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MAXIMUM GENUS, INDEPENDENCE NUMBER AND GIRTH***

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Abstract

It is known (for example see [2]) that the maximum genus of a graph is mainly determined by the Betti deficiency of the graph. In this paper, the authors establish an upper bound on the Betti deficiency in terms of the independence number as well as the girth of a graph, and thus use the formulation in [2] to translate this result to lower bound on the maximum genus. Meantime it is shown that both of the bounds are best possible.

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§1. Introduction

The graph considered here is connected and simple unless stated otherwise. For terminology and notation without explanation, we refer to [1].

In this paper we study the maximum genus of a graph, an invariance that characterizes the graph cellularly embedded in an orientable surface. Recall that the maximum genus, denoted by $\gamma_M(G)$, of the graph G is the maximum integer number k with the property that there exists a cellular embedding of G on the orientable surface S of genus k. The Euler polyhedral equation shows that the maximum genus of any graph G satisfies the following inequality

$$\gamma_M(G) \leq \Big\lfloor \frac{\beta(G)}{2} \Big\rfloor,$$

where $\beta(G) = |E(G)| - |V(G)| + 1$ is known as the cycle rank of the graph G. A graph G is said to be upper embeddable if $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$.

For details concerning the maximum genus of graphs, the reader may refer to [2] or [3].

The maximum genus (also, the upper embeddability) of graphs has received considerable attention for these years. Particularly, one of the most interesting questions is to give a better lower bound on the maximum genus of a graph in terms of other invariances of the graph. Concerning these results, the interested reader may refer to papers [6–14].

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Let G be a graph. Denote by $\alpha(G)$ the independence number of G, and by $g(G)$ the girth of G. If G has no circuit, we define $g(G) = \infty$.

In this paper we provide a better upper bound on the Betti deficiency $\xi(G)$ of a graph G in terms of the independence number as well as the girth (the definition of the Betti deficiency will be given in the next section), and thus obtain a better lower bound on the maximum genus. Our main results are the following two theorems.

Theorem 1.1. *Let G be a graph. Then $\xi(G) \leq 2\alpha(G)/(g(G)-1)$.*

Theorem 1.2. *Let G be a graph. Then $\gamma_M(G) \geq (\beta(G)-m)/2$, where*

$$m = 2\alpha(G)/(g(G)-1).$$

Since $g(G) \geq 3$ for any simple graph G, the following result is a subsequence of Theorems 1.1 and 1.2 above.

Corollary 1.1. *Let G be a graph. Then*

$$\xi(G) \leq \alpha(G) \ \text{ and } \ \gamma_M(G) \geq \frac{1}{2}\Big(\beta(G) - \alpha(G)\Big).$$

Moreover the last section of this paper gives examples of graphs to show that the bounds in the two theorems are best possible in the sense.

§2. Two Basic Results on $\gamma_M(G)$

Suppose T is a spanning tree of a graph G, and denote by $\xi(G,T)$ the number of components of $G\backslash E(T)$ with odd number of edges. We call $\xi(G) = \min_T \xi(G,T)$ the Betti deficiency of G, where T is taken over all the spanning trees of G. Again, for any subset $A \subseteq E(G)$, denote by $c(G\backslash A)$ the number of components of $G\backslash A$, and by $b(G\backslash A)$ the number of components of $G\backslash A$ with odd cycle rank.

The following two theorems are basic for the study of the maximum genus of graphs.

