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EQUIVALENCE BETWEEN EXACT INTERNAL CONTROLLABILITY OF THE KIRCHHOFF PLATE-LIKE EQUATION AND THE WAVE EQUATION**

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Abstract

When the rotatory inertia is taken into account, vibrations of a linear plate can be described by the Kirchhoff plate equation. Consider this equation with locally distributed control forces and some boundary condition which is the simply supported boundary condition for a rectangular plate. In this paper, the authors establish exact controllability of the system in terms of the equivalence to exact internal controllability of the wave equation, by means of a frequency domain characterization of exact controllability introduced recently in [11].

Keywords Kirchhoff plate equation, Locally distributed control, Exact controllability, Wave equation, Frequency domain condition

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§1. Introduction

Consider the following initial-boundary value problem

$$\begin{cases} y_{tt} - \gamma\Delta y_{tt} + \Delta^2 y = b(x)u(x,t) & \text{in} \quad \Omega \times (0,T), \\ y = \Delta y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\gamma > 0$ is a constant, $b(x)u(x,t)$ is the control force with distributed function $b(x) \geq 0$ for all $x \in \Omega$, Ω is a bounded open set in $\mathbb{R}^n$ with the Lipschitz boundary $\partial\Omega$. When Ω is a rectangle, (1.1) is the well-known Kirchhoff plate equation with the simply supported boundary condition. The one-dimensional version of (1.1) is the Rayleigh beam equation. For various dynamical plate models, see [7,8].

We will establish exact controllability of (1.1) in terms of the equivalence to exact internal controllability of the wave equation

$$\begin{cases} w_{tt} - \Delta w = b(x)h(x,t) & \text{in} \quad \Omega \times (0,T), \\ w = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & x \in \Omega. \end{cases} \tag{1.2}$$

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Let $D(\Delta) = \{v \in H_0^1(\Omega) \mid \Delta v \in L^2(\Omega)\}$, $V = H_0^1(\Omega)$, $H = L^2(\Omega)$. Then the finite energy state spaces of the systems (1.1), (1.2) are, respectively, $\mathcal{H}_k = D(\Delta) \times V$, $\mathcal{H}_w = V \times H$. Consider the generalized function space $H^{-1} = V' \supset H \supset V$. For $b \in C^1(\bar{\Omega})$ and $u \in H^{-1}$, we define $bu \in H^{-1}$ by

$$(bu)(\xi) = u(b\xi), \quad \forall \xi \in H_0^1(\Omega). \tag{1.3}$$

We have used the Sobolev spaces. We refer the readers to [1] for the knowledge of those spaces.

Definition 1.1. *The system* (1.1) *is said to be exactly controllable in* $\mathcal{H}_k$ *by* H^{-1}*-controls if for every* $(y_0, y_1) \in \mathcal{H}_k$*, there exist a* $T > 0$ *and a control* $u(\cdot) \in L^2(0, T; H^{-1})$*, such that the solution of* (1.1) *satisfies* $(y(\cdot, T), y_t(\cdot, T)) = 0$ *in* $\mathcal{H}_k$.

Exact controllability of (1.2) in $\mathcal{H}_w$ by L^2-controls can be defined similarly.

In this paper, we will prove

Theorem 1.1. *Suppose systems* (1.1) *and* (1.2) *have the same* Ω *and* $0 \le b(\cdot) \in C^2(\bar{\Omega})$*. Then, the system* (1.1) *is exactly controllable in* $\mathcal{H}_k$ *by* H^{-1}*-controls if and only if the system* (1.2) *is exactly controllable in* $\mathcal{H}_w$ *by* L^2*-controls.*

We point out that exact internal controllability of the Petrovsky Equation ((1.1) with neglect of the rotatory inertia, i.e., $\gamma = 0$) has been extensively studied (see [5, 6, 9, 10, 11, 13]). However, to the authors best knowledge, exact internal controllability of (1.1), even for the one-dimensional case, was not considered before this work. Our controllability result is sufficient for exponential stabilizability and solvability of the regulator problem over infinite time, although it does not give a description of control time duration on which every initial state can be steered to the zero state.

The remainder of this paper is composed of two sections. In §2, we prove Theorem 1.1 by means of a frequency domain characterization of exact controllability of conservative systems, which is introduced in [11]. In §3, we give some conclusions following from our main theorem and make some comments. These results provide useful information for designing the location of controllers/dampers for a vibrating plate when the rotatory inertia is taken into account.

