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ON THE DIFFUSION PHENOMENON OF QUASILINEAR HYPERBOLIC WAVES

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Abstract

The authors consider the asymptotic behavior of solutions of the quasilinear hyperbolic equation with linear damping

$$u_{tt} + u_t - \operatorname{div}(a(\nabla u)\nabla u) = 0,$$

and show that, at least when $n \le 3$, they tend, as $t \to +\infty$, to those of the nonlinear parabolic equation

$$v_t - \operatorname{div}(a(\nabla v)\nabla v) = 0,$$

in the sense that the norm $\|u(.,t) - v(.,t)\|_{L^\infty(\mathrm{R}^n)}$ of the difference $u - v$ decays faster than that of either u or v. This provides another example of the diffusion phenomenon of nonlinear hyperbolic waves, first observed by Hsiao, L. and Liu Taiping (see [1, 2]).

Keywords Asymptotic behavior of solutions, Quasilinear hyperbolic and parabolic equations, Diffusion phenomenon
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§1. Introduction

1.1. Consider the following quasilinear hyperbolic Cauchy problem with linear damping

$$\begin{cases} u_{tt} + u_t - \operatorname{div}(a(\nabla u)\nabla u) = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{cases} \tag{1.1}$$

where $x \in \mathrm{R}^n$, $t \ge 0$, and $a(\cdot)$ is a smooth function satisfying

$$a(y) = 1 + O(|y|^\alpha) \quad \text{as } |y| \to 0, \quad \alpha \in \mathrm{N}. \tag{1.2}$$

The purpose of this paper is to show that, at least when $n \le 3$, the asymptotic profile of the solution $u(x,t)$ of (1.1) is given by the solution $v(x,t)$ of the corresponding parabolic problem

$$\begin{cases} v_t - \operatorname{div}(a(\nabla v)\nabla v) = 0, \\ v(x,0) = u_0(x) + u_1(x), \end{cases} \tag{1.3}$$

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in the sense that the estimate

$$\|u(\cdot,t)-v(\cdot,t)\|_{L^\infty(\mathrm{R}^n)} = 0(t^{-\min\{\frac{n+1}{2},\frac{n}{4}+1\}}) \quad \text{as} \ \ t\to\infty \tag{1.4}$$

holds for suitably small, smooth initial data u_0, u_1. Since both $\|u(.,t)\|_{L^\infty(\mathrm{R}^n)}$ and $\|v(.,t)\|_{L^\infty(\mathrm{R}^n)}$ are known to decay like $t^{-n/2}$, (1.4) implies that the difference $u-v$ decays faster than either u or v: this result, known as the "diffusion phenomenon", indicates that problem (1.1) has an asymptotically parabolic structure.

1.2. The diffusion phenomenon was originally observed by Hsiao and Liu[1], for the system of hyperbolic conservation laws with damping

$$\begin{cases} v_t-u_x=0, \\ u_t+p(v)_x=-\alpha u, \end{cases} \tag{1.5}$$

with smooth initial data $u(x,0)=u_0(x)$, $v(x,0)=v_0(x)$, that are asymptotically constant, that is,

$$\{u_0(x),v_0(x)\}\to\{u_\pm,v_\pm\} \quad \text{as} \quad x\to\pm\infty. \tag{1.6}$$

In (1.5), it is assumed that $p(v)>0$, $p'(v)<0$ for $v>0$, and v_0, $v_\pm>0$. Hsiao and Liu showed that, for $v_+\neq v_-$, the solution $\{u,v\}$ behaves asymptotically like the diffusion wave $\{\bar u,\bar v\}$, solution of the parabolic system

$$\begin{cases} \bar v_t=-\frac{1}{\alpha}p(\bar v)_{xx}, \\ p(\bar v)_x=-\alpha\bar u, \\ \bar v(x,0)\to v_\pm \quad \text{as} \ \ x\to\pm\infty. \end{cases} \tag{1.7}$$

The same problem was also considered by Li[4], who obtained better estimates than those of [1]. Note that for special initial data $\{u_0,v_0\}$ satisfying

$$\begin{cases} \{u_0,v_0\}(\pm\infty)=(0,\bar{v_0}), \quad \bar{v_0}\in\mathrm{R}^+, \\ \int_{-\infty}^{\infty}(v_0(x)-\bar{v_0})dx=0, \end{cases} \tag{1.8}$$

system (1.5) is reduced to the quasilinear hyperbolic equation with damping

$$\begin{cases} v_{tt}+\alpha v_t-(p(\bar{v_0}+v_x)-p(\bar{v_0}))_x=0, \\ v(x,0)=\int_{-\infty}^{x}(v_0(y)-\bar{v_0})dy, \qquad v_t(x,0)=u_0(x). \end{cases} \tag{1.9}$$