Theorem 2.1.[2,4] *Let G be a graph. Then $\gamma_M(G) = (\beta(G) - \xi(G))/2$.*

Theorem 2.2.[5] *Let G be a graph. Then*

$$\xi(G) = \max_{A \subseteq E(G)} \{c(G\backslash A) + b(G\backslash A) - |A| - 1\}.$$

§3. Some Lemmas

Lemma 3.1. *Let e be a cut-edge of a graph G, and let G_1 and G_2 be two components of $G\backslash e$. Then $\xi(G) = \xi(G_1) + \xi(G_2)$.*

Proof. Let T be any spanning tree of G. Since e is a cut-edge of G, $T\backslash e$ has exactly two subgraphs, say T_1 and T_2, which are viewed as two spanning trees of G_1 and G_2, respectively. By a direct application of the definition of the Betti deficiency, it can be induced that $\xi(G_1) + \xi(G_2) \leq (G)$. Analogously, the reverse inequality $\xi(G_1) + \xi(G_2) \geq (G)$ is easily obtained.

Lemma 3.2. *Let G be a graph with $\xi(G) \geq 2$. Then there exists an edge e of G satisfying one of the following two properties:*

(1) *if $G\backslash e$ is connected, then $\xi(G) \leq \xi(G\backslash e) - 1$;*

(2) *if $G\backslash e$ is disconnected, that is, e is a cut-edge of G, then*

$$\xi(G) = \xi(G') + \xi(F) \quad \textit{and} \quad \xi(F) \leq 1,$$

where G' and F are the two components of $G\backslash e$.

Proof. By Theorem 2.2, there exists a subset $A \subseteq E(G)$ such that

$$\xi(G) = c(G\backslash A) + b(G\backslash A) - |A| - 1. \tag{$*$}$$

We first note that $c(G\backslash A) \geq 2$. If, otherwise, it is not the case, from the fact that $b(G\backslash A) \leq c(G\backslash A)$ it is easy to deduce from $(*)$ that $\xi(G) \leq 1$. It contradicts the assumption that $\xi(G) \geq 2$. Meantime we also see that $A \neq \emptyset$ because G is connected and $c(G\backslash A) \geq 2$. We now consider the following two cases.

Case 1. There exists an edge $e \in A$ such that $G\backslash e$ is connected. In this case we set $A' = A\backslash\{e\}$. Clearly,

$$A' \subseteq E(G\backslash e), \quad |A'| = |A| - 1,$$

and furthermore $(G\backslash e)\backslash A' = G\backslash A$. Therefore

$$c\Big((G\backslash e)\backslash A'\Big) = c(G\backslash A) \quad \text{and} \quad b\Big((G\backslash e)\backslash A'\Big) = b(G\backslash A).$$

Combining Theorem 2.2 with $(*)$ above, we can obtain that $\xi(G\backslash e) \geq \xi(G) + 1$. This shows that the property (1) is satisfied.

Case 2. For any edge $e \in A$, $G\backslash e$ is disconnected. In this case we first prove the following two claims.

Claim 1. *For any edge $e \in A$, the two end vertices of e must belong to two distinct components of $G\backslash A$.*

Subproof. By reduction to absurdity. Assume that there exists an edge $e \in A$ such that the two end vertices x_1 and x_2 of e belong to some component of $G\backslash A$, say H. Since H is connected, there is a path of H connecting x_1 with x_2, and thus the edge e lies on a circuit of G, which implies that $G\backslash e$ is connected. It contradicts the assumption that $G\backslash e$ is disconnected.

Claim 2. *There exists an edge $e \in A$ and a component F of $G\backslash e$ such that F is also a component of $G\backslash A$.*

Subproof. We construct a new graph G_A as follows. The vertices of G_A are all the components of $G\backslash A$. For each edge in A make an edge in G_A joining the corresponding vertices. First, G_A is connected since G is connected, and $|V(G_A)| \geq 2$ since $c(G\backslash A) \geq 2$. Second, it follows from the above Claim 1 that G_A has no loop and all the edges of G_A correspond to A. Finally, each edge of G_A is a cut-edge from the assumption of Case 2 that $G\backslash e$ is disconnected for each edge $e \in A$. Therefore, G_A is a tree, and then has an edge e' incident to a vertex v' of degree one. Let $e \in A$ be the edge of G corresponding to e' and F be the component of $G\backslash A$ corresponding to v'. It is easy to see that both e and F are desired.