§2. Proof of Theorem 1.1

Denote the norm and the inner product in H by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Define in H

$$A = -\Delta, \ \ D(A) = \{v \in H_0^1(\Omega) \mid \Delta v \in L^2(\Omega)\}; \quad By = by, \quad \forall\, y \in H.$$

Then A is positive-definite self-adjoint and $B \in \mathcal{L}(H)$. It is well known that $V = D(A^{\frac{1}{2}})$, $\|A^{\frac{1}{2}}v\| = \||\nabla v|\|$ for $v \in V$, and H^{-1} is the completion of H under the norm $\|A^{-\frac{1}{2}} \cdot \|$. Therefore, both $(V, \|\cdot\|_V)$ and $(H^{-1}, \|A^{-\frac{1}{2}} \cdot \|)$ are Hilbert spaces, where

$$\|v\|_V = \left(\|v\|^2 + \gamma \|A^{\frac{1}{2}}v\|^2\right)^{\frac{1}{2}}.$$

Let $S = I + \gamma A$. Then S is also a positive-definite self-adjoint operator in H. And we have $D(S^{\frac{1}{2}}) = V$, $\|\cdot\|_V = \|S^{\frac{1}{2}} \cdot \|$.

The operator A can be extended as a positive-definite self-adjoint operator in H^{-1} and the defined domain of the extension is V. B can also be extended as a bounded operator on H^{-1} by (1.3). Denote their extensions in H^{-1} also by A, B, S. Then (1.1), (1.2) can be rewritten in the following standard form of the second order conservative system with control:

$$\ddot{y} + A_1 y = B_1 u \quad \text{ in } V, \tag{2.1}$$

$$\ddot{w} + Aw = Bh \quad \text{ in } H \tag{2.2}$$

plus the corresponding initial condition, where $A_1 = A^2 S^{-1}$ is a positive-definite self-adjoint operator in V with the defined domain $D(A_1) = D(A^{\frac{3}{2}})$, and $B_1 = S^{-1}B \in \mathcal{L}(H^{-1},\ V)$. According to Theorem 3.4 in [11], the system (2.1) is exactly controllable in $\mathcal{H}_k$ if and only if there exists a constant $\delta_1 > 0$ such that

$$\|(\omega^2 - A_1)y\|_V + \|\omega B_1 B_1^* y\|_V \geq \delta_1 \|\omega y\|_V, \quad \forall \omega \in \mathbb{R},\ y \in D(A_1), \tag{2.3}$$

the system (2.2) is exactly controllable in $\mathcal{H}_w$ if and only if there exists a constant $\delta > 0$ such that

$$\|(\omega^2 - A)y\| + \|\omega BB^* y\| \geq \delta \|\omega y\|, \quad \forall \omega \in \mathbb{R},\ y \in D(A). \tag{2.4}$$

Lemma 2.1. *If* $b \in C^2(\bar{\Omega})$, *then* $BD(A) \subset D(A),\ BV \subset V$, *and* $B_1^* = AB \in \mathcal{L}(V;\ H^{-1})$.

Proof. Since $b \in C^2(\bar{\Omega})$, obviously we have $BD(A) \subset D(A),\ BV \subset V$. For $u \in H,\ y \in V$,

$$\begin{aligned}\langle B_1 u, y\rangle_V &= \langle S^{\frac{1}{2}} S^{-1} Bu,\ S^{\frac{1}{2}} y\rangle = \langle u, By\rangle \\ &= \langle A^{-\frac{1}{2}} u, A^{-\frac{1}{2}} ABy\rangle = \langle u, ABy\rangle_{H^{-1}}.\end{aligned}$$

Thus, we obtain $B_1^* = AB$.

Lemma 2.2. *The inequality* (2.3) *holds if and only if there exists a constant* $\delta' > 0$ *such that*

$$\|(\omega^2 - A^2 S^{-1})y\| + \|\omega A^{-\frac{1}{2}} BABA^{-\frac{1}{2}} y\| \geq \delta' \|\omega y\|, \quad \forall \omega \in \mathbb{R},\ y \in D(A). \tag{2.5}$$

