(u)>0, u\ne 0, L(u)=\int_{0^+}^{u}\frac{dt}{l(t)}$ for $u>0; L^{-1}(\cdot)$ is the inverse function of $L(\cdot)$. And suppose $|F(x)|>0$ for $x\ne 0$ and the following inequality holds:*

$$-\lambda\int_{-x}^{0}v(u)l(|F(u)|)du\le F(x)\le\lambda\int_0^x v(u)l(|F(u)|)du,\quad 0\le x<+\infty. \tag{3.8}$$

*Then*

$$|F(x)| \le L^{-1}\Big(\lambda \int_0^{|x|} v(u)du\Big) \tag{3.9}$$

*and there exists a* $\lambda_0 > 0$ *such that for* $0 \le \lambda < \lambda_0$ *Equation* (1.1) *is oscillatory.*

**Proof.** From Lemma 2.1, letting

$$k = 0, \quad m = \lambda, \quad u(t) = |F(x)|, \quad a = 0 = u_0,$$

we obtain (3.9). The rest of Theorem 3.3 can be proved by using the same methods used in the proof of Lemma 2.2 and Theorem 3.1.

If we take $v(x) = g(x),\ l(u) = u^\alpha,\ \alpha < 1$ in Theorem 3.3, then from (3.8),

$$|F(x)| \le [(1+\alpha)\lambda G(x)]^{\frac{1}{1-\alpha}} G^{\frac{1}{1-\alpha}}(x). \tag{3.10}$$

From above analysis, we can prove the following

**Example 3.1.** If $h(y) = m|y|^p \operatorname{sgn} y, \lambda_1 = m^{\frac{1}{p}}(1+p)^{(1+p)/p}, |F(x)| > 0, x \ne 0$, and

$$-\lambda \int_{-x}^0 |g(u)||F(u)|^{\frac{-1}{p}} du \le F(x) \le \lambda \int_0^x g(u)|F(u)|^{\frac{-1}{p}} du, \quad x > 0, \tag{3.11}$$

where $0 \le \lambda < \lambda_1$, then Equation (1.1) is oscillatory.

Let $\alpha = -\frac{1}{p}$, then $\frac{1}{1-\alpha} = \frac{p}{1+p}$. From (3.8), (3.9) and (3.11) we obtain (3.10). From (3.7), after some calculation, we get

$$\lambda_1 = m^{\frac{1}{p}}(1+p)^{\frac{1+p}{p}}.$$

**Remark 3.2.** Let $m = p = 1$; then $\alpha = -1, \lambda_1 = 4$. We obtain the results of (4)' and (5)'.

**Example 3.2.** Let $h(y) = y^{2n-1}$, where $n \ge 1$ is a positive integer, $\lambda_2 = (2n)^{\frac{2n}{2n-1}}, |F(x)| > 0, x \ne 0$. If

$$-\lambda \int_{-x}^0 |g(u)||F(u)|^{-\frac{1}{2n}} du \le F(x) \le \lambda \int_0^x g(u)|F(u)|^{\frac{-1}{2n-1}} du, \tag{3.12}$$

where $0 \le \lambda < \lambda_2$, then Equation (1.1) is oscillatory.

Letting $p = 2n-1, m = 1$ in Example 3.1, we obtain Example 3.2.

**Remark 3.3.** The results of (4)' and (5)' are special cases of Example 3.2, that is, when $n = 1$.

**Example 3.3.** Let $v(x) = |g(x)|^k, k > 0; l(u) = u^{-n}, n \ge 0; h(y) = y^{2m-1}, m \ge 1$. Then (3.8) becomes

$$-\lambda \int_{-x}^0 (|g(u)|^k/|F(u)|^n)du \le F(x) \le \lambda \int_0^x (|g(u)|^k/|F(u)|^n)du, \quad x \ge 0,$$

and we have

$$|F(x)| \le [(n+1)\lambda \int_0^{|x|} |g(u)|^k du]^{\frac{1}{n+1}}. \tag{3.13}$$

By Theorem 3.2. we can assume without loss of generality that $g(x) = x$. Then (3.13) becomes

$$|F(x)| \le [(n+1)\lambda]^{\frac{1}{n+1}} |x|^{k+1}/(k+1). \tag{3.14}$$

From (2.10)–(2.12), it is not difficult to obtain, if $2m+2=k$, that

$$Q(r)=2mr^{2m-1},$$

$$P(r)=\max_{0\le\theta\le 2\pi}\frac{|\sin\theta||\cos\theta|^{k+1}r^{k+1}}{k+1}=\frac{(k+1)^{\frac{k-1}{2}}}{(k+2)^{\frac{(k+2)}{2}}}r^{k+1},$$

$$\lambda_0=\left(\frac{2m(k+2)^{(k+2)/2}}{(k+1)^{(k-1)/2}}\right)^{n+1}\Big/(n+1).$$

**Example 3.4.** Let $\int_0^x v(u)du=M(G(x)), l(u)=u^\alpha, \alpha<1$ in (3.8), $M(r)>0$ for $r>0$. Then

$$|F(x)|\le[(1-\alpha)\lambda]^{\frac{1}{1-\alpha}}[M(G(x))]^{\frac{1}{1-\alpha}},$$

and $\lambda_0$ is defined as

$$\lambda_0=\inf_{r>0}\frac{Q(r)^{1-\alpha}}{(1-\alpha)P(r)},$$

where $Q(r)$ is defined by (2.11),

$$P(r)=\max_{0\le\theta\le T}\Big|\frac{(g(r)|S(\theta)|)^{1-\alpha}M(G(x))}{[4G(r)G(1)]^{\frac{1-\alpha}{2}}}\Big|_{x=f_1(r)C(\theta)}.$$

**Remark 3.4.** All we discussed above are global properties. If we restrict our attention to local properties of (1.1), we get $\lambda_0=\lambda_0(\epsilon)$, where $\epsilon>0$ and $0\le r\le r(\epsilon)<+\infty$.

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