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EQUIVALENCE OF COMPLETE CONVERGENCE AND LAW OF LARGE NUMBERS FOR B-VALUED RANDOM ELEMENTS**

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Abstract

Under some conditions on probability, this note discusses the equivalence between the complete convergence and the law of large number for B-valued independent random elements. The results of [10] become a simple corollary of the results here. At the same time, the author uses them to investigate the equivalence of strong and weak law of large numbers, and there exists an example to show that the conditions on probability are weaker.

Keywords Complete convergence, B-valued random element, Law of large number, Equivalence
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§1. Introduction and Main Results

A sequence $\{X_n, n \geq 1\}$ of real valued random variables is said to satisfy the law of large numbers of Hsu-Robbins (1947) type with a sequence $\{b_n\}$ of real numbers if

$$\sum_{n=1}^{\infty} P(|S_n - b_n| \geq n\epsilon) < +\infty, \quad \forall \epsilon > 0, \tag{1.1}$$

where $S_n = \sum\limits_{i=1}^{n} X_i$. Conditions under which (1.1) holds were discussed in many papers. A more general result for iid real valued random variables is given in [2].

Let B be a real separable Banach space with norm $\|\cdot\|$ and $\{X_n\}$ a sequence of B-valued random elements and put $S_n = \sum\limits_{i=1}^{n} X_i, n \geq 1$. C and c denote positive finite constants which may change from one place to another.

The Banach space B is called p-type space $(1 \leq p \leq 2)$ if for any zero mean B-valued independent random element sequence $\{X_n, n \geq 1\}$, there exists $C = C_p > 0$ such that

$$E\Big\|\sum_{i=1}^{n} X_i\Big\|^p \leq C \sum_{i=1}^{n} E\|X_i\|^p, \quad n \geq 1. \tag{1.2}$$

Let S be the class of positive non-decreasing function $\varphi(x)$ on $R^+ = [0, +\infty)$ satisfying the following conditions:

(i) There exists a constant $k = k(\varphi) > 0$ such that

$$\varphi(xy) \leq k(\varphi(x) + \varphi(y)), \quad \forall x, y \in R^+.$$

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(ii) $x/\varphi(x)$ is non-decreasing for sufficiently large x.

Recently, Yang[10] studied the complete convergence for independent but not necessary identically distributed random elements in a Banach space of type 2, her results are as follows:

Theorem A. *Suppose that* $0 < t < 2, \varphi(\cdot) \in S, \delta > 0, d = 1$ *or* -1. *Let* $\{X_n\}$ *be a sequence of independent random elements in a Banach space of type* 2, *if* $\sum\limits_{i=1}^{n} E[\|X_i\|^t(\varphi(\|X_i\|^t))^{-d}]^{1+\delta} = O(n)$, *then* $\forall\epsilon > 0$, $\sum\limits_{n=1}^{\infty} \frac{1}{n} P(\|S_n\| \geq \epsilon \cdot (n(\varphi(n))^d)^{1/t}) < +\infty$;

$$\sum_{n=1}^{\infty} \frac{1}{n} P\Big(\max_{1\leq k\leq n} \|S_k\| \geq \epsilon \cdot (n(\varphi(n))^d)^{1/t}\Big) < +\infty.$$

The aim of this paper is to discuss the complete convergence of Banach space valued independent but not necessary identically distributed random elements under some conditions on probability. As corollary, we give Theorem A. We use them to obtain characterization for law of large numbers of B-valued independent random elements, and furthermore investigate the equivalence for strong and weak law of large numbers of B-valued random elements. Meanwhile, there exsits an example to show that the conditions on probability here are weaker.