Proof. Since $\|\cdot\|_V = \|S^{\frac{1}{2}} \cdot\|$, by Lemma 1.1 the inequality (2.3) is

$$\|(\omega^2 - A^2 S^{-1}) S^{\frac{1}{2}} y\| + \|\omega S^{\frac{1}{2}} S^{-1} BABy\| \geq \delta_1 \|\omega S^{\frac{1}{2}} y\|, \quad \forall \omega \in \mathbb{R},\ y \in D(A^{\frac{3}{2}}). \tag{2.6}$$

Obviously, (2.6) is equivalent to

$$\begin{aligned}&\|(\omega^2 - A^2 S^{-1})y\| + \|\omega (A^{\frac{1}{2}} S^{-\frac{1}{2}}) A^{-\frac{1}{2}} BABA^{-\frac{1}{2}} (A^{\frac{1}{2}} S^{-\frac{1}{2}}) y\| \\ &\geq \delta_1 \|\omega y\|, \quad \forall \omega \in \mathbb{R},\ y \in D(A).\end{aligned} \tag{2.7}$$

Observing that both $A^{\frac{1}{2}} S^{-\frac{1}{2}}$ and $\big(A^{\frac{1}{2}} S^{-\frac{1}{2}}\big)^{-1}$ belong to $\mathcal{L}(H)$ by the closed graph theorem, we conclude that (2.5) is true if and only if (2.7) is true.

Proof of Theorem 1.1. We will prove the equivalence between (2.4) and (2.5). If (2.4) does not hold, then there exist $\omega_n \in \mathbb{R},\ y_n \in D(A)$ with $|\omega_n| > \frac{1}{2} \|A^{-\frac{1}{2}}\|^{-1}$ and

$$\|\omega_n y_n\| = 1 \tag{2.8}$$

such that

$$\|\omega_n BB^* y_n\| \to 0, \tag{2.9}$$

$$\|(\omega_n^2 - A) y_n\| \to 0. \tag{2.10}$$

Since $B^* = B$, (2.8) and (2.9) imply

$$\|\omega_n B y_n\| \to 0. \tag{2.11}$$

We multiply (2.10) by $\|y_n\|$ to get

$$\|A^{\frac{1}{2}} y_n\| \to 1. \tag{2.12}$$

This implies that $\{y_n\}$ is a bounded sequence in V. Thus, by the compactness of the imbedding $V \hookrightarrow H$ there exists a subsequence of $\{y_n\}$, denoted still by $\{y_n\}$, which converges to some y_0 in H.

(i) Case $y_0 \neq 0$: By (2.8), $|\omega_n| \to \|y_0\|^{-1}$. We can assume without loss of generality that $\omega_n \to \omega_0 (\neq 0)$. Then, (2.11) implies

$$By_0 = 0. \tag{2.13}$$

Moreover, from (2.10) and the closedness of A we have

$$y_0 \in D(A), \quad Ay_0 = \omega_0^2 y_0. \tag{2.14}$$

Thus

$$BA^{-\frac{1}{2}}y_0 = \frac{1}{|\omega_0|}By_0 = 0, \tag{2.15}$$

$$A^2S^{-1}A^{-\frac{1}{2}}y_0 = \frac{\omega_0^4}{1+\gamma\omega_0^2}A^{-\frac{1}{2}}y_0. \tag{2.16}$$

Combining (2.15), (2.16) and $\|A^{-\frac{1}{2}}y_0\| \neq 0$, we conclude that (2.5) does not hold.

(ii) Case $y_0 = 0$: Since $AS^{-1} = \frac{1}{\gamma}(1 - S^{-1}) \in \mathcal{L}(H)$, by (2.10) we have

$$\Big\|\Big(\frac{\omega_n^2}{\gamma} - A^2S^{-1}\Big)y_n\Big\| = \Big\|\frac{1}{\gamma}(\omega_n^2 - A)y_n + \frac{1}{\gamma}AS^{-1}y_n\Big\| \to 0, \tag{2.17}$$

$$f_n \equiv (|\omega_n| - A^{\frac{1}{2}})y_n = (|\omega_n| + A^{\frac{1}{2}})^{-1}(\omega_n^2 - A)y_n \to 0 \quad \text{in} \quad H, \tag{2.18}$$

$$\|\omega_n^2 A^{-\frac{1}{2}}y_n\| \to 1. \tag{2.19}$$

Equation (2.18) implies

$$\|\omega_n A^{-\frac{1}{2}}y_n\| \to 0. \tag{2.20}$$

We now write

$$\begin{aligned}\|\omega_n A^{\frac{1}{2}}BA^{-\frac{1}{2}}y_n\|^2 &= |\omega_n|^2\langle A(BA^{-\frac{1}{2}}y_n)\ ,\ BA^{-\frac{1}{2}}y_n\rangle \\ &= |\omega_n|^2\langle bA^{\frac{1}{2}}y_n - 2\nabla b\cdot\nabla(A^{-\frac{1}{2}}y_n) - \Delta bA^{-\frac{1}{2}}y_n\ ,\ bA^{-\frac{1}{2}}y_n\rangle.\end{aligned} \tag{2.21}$$