Now we choose such e and F as described in Claim 2. Let G' be the other component of $G\backslash e$. Since F and G' are the two components of $G\backslash e$, it follows from Lemma 3.1 that $\xi(G) = \xi(G') + \xi(F)$. In the following we shall prove $\xi(F) \leq 1$. Set $A' = E(G') \bigcap A$. By Claim 1, $E(F) \bigcap A = \emptyset$, and thus

$$|A'| = |A\backslash\{e\}| = |A| - 1.$$

Meantime we note that all the components of $G'\backslash A'$ are the same as those of $G\backslash A$ except for F. It thus implies that $c(G'\backslash A') = c(G\backslash A) - 1$, and $b(G'\backslash A') \geq b(G\backslash A) - 1$ no matter whether $\beta(F)$ is odd or even. Using Theorem 2.2 together with $(*)$ above, we easily get that

$\xi(G') \geq \xi(G) - 1$, and thus $\xi(F) \leq 1$. The above arguments show that the property (2) of the lemma holds.

Lemma 3.3. *Let G be a graph. For any edge e of G, we have*

(1) $g(G) \leq g(G\backslash e)$; (2) $\alpha(G\backslash e) \leq \alpha(G) + 1$.

Proof. By the definition of girth, (1) is straightforward. The truth of (2) is also clear from the fact that adding an edge joining two vertices in a graph leads to the independence number decreasing at most one.

Lemma 3.4. *Let G be a graph but not a tree. We have*

(1) *for any vertex $x \in V(G)$, there exists an independent set $J \subseteq V(G)$ such that $x \notin J$ and $2|J|/(g(G) - 1) \geq 1$;*

(2) $2\alpha(G)/(g(G) - 1) \geq 1$.

Proof. We only prove (1) because (2) is a direct result of (1). Since G is not a tree, G must have a circuit C with the length of $g(G)$. Let $C = y_1y_2 \cdots y_ky_1$, where $y_i \in V(G)$, and $1 \leq i \leq k = g(G)$. By the definition of girth, any two not successive vertices lying on C are not adjacent. If $x \notin V(C)$, we can find an independent set $J = \{y_i \in V(C) | \, i \text{ is odd}\}$. It then is easy to check that J is just what we need because $|J| \geq (k - 1)/2$ and $x \notin J$. If $x \in V(C)$, the conclusion is also true as long as we relabel the vertex on C such that the index of x is even. This proves the lemma.

§4. The Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. The method is by induction on the number of edges of the graph G. We first note that if G is a tree, clearly $\xi(G) = 0$ and $g(G) = \infty$ by the definitions, and thus the conclusion is true. Now we assume that the conclusion is true for a graph of less number of edges than that of G, and we shall prove that the conclusion holds for G. The following three cases are considered.

Case 1. $\xi(G) = 0$. The conclusion is trivial.

Case 2. $\xi(G) = 1$. Obivously G is not a tree, and thus the conclusion is immediate by Lemma 3.4 (2).

Case 3. $\xi(G) \geq 2$. In this case, according to Lemma 3.2 we deal with the following two subcases.

Subcase 3.1. The property (1) of Lemma 3.2 holds. Then we have the following inequalities

$$
\begin{aligned}
\xi(G) &\leq \xi(G\backslash e) - 1 \quad \text{(by the property (1) of Lemma 3.2)} \\
&\leq \frac{2\alpha(G\backslash e)}{g(G\backslash e) - 1} - 1 \quad \text{(by the inductive hypothesis)} \\
&\leq \frac{2\alpha(G) + 2}{g(G) - 1} - 1 \quad \text{(by Lemma 3.3)} \\
&= \frac{2\alpha(G) + 3 - g(G)}{g(G) - 1} \\
&\leq \frac{2\alpha(G)}{g(G) - 1} \quad \text{(because } G \text{ is simple and } g(G) \geq 3)
\end{aligned}
$$