Our main results are as follows:

Theorem 1.1. *Let* $1 \leq t < 2, r > 1, \varphi(\cdot) \in S, \delta > 0, d = 1$ *or* -1 *and let* $\{X_n\}$ *be a sequence of* B*-valued independent random elements. If* $\sum\limits_{i=1}^{n} P(\|X_i\|^t(\varphi(\|X_i\|))^{-d} > x) \leq Cnx^{-(r+\delta)}$ *for sufficiently large* x *and* n, *then the following two statements are equivalent:*

(a) $S_n/(n(\varphi(n))^d)^{1/t} \to 0$ *in Probability;*

(b) *Suppose that* $l(x) > 0$ *is a slowly varying function as* $x \to +\infty$ *and* $EX_n = 0$, *then* $\forall\epsilon > 0$, *the following statements are equivalent and hold:*

(i) $\displaystyle\sum_{n=1}^{\infty} n^{r-2} l(n) P(\|S_n\| \geq \epsilon \cdot (n(\varphi(n))^d)^{1/t}) < +\infty;$

(ii) $\displaystyle\sum_{n=1}^{\infty} n^{r-2} l(n) P\Big(\max_{1\leq k\leq n} \|S_k\| \geq \epsilon \cdot (n(\varphi(n))^d)^{1/t}\Big) < +\infty;$

(iii) $\displaystyle\sum_{n=1}^{\infty} n^{r-2} l(n) P\Big(\sup_{k\geq n}(\|S_k\|/(k(\varphi(k))^d)^{1/t}) \geq \epsilon\Big) < +\infty.$

Theorem 1.2. *Let* $1 \leq t < 2, \varphi(\cdot) \in S, \delta > 0, d = 1$ *or* -1 *and let* $\{X_n\}$ *be a sequence of* B*-valued independent random elements. If* $\sum\limits_{i=1}^{n} P(\|X_i\|^t(\varphi(\|X_i\|))^{-d} > x) \leq Cnx^{-(1+\delta)}$ *for sufficiently large* x *and* n, *then the following statements are equivalent:*

(a) $S_n/(n(\varphi(n))^d)^{1/t} \to 0$ *in Probability;*

(b) *Suppose that* $l(x) > 0$ *is a non-decreasing slowly varying function as* $x \to +\infty$ *and* $EX_n = 0$, *then* $\forall\epsilon > 0$, *the following statements are equivalent and hold:*

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\|S_n\| \geq \epsilon \cdot (n(\varphi(n))^d)^{1/t}) < +\infty;$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{l(n)}{n} P\Big(\max_{1\leq k\leq n} \|S_k\| \geq \epsilon \cdot (n(\varphi(n))^d)^{1/t}\Big) < +\infty.$

(c) *Suppose that* $l(x) > 0$ *is a non-decreasing slowly varying function as* $x \to +\infty$ *and*

$EX_n = 0$, then $\forall\epsilon > 0$ we have

$$\sum_{n=1}^{\infty}\frac{l(n)}{n}P\Big(\sup_{k\ge n}(\|S_k\|/(k(\varphi(k))^d)^{1/t})\ge\epsilon\Big)<+\infty.$$

Corollary 1.1. *Suppose that $0 < t < 2, r \ge 1, \varphi(\cdot)\in S, \delta > 0, d = 1$ or -1, $l(x) > 0$ is a slowly varying function as $x \to +\infty$ and when $r = 1, l(x) > 0$ is non-decreasing. Let $\{X_n\}$ be a sequence of B-valued independent random elements; when $1 \le t < 2$, further let B be of type p for some $t < p \le 2$ and $EX_n = 0$. If*

$$\sum_{i=1}^{n}E[\|X_i\|^t(\varphi(\|X_i\|))^{-d}]^{r+\delta}=O(n), \tag{1.3}$$

then $\forall\epsilon > 0$

(i) $\displaystyle\sum_{n=1}^{\infty}n^{r-2}l(n)P(\|S_n\|\ge\epsilon\cdot(n(\varphi(n))^d)^{1/t})<+\infty;$

(ii) $\displaystyle\sum_{n=1}^{\infty}n^{r-2}l(n)P\Big(\max_{1\le k\le n}\|S_k\|\ge\epsilon\cdot(n(\varphi(n))^d)^{1/t}\Big)<+\infty;$