System (1.5) was also considered by Nishihara in [6], with an improvement of Hsiao and Liu's estimates; in [7] he also considered the equivalent second order formulation in one space dimension

$$\begin{cases} u_{tt}+\alpha u_t-(a(u_x))_x=0, \\ u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x), \end{cases} \tag{1.10}$$

and proved that solutions of (1.10) behave asymptotically as those of the linear parabolic problem

$$\begin{cases} \alpha\phi_t-a'(0)\phi_{xx}=0, \\ \phi(x,0)=u_0(x)+\frac{1}{\alpha}u_1(x), \end{cases} \tag{1.11}$$

as described by the estimates

$$\begin{aligned} \|u(\cdot,t)-\phi(\cdot,t)\|_{L^\infty(\mathrm{R}^n)} &= O(t^{-1}), \\ \|u_x(\cdot,t)-\phi_x(\cdot,t)\|_{L^\infty(\mathrm{R}^n)} &= O(t^{-3/2}), \\ \|u_t(\cdot,t)-\phi_t(\cdot,t)\|_{L^\infty(\mathrm{R}^n)} &= O(t^{-2}). \end{aligned} \tag{1.12}$$

1.3. In this paper we consider problem (1.1) in R^n for $n\le 3$, and extend Nishihara's result, showing that, as $t\to+\infty$, the solutions of (1.1) converge to those of (1.3), in the sense described by the asymptotic estimate (1.4), which corresponds to the first of estimates (1.12) for $n = 1$. Rather than comparing the hyperbolic problem with a linearized parabolic one, we follow a possibly more natural approach, considering both nonlinear problems (1.1) and (1.3) directly: we do so, by resorting to fairly well-known decay estimates for the nonlinear equations, which in turn depend on classical decay estimates on the linear, dissipative equations. On the other hand, we remark that the special form of the initial value in the reduced parabolic problem (1.3) seems to be essential, because we are not able to obtain the faster decay estimates for different initial values.

This paper is organized as follows: after recalling from [5], [8], and [9] some results on the global existence and asymptotic behavior of solutions to nonlinear hyperbolic and parabolic problems, we proceed in Section 2 to the study of the diffusion phenomenon in the linear case; this is of interest not only in itself, but also because, as expected, it provides us with the necessary basis to deal with the nonlinear case, which we consider in Section 3. Throughout this paper we denote different constants by the same C, and by L^p, H^s, $W^{s,p}$ the usual Sobolev spaces $L^p(\mathrm{R}^n)$, $H^s(\mathrm{R}^n)$, $W^{s,p}(\mathrm{R}^n)$.

1.4. We conclude this section by recalling some global existence results and asymptotic estimates for classical solutions to nonlinear hyperbolic and parabolic problems. We start with the Cauchy problem

$$\begin{cases} u_{tt} + u_t - \Delta u = F(Du, D^2u), \\ u(x,0) = \epsilon u_0(x), \qquad u_t(x,0) = \epsilon u_1(x), \end{cases} \tag{1.13}$$

where $D \doteq \{\partial_1, \cdots, \partial_n\}$, $\partial_i = \partial/\partial x_i$, and $\epsilon > 0$ is a small parameter. Setting

$$\hat\lambda \doteq \{(\lambda_i),\ i = 1, \cdots, n;\ \ (\lambda_{ij}),\, i,j = 1, \cdots, n\,\},$$

we assume that in a neighborhood of $\hat\lambda = 0$ the nonlinear term $F = F(\hat\lambda)$ in (1.13) is a sufficiently smooth function, satisfying

$$F(\hat\lambda) = O(|\hat\lambda|^{1+\alpha}), \qquad \alpha \in \mathrm{N}; \tag{1.14}$$

that is, the Taylor expansion of $F(\hat\lambda)$ at $\hat\lambda = 0$ starts with a term of order $1 + \alpha$. The following result on problem (1.13) is proven in [9]:

Theorem 1.1. *For all $n \geq 1$, given integer $s \geq n + 7$, assume* (1.14) *and that*

$$u_0 \in H^{s+1} \cap L^1, \qquad u_1 \in H^s \cap L^1. \tag{1.15}$$

If ϵ is sufficiently small, problem (1.13) *admits a unique, globally defined classical solution u, satisfying the estimates*

$$\|u(\cdot,t)\|_{L^\infty} \leq C(1+t)^{-\frac{n}{2}}, \tag{1.16}$$

$$\|\nabla u(\cdot,t)\|_{W^{[\frac{s}{2}]+1,\infty}} \leq C(1+t)^{-\frac{n+1}{2}}, \tag{1.17}$$

$$\|\nabla u(\cdot,t)\|_{W^{[\frac{n}{2}]+[\frac{s}{2}]+3,2}} \leq C(1+t)^{-\frac{n+2}{4}}. \tag{1.18}$$

Analogously, for the nonlinear parabolic problem

$$\begin{cases} v_t - \Delta v = F(Dv, D^2v), \\ v(x,0) = \epsilon v_0(x), \end{cases} \tag{1.19}$$

we have from [5] and [8] the following result:

Theorem 1.2. *For any $n \geq 1$ and integer $s \geq n + 7$, assume F is as above, and that $v_0 \in H^{s+1} \cap L^1$. Then, if ϵ is sufficiently small, problem* (1.19) *has a unique, globally defined* *classical solution, satisfying the estimates*

$$\|v(.,t)\|_{L^\infty} \le C(1+t)^{-\frac{n}{2}}, \tag{1.20}$$

$$\|\nabla v(.,t)\|_{W^{[\frac{s}{2}],\infty}} \le C(1+t)^{-\frac{n+1}{2}}, \tag{1.21}$$

$$\|\nabla v(.,t)\|_{W^{s,2}} \le C(1+t)^{-\frac{n+2}{4}}. \tag{1.22}$$

We remark that a comparison of the decay rates given by Theorems 1.1 and 1.2 already shows the asymptotically parabolic nature of problem (1.13).

§2. The Linear Case

2.1. Consider the linear problem

$$\begin{cases} u_{tt}+u_t-\Delta u=0, \\ u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x), \end{cases} \tag{2.1}$$

we can represent its solution as

$$u(x,t)=(k_1(.,t)*u_1)(x)+(k_2(.,t)*u_0)(x), \tag{2.2}$$

where $*$ denotes convolution with respect to the space variables, and $\hat{k}_i(\xi,\cdot)$ (Fourier transform with respect to space variables) solves the ODE

$$\frac{d^2}{dt^2}\hat{k}_i+\frac{d}{dt}\hat{k}_i+|\xi|^2\hat{k}_i=0, \tag{2.3}$$

with initial values respectively

$$\begin{cases} \hat{k}_1(\xi,0)=0, \qquad \frac{d\hat{k}_1}{dt}(\xi,0)=1, \\ \hat{k}_2(\xi,0)=1, \qquad \frac{d\hat{k}_2}{dt}(\xi,0)=0. \end{cases} \tag{2.4}$$

We have then from [10, Lemma 1]

Lemma 2.1. *For any multi-index β and integer $i\ge 0$, if $u_0\in H^{[\frac{n}{2}]+|\beta|+i+1}(\mathrm{R}^n)\cap L^1(\mathrm{R}^n)$ and $u_1\in H^{[\frac{n}{2}]+|\beta|+i}(\mathrm{R}^n)\cap L^1(\mathrm{R}^n)$, then*

$$\left\|(D_x)^\beta(\partial_t)^i(k_1(.,t)*u_1)\right\|_{L^2} \le C(1+t)^{-\frac{n+2|\beta|}{4}-i}\{\|u_1\|_{L^1}+\|u_1\|_{H^{|\beta|+i-1}}\}, \tag{2.5}$$

$$\left\|(D_x)^\beta(\partial_t)^i(k_2(.,t)*u_0)\right\|_{L^2} \le C(1+t)^{-\frac{n+2|\beta|}{4}-i}\{\|u_0\|_{L^1}+\|u_0\|_{H^{|\beta|+i}}\}, \tag{2.6}$$

$$\left\|(D_x)^\beta(\partial_t)^i(k_1(.,t)*u_1)\right\|_{L^\infty} \le C(1+t)^{-\frac{n+|\beta|}{2}-i}\{\|u_1\|_{L^1}+\|u_1\|_{H^{[\frac{n}{2}]+|\beta|+i}}\}, \tag{2.7}$$

$$\left\|(D_x)^\beta(\partial_t)^i(k_2(.,t)*u_0)\right\|_{L^\infty} \le C(1+t)^{-\frac{n+|\beta|}{2}-i}\{\|u_0\|_{L^1}+\|u_0\|_{H^{[\frac{n}{2}]+|\beta|+i+1}}\}. \tag{2.8}$$