By (2.18), (2.19) and (2.11), we have

$$|\omega_n|^2\langle bA^{\frac{1}{2}}y_n, bA^{-\frac{1}{2}}y_n\rangle = \langle|\omega_n|By_n - bf_n, |\omega_n|^2bA^{-\frac{1}{2}}y_n\rangle \to 0, \tag{2.22}$$

$$\begin{aligned}|\omega_n|^2\big|\langle\nabla b\cdot\nabla(A^{-\frac{1}{2}}y_n), bA^{-\frac{1}{2}}y_n\rangle\big| &\le |\nabla b|_\infty\,\|\omega_n|\nabla(A^{-\frac{1}{2}}y_n)|\|\,|b|_\infty\,\|\omega_nA^{-\frac{1}{2}}y_n\| \\ &= |\nabla b|_\infty\,|b|_\infty\,\|\omega_nA^{-\frac{1}{2}}y_n\| \to 0,\end{aligned} \tag{2.23}$$

$$|\omega_n|^2|\langle\Delta bA^{-\frac{1}{2}}y_n, bA^{-\frac{1}{2}}y_n\rangle| \le |\Delta b|_\infty\,|b|_\infty\,\|\omega_nA^{-\frac{1}{2}}y_n\|^2 \to 0, \tag{2.24}$$

where $|\cdot|_\infty$ is the norm in $L^\infty(\Omega)$ or $L^\infty(\Omega;\mathbb{R}^n)$.

It follows from (2.21)–(2.24) that

$$\|\omega_nA^{\frac{1}{2}}BA^{-\frac{1}{2}}y_n\| \to 0. \tag{2.25}$$

Since $BV \subset V$, by the closed graph theorem we have $A^{\frac{1}{2}}BA^{-\frac{1}{2}} \in \mathcal{L}(H)$. Thus, we obtain

$$\|\omega_nA^{-\frac{1}{2}}BABA^{-\frac{1}{2}}y_n\| = \|\omega_n(A^{\frac{1}{2}}BA^{-\frac{1}{2}})^*(A^{\frac{1}{2}}BA^{-\frac{1}{2}})y_n\| \to 0. \tag{2.26}$$

Combination of (2.8), (2.17) and (2.26) contradicts (2.5).

On the other hand, if (2.5) is false, then there exist $\omega_n \in \mathbb{R},\ y_n \in D(A)$ with

$$\|\omega_n y_n\| = 1 \tag{2.27}$$

such that

$$\|\omega_n A^{-\frac{1}{2}} BABA^{-\frac{1}{2}} y_n\| \to 0, \tag{2.28}$$

$$\|(\omega_n^2 - A^2 S^{-1}) y_n\| \to 0. \tag{2.29}$$

Combination of (2.27) and (2.28) yields

$$\|\omega_n A^{\frac{1}{2}} BA^{-\frac{1}{2}} y_n\| \to 0. \tag{2.30}$$

Multiplying (2.29) by $\|y_n\|$, we get

$$\|AS^{-\frac{1}{2}} y_n\| = \|(AS^{-1}) S^{\frac{1}{2}} y_n\| \to 1. \tag{2.31}$$

Since $(AS^{-1})^{-1} = \gamma I + A^{-1} \in \mathcal{L}(H)$, from (2.31) we see that $\{y_n\}$ is a bounded sequence in V. Therefore, there exists again a subsequence of $\{y_n\}$, denoted still by $\{y_n\}$, which converges to some y_0 in H.

(1) Case $y_0 \neq 0$: The proof is simple and quite similar to that in (i).