Subcase 3.2. The property (2) of Lemma 3.2 holds. Since both G' and F are the two components of $G\backslash e$, we have that $g(G) \leq g(G')$ and $g(G) \leq g(F)$. Furthermore, we also note that $\alpha(G') \le \alpha(G)$ because a maximum independent set in G' is an independent set in G. By the inductive hypothesis we first have

$$\xi(G') \le \frac{2\alpha(G')}{g(G')-1}.$$

If $\xi(F) = 0$, then

$$\xi(G) = \xi(G') + \xi(F) = \xi(G') \le \frac{2\alpha(G')}{g(G')-1} \le \frac{2\alpha(G)}{g(G)-1}.$$

If $\xi(G) = 1$, certainly G is not a tree. Let $x \in V(F)$ be one end vertex of e. By Lemma 3.4 (1) there exists an independent set J of F such that $x \notin J$ and $1 \le 2|J|/(g(F)-1)$. Since $x \notin J$, we easily get that $\alpha(G') + |J| \le \alpha(G)$. So

$$\xi(G) = \xi(G') + \xi(F) = \xi(G) + 1 \le \frac{2\alpha(G')}{g(G')-1} + \frac{2|J|}{g(F)-1} \le \frac{2\alpha(G)}{g(G)-1}.$$

All the cases covered above show that the conclusion is true for G. Therefore the inductive hypothesis finishes the proof.

Proof of Theorem 1.2. Combining Theorem 2.1 with Theorem 1.1, we see that the proof is straightforward.

§5. The Bounds in Theorems 1.1 and 1.2

In this section we shall show that the bounds in Theorems 1.1 and 1.2 are both best possible. Because the lower bound on the maximum genus in Theorem 1.2 is just a translation of the upper bound on $\xi(G)$ in Theorem 1.1 by using Theorem 2.1, we only consider the sharpness of the upper bound on $\xi(G)$ in Theorem 1.1.

Fact 1. *The bound in Theorem* 1.1 *is achieved by a circuit C of odd length k.*

Clearly, $\xi(C) = 1$. Furthermore $\alpha(C) = (g(C)-1)/2$ because k is odd. Therefore

$$\xi(C) = 2\alpha(C)/(g(C)-1) = 1.$$

We now give another fact. Let H_m be a star graph with $m+1$ vertices, that is to say, H_m is a tree of $m+1$ vertices with one vertex v of H_m being adjacent to all the other m vertices $v_1, v_2, \cdots, v_m$. Now we obtain a new graph from H_m by replacing each vertex v_i $(1 \le i \le m)$ with a circuit C_k of odd length $k \ge 3$, and putting the edge formerly incident with v_i to be incident with some vertex in C_k. Denote the resulting graph by G_k^m. For example, the following figure helps us to understand the graph G_k^m.

A graph G_k^m for $m = 4$ and $k = 3$

Fact 2. *For any small real number $\varepsilon > 0$, there exists infinitely many graphs G_k^m such that*

$$\xi(G_k^m) + \varepsilon > \frac{\alpha(G_k^m)}{g(G_k^m) - 1}.$$

From a simple observation we see that for any spanning tree T of G_k^m, all the components of $G \backslash E(T)$ with odd number of edges are exactly composed of m edges, each of which lies on a different circuit C_k. Therefore, $\xi(G_k^m, T) = m$, and so $\xi(G_k^m) = m$ by the definition. On the other hand, it is clear that the girth $g(G_k^m) = k$, and furthermore it is easy to check that the independence number $\alpha(G_k^m) = m(k-1)/2 + 1$ (noting that k is odd). Thus we have

$$\frac{2\alpha(G_k^m)}{g(G_k^m) - 1} = m + \frac{2}{k-1} = \xi(G_k^m) + \frac{2}{k-1}.$$

Since $\lim\limits_{k\to\infty} 2/(k-1) = 0$, the statement of the fact is clear.

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