In particular, we deduce from Lemma 2.1 that, as $t\to\infty$,

$$\|u_t(.,t)\|_{L^2}, \quad \|\Delta u(.,t)\|_{L^2}=O((1+t)^{-\frac{n}{4}-1}), \tag{2.9}$$

$$\|u_{tt}(.,t)\|_{L^2}=O((1+t)^{-\frac{n}{4}-2}), \tag{2.10}$$

$$\|u_t(.,t)\|_{L^\infty}, \quad \|\Delta u(.,t)\|_{L^\infty}=O((1+t)^{-\frac{n}{2}-1}), \tag{2.11}$$

$$\|u_{tt}(.,t)\|_{L^\infty}=O((1+t)^{-\frac{n}{2}-2}), \tag{2.12}$$

and the crucial remark is that, since the term u_{tt} decays faster than u_t and Δu, it can be neglected as $t\to\infty$. This is at the basis of the diffusion phenomenon exhibited by the dissipative problem (2.1), namely that, asymptotically, the behavior of u, being determined by u_t and Δu, is essentially parabolic.

2.2. We now describe the diffusion phenomenon in the linear case: letting v be the solution of

$$\begin{cases} v_t - \Delta v = 0, \\ v(x,0) = u_0(x) + u_1(x), \end{cases} \tag{2.13}$$

and setting $w = u - v$, we obtain

$$\begin{cases} w_t - \Delta w = -u_{tt}, \\ w(x,0) = -u_1(x). \end{cases} \tag{2.14}$$

We claim:

Theorem 2.1. *Assume that u_0, u_1 are as in Lemma 2.1. Then for all $n \geq 1$,*

$$\|w(.,t)\|_{L^\infty} = O(t^{-\frac{n}{4}-1}) \qquad as\ \ t \to \infty; \tag{2.15}$$

consequently, the diffusion principle holds when $n \leq 3$ for the linear dissipative wave equation.

Proof. We first note that (2.17) does imply the diffusion principle, since it implies that when $n \leq 3$ the difference $u - v$ of the solutions to problems (2.1) and (2.13) decays faster than either u or v, as described by (1.16) and (1.20). To prove (2.15), we start by solving (2.14) exactly, with

$$w(x,t) = \int_{\mathrm{R}^n} E(x-y,t)(-u_1(y))dy + \int_0^t \int_{\mathrm{R}^n} E(x-y,t-\tau)(-u_{tt}(y,\tau))dyd\tau, \tag{2.16}$$

where E is the heat kernel

$$E(x,t) = \frac{1}{(2\sqrt{\pi t})^n} e^{-|x|^2/4t}, \tag{2.17}$$

which satisfies the asymptotic decay estimates

$$\|E(.,t)\|_{L^1} \leq O(1), \tag{2.18}$$

$$\|\partial_t^k E(.,t)\|_{L^2} \leq O(t^{-\frac{n}{4}-k}), \tag{2.19}$$

$$\|\partial_t^k E(.,t)\|_{L^\infty} \leq O(t^{-\frac{n}{2}-k}), \tag{2.20}$$

$$\|\nabla E(.,t)\|_{L^\infty} \leq O(t^{-\frac{n+1}{2}}) \tag{2.21}$$

(see e.g. [10, §11.2]). We then split the integral in (2.16) into integrals over the intervals $[0, t/2]$ and $[t/2, t]$; by integration by parts in τ, we compute that

$$\begin{aligned} &-\int_0^{\frac{t}{2}} \int_{\mathrm{R}^n} E(x-y,t-\tau)u_{tt}(y,\tau)dyd\tau \\ =& -\int_{\mathrm{R}^n} E(x-y,t/2)u_t(y,t/2)dy + \int_{\mathrm{R}^n} E(x-y,t)u_1(y)dy \\ &-\int_0^{\frac{t}{2}} \int_{\mathrm{R}^n} E_t(x-y,t-\tau)u_t(y,\tau)dyd\tau, \end{aligned} \tag{2.22}$$

therefore (this is where the special form of the initial value $v(.,0)$ in (2.13) is needed), (2.16) can be rewritten as

$$w(x,t) = -\int_{\mathrm{R}^n} E(x-y,t/2)u_t(y,t/2)dy - \int_0^{\frac{t}{2}}\int_{\mathrm{R}^n} E_t(x-y,t-\tau)u_t(y,\tau)dyd\tau$$
$$-\int_{\frac{t}{2}}^{t}\int_{\mathrm{R}^n} E(x-y,t-\tau)u_{tt}(y,\tau)dyd\tau \equiv I_1(x,t)+I_2(x,t)+I_3(x,t). \tag{2.23}$$