(2) Case $y_0 = 0$: Since $AS^{-1} \in \mathcal{L}(H)$, from (2.29) we derive

$$\|\gamma \omega_n^2 y_n - A y_n\| \to 0, \tag{2.32}$$

which implies

$$\|\sqrt{\gamma} |\omega_n| y_n - A^{\frac{1}{2}} y_n\| \to 0. \tag{2.33}$$

Observing that $\|A^{\frac{1}{2}} y_n\|$ is bounded, by (2.30) we have

$$\begin{aligned} &\omega_n \langle BA^{\frac{1}{2}} y_n - 2\nabla b \cdot \nabla (A^{-\frac{1}{2}} y_n) - \Delta b A^{-\frac{1}{2}} y_n \ , \ y_n \rangle \\ &= \omega_n \langle ABA^{-\frac{1}{2}} y_n \ , \ y_n \rangle = \omega_n \langle A^{\frac{1}{2}} BA^{-\frac{1}{2}} y_n \ , \ A^{\frac{1}{2}} y_n \rangle \to 0. \end{aligned} \tag{2.34}$$

It is easy to see

$$\omega_n \langle \nabla b \cdot \nabla (A^{-\frac{1}{2}} y_n) \ , \ y_n \rangle \to 0, \qquad \omega_n \langle A^{-\frac{1}{2}} y_n \Delta b \ , \ y_n \rangle \to 0. \tag{2.35}$$

Thus, from (2.34) we obtain

$$\omega_n \langle A^{\frac{1}{2}} y_n \ , \ B y_n \rangle = \omega_n \langle BA^{\frac{1}{2}} y_n \ , \ y_n \rangle \to 0. \tag{2.36}$$

Combination of (2.33) and (2.36) yields

$$\langle \omega_n^2 B y_n \ , \ y_n \rangle \to 0. \tag{2.37}$$

Since $0 \leq B = B^* \in \mathcal{L}(H)$, we conclude

$$\omega_n B^* B y_n \to 0 \quad \text{in} \quad H. \tag{2.38}$$

Equations (2.27), (2.32) and (2.38) contradict (2.4).

§3. Conclusions and Comments

Let $0 \leq b(\cdot) \in C^2(\bar{\Omega})$ and $G = \{x \in \Omega \mid b(x) > 0\}$. Then G is an open subset of Ω. We call it control region. It expresses geometric characteristics (such as location, measure and shape) of the controller of the system (1.1)/(1.2). In design of the controller of (1.1)/(1.2), an important problem is how to choose G so that the exact controllability of (1.1)/(1.2) can be achieved. For the wave equation (1.2), this problem has been well solved (see [2,3,4] for at least C^3 smooth Ω, and [11] for $C^{1,\alpha}$ smooth or convex Ω). Therefore, for the system (1.1) the answer to the problem can follow readily from our Theorem 1.1 and those existing results. Here we only present the conclusions concerning the one-dimensional and rectangular Ω because in the two cases the boundary condition is natural. By Theorem 1.1 here, Theorems 3.2, 4.1, 4.2, 4.5 and Remark 4.3 in [11] we have the following results:

(1) Let Ω be an interval. Then, the system (1.1) is exactly controllable in $\mathcal{H}_k$ by H^{-1} controls if and only if there exists $x_0 \in \bar{\Omega}$ such that $b(x_0) > 0$.

(2) Let Ω be a rectangle. Then, the system (1.1) is exactly controllable in $\mathcal{H}_k$ by H^{-1} controls if either one of the following holds:

(i) G contains either one of the diagonal lines of Ω.

(ii) G contains two open straight lines satisfying that each side of $\bar{\Omega}$ meets at least one of the four end points and the two lines intersect at a point in Ω.

But the system (1.1) is never exactly controllable if G is contained in a proper subrectangle of Ω (compare with the result in [9]).

When Ω is a general multi-dimensional region, for the Kirchhoff plate-like equation we only need the same regularity condition on Ω as in the wave equation, say, that Ω is $C^{1,\alpha}$ smooth or convex. The C^3 smooth $\partial\Omega$ is required in [13], for the Petrovsky equation.

The concentrated forces are allowed to apply to the Kirchhoff plate models and result in the finite energy states. If we use the L^2, instead of H^{-1}, control forces, the system (1.1) is never exactly controllable in $\mathcal{H}_k$ because of the compactness of the control operator.

For the Petrovsky system, the time duration on which every initial state can be steered to the zero state is allowed to be arbitrarily small (see [9,13] for example). This property never occurs to the system (1.1) for the finite speed of propagation of waves. If the system (1.1) is exactly controllable, then there exists a uniform control time duration on which every initial state can be steered to the zero state (see [12]). The problem about determination of such a time duration remains open.

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