By (2.9), (2.12), (2.18), and (2.19), we can estimate

$$|I_1(x,t)| \le \|E(\cdot,t/2)\|_{L^2}\|u_t(\cdot,t/2)\|_{L^2} \le Ct^{-\frac{n}{4}}(1+t)^{-1-\frac{n}{4}} = O(t^{-\frac{n}{2}-1}), \tag{2.24}$$

$$|I_2(x,t)| \le \int_0^{\frac{t}{2}} \|E_t(\cdot,t-\tau)\|_{L^2}\|u_t(\cdot,\tau)\|_{L^2}d\tau$$
$$\le C\,t^{-\frac{n}{4}-1}\int_0^{\frac{t}{2}}(1+\tau)^{-1-\frac{n}{4}}d\tau = O(t^{-\frac{n}{4}-1}); \tag{2.25}$$

$$|I_3(x,t)| \le \int_{\frac{t}{2}}^{t} \|E(\cdot,t-\tau)\|_{L^1}\|u_{tt}(\cdot,\tau)\|_{L^\infty}d\tau$$
$$\le C\int_{\frac{t}{2}}^{t}\|u_{tt}(\cdot,\tau)\|_{L^\infty}d\tau \le C\int_{\frac{t}{2}}^{t}(1+\tau)^{-\frac{n}{2}-2}d\tau \le O(t^{-\frac{n}{2}-1}). \tag{2.26}$$

Theorem 2.1 follows then from (2.24), (2.25) and (2.26).

§3. The Nonlinear Case

3.1. We now describe the diffusion phenomenon for the nonlinear problem (1.1), which we rewrite as

$$\begin{cases} u_{tt}+u_t-\Delta u = \operatorname{div}((a(\nabla u)-1)\nabla u), \\ u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x). \end{cases} \tag{3.1}$$

From Theorem 1.1 we know that, if u_0 and u_1 are sufficiently small, problem (3.1) has a unique global classical solution, which satisfies estimates (1.16), (1.17) and (1.18). In fact, we also have

Lemma 3.1. *Assume $s \ge n+7$,* (1.2) *and* (1.15). *Then the solution of* (3.1) *satisfies the following estimates*

$$\|u_t(.,t)\|_{L^2} \le C\,(1+t)^{-\frac{n}{4}-1}$$

and

$$\|u_{tt}(.,t)\|_{L^\infty} \le C\,(1+t)^{-\frac{n}{2}-\frac{3}{2}}.$$

Proof. These estimates can be established as in [3], by means of a straightforward application of Lemma 2.1 and Theorem 1.1, using Duhamel's formula to write the solution of (3.1) as the following

$$u(x,t) = (k_1(.,t)*u_1)(x) + (k_2(.,t)*u_0)(x) + \int_0^t k_1(.,t-\tau)*\operatorname{div}((a(\nabla u)-1)\nabla u)d\tau.$$

Let $v(x,t)$ be the solution of (1.3), and $\phi = u - v$. Then ϕ satisfies

$$\begin{cases} \phi_t - \Delta\phi = -u_{tt} + \operatorname{div}[a(\nabla u)-1)\nabla u - (a(\nabla v)-1)\nabla v] \equiv -u_{tt} + \operatorname{div}G, \\ \phi(x,0) = -u_1(x). \end{cases} \tag{3.2}$$

To estimate G, we shall need

Lemma 3.2. *Let a be as in* (1.3), *and w a smooth function. Then*

$$\|(a(\nabla w)-1)\nabla w\|_{L^1} \le C\,\|\nabla w\|_{L^\infty}^{\alpha-1}\|\nabla w\|_{L^2}^2, \tag{3.3}$$

$$\|\mathrm{div}((a(\nabla w)-1)\nabla w)\|_{L^\infty} \le C\,\|\nabla w\|_{L^\infty}^{\alpha}\|D^2 w\|_{L^\infty}. \tag{3.4}$$

Proof. These estimates are an immediate consequence of assumption (1.2), here we omit the detail.

As in the linear case, the solution of (3.2) can be represented as

$$\begin{aligned}\phi(x,t) = &\int_{\mathrm{R}^n} E(x-y,t)(-u_1(y))dy \\ &+\int_0^t\int_{\mathrm{R}^n} E(x-y,t-\tau)(-u_{tt}(y,\tau)+\mathrm{div}\,G(y,\tau))dyd\tau,\end{aligned} \tag{3.5}$$

and the diffusion phenomenon is described by the analogous of Theorem 2.1, namely

Theorem 3.1. *Assume that u_0, u_1 are as in Theorem* 2.1. *Then*

$$\|\phi(.,t)\|_{L^\infty} = O(t^{-\min\{\frac{n+1}{2},\frac{n}{4}+1\}}) \qquad \text{as } t\to\infty; \tag{3.6}$$

therefore, the diffusion phenomenon holds when $n \le 3$ for system (3.1).

Proof. Proceeding exactly as in the proof of Theorem 2.1, we decompose

$$\begin{aligned}\phi(x,t) = &-\int_{\mathrm{R}^n} E(x-y,t/2)u_t(y,t/2)dy \\ &-\int_0^{\frac{t}{2}}\int_{\mathrm{R}^n} E_t(x-y,t-\tau)u_t(y,\tau)dyd\tau - \int_{\frac{t}{2}}^t\int_{\mathrm{R}^n} E(x-y,t-\tau)u_{tt}(y,\tau)dyd\tau \\ &+\Big(\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t\Big)\int_{\mathrm{R}^n} E(x-y,t-\tau)\mathrm{div}\,G(y,\tau)dyd\tau v \\ \equiv{}& I_1'(x,t)+I_2'(x,t)+I_3'(x,t)+I_4'(x,t)+I_5'(x,t).\end{aligned} \tag{3.7}$$

To estimate the right-hand side of (3.7), we use the results of Lemmas 3.1, 3.2 and Theorems 1.1, 1.2. Since the solution of problem (3.1) decays like that of the linear problem (2.1), we have at first that, for $i=1,2,3$,

$$|I_i'(x,t)| \le O(t^{-\min\{\frac{n}{4}+1,\frac{n+1}{2}\}}), \tag{3.8}$$

this is exactly as in the linear case. The estimate of $|I_4'|$ and $|I_5'|$ is similar: by (2.21), (3.3), (1.18) and (1.22), we have

$$\begin{aligned}|I_4'(x,t)| &\le \int_0^{\frac{t}{2}} \|\nabla E(\cdot,t-\tau)\|_{L^\infty}(\|((a(\nabla u)-1)\nabla u)(.,\tau)\|_{L^1} \\ &\quad + \|((a(\nabla v)-1)\nabla v)(.,\tau)\|_{L^1})d\tau \\ &\le C\,t^{-\frac{n+1}{2}}\int_0^{\frac{t}{2}} \big(\|\nabla u(.,\tau)\|_{L^2}^2+\|\nabla v(.,\tau)\|_{L^2}^2\big)\,d\tau \\ &\le C\,t^{-\frac{n+1}{2}}\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{n}{2}-1}d\tau = O(t^{-\frac{n+1}{2}}),\end{aligned} \tag{3.9}$$

and, by (2.18), (3.4), (1.17) and (1.21),

$$\begin{aligned}|I_5'(x,t)| &\le \int_{\frac{t}{2}}^{t} \|E(\cdot,t-\tau)\|_{L^1}\|\mathrm{div}\, G(.,\tau)\|_{L^\infty} d\tau \\ &\le C\int_{\frac{t}{2}}^{t}(1+\tau)^{-(n+1)}d\tau \le O(t^{-n}). \end{aligned}\tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we can easily conclude the proof of Theorem 3.1.

3.2 Remarks. (1) In a similar way, it is also possible to obtain faster decay estimates of the differences $\|\nabla u(\cdot,t)-\nabla v(\cdot,t)\|_{L^\infty}$ and $\|u_t(\cdot,t)-v_t(\cdot,t)\|_{L^\infty}$, as in (1.12).

(2) If $\|u_t(.,t)\|_{L^1}$ satisfies the same estimate as $\|v_t(.,t)\|_{L^1}$, that is,

$$\|u_t(.,t)\|_{L^1} \le O((1+t)^{-1}), \tag{3.11}$$

then the limitation $n\le 3$ would be unnecessary, for estimate (2.15) could be improved to

$$\|w(.,t)\|_{L^\infty} \le O(t^{-\frac{n}{2}-1+\delta}), \tag{3.12}$$

with arbitary small $\delta>0$ and $n\ge 1$. However, we do not know if estimate (3.11) holds.

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