(iii) $\displaystyle\sum_{n=1}^{\infty}n^{r-2}l(n)P\Big(\sup_{k\ge n}(\|S_k\|/(k(\varphi(k))^d)^{1/t})\ge\epsilon\Big)<+\infty.$

Corollary 1.2. *Let $0 < t < 2, \varphi(\cdot)\in S, \delta > 0, d = 1$ or -1 and $\{X_n\}$ be a sequence of B-valued independent random elements. If*

$$\sum_{i=1}^{n}P(\|X_i\|^t(\varphi(\|X_i\|))^{-d}>x)\le Cnx^{-(1+\delta)} \tag{1.4}$$

for sufficiently large x and n, then the following statements are equivalent:

(a) $S_n/(n(\varphi(n))^d)^{1/t}\to 0$ *in Probability;* (b) $S_n/(n(\varphi(n))^d)^{1/t}\to 0$ *a.s.*

Remark 1.1. Obviously, Corollary 1.1 is a general result. For example, taking $r = 1,\ l(x) = 1,\ p = 2$, we see that Corollary 1.1 becomes Theorem A.

Remark 1.2. Let $1 \le t < 2$, and $\{X_n\}$ be a sequence of iid B-valued random elements with $E\|X_1\|^t < +\infty$. de Acosta[3] proved the following results:

$$S_n/n^{1/t}\to 0 \ \text{ in Probability } \iff S_n/n^{1/t}\to 0 \ \text{ a.s.}$$

In 1993, Wang, Bhaskara Rao and Yang[7] extended the above results to the B-valued independent random element sequence $\{X_n\}$ which is uniformly stochastic bounded by a non-negative real random variable X (i.e. $\sup_n P(\|X_n\| > x) \le P(X > x)$) and satisfies $EX^t < +\infty$. The random element in Corollary 1.2 is not necessarily uniformly stochastic bounded and the constant δ in condition (1.4) is necessary.

Example. In Corollary 1.2, taking $\delta = 0,\ t = 1\ \varphi(x) = 1$, we see that there exists iid symmetric real random variable sequence $\{X_n\}$, which satisfies $\lim_{x\to\infty} xP(|X_1| > x) = 0$ and $S_n/n \to 0$ in Probability, but $S_n/n \to 0$ a.s. is not true[6].

Remark 1.3. By a theorem of [1] and the above example, we know that the constant δ in Theorem 1.2 can not be dropped.

§2. Proofs of Main Results

It is well known that if $l(x) > 0$ is a slowly varying function as $x \to +\infty$, then

(1) $\lim_{x\to+\infty}\frac{l(tx)}{l(x)}=1,\ \forall t>0;\quad \lim_{x\to+\infty}\frac{l(x+u)}{l(x)}=1,\ \forall u\ge 0.$

(2) $\lim_{k\to+\infty}\sup_{2^k\le x\le 2^{k+1}}\frac{l(x)}{l(2^k)}=1.$

(3) $\lim_{x\to+\infty}x^{\delta}l(x)=+\infty,\ \lim_{x\to+\infty}x^{-\delta}l(x)=0,\ \forall\delta>0.$

Lemma 2.1.[9,Lemma 1] *Let $\varphi(\cdot) \in S, \delta > 0$, then for any $x \geq 0$, there exists positive constant C such that*

$$C\varphi(x) \leq \varphi(x\varphi(x)) \leq C\varphi(x),\ C\varphi(x) \leq \varphi(x/\varphi(x)) \leq C\varphi(x),\ C\varphi(x) \leq \varphi(x^{\delta}) \leq C\varphi(x).$$

Proof of Theorem 1.1. We prove only the case for $d = 1$, the proof of the case for $d = -1$ is analogous.

(a) $\Rightarrow$ (b). By the condition $S_n/(n\varphi(n))^{1/t} \xrightarrow{P} 0$ and using the Ottaviani inequality[8, p.15], we have

$$P\Big(\max_{1\leq k\leq n} \|S_k\| \geq \epsilon \cdot (n\varphi(n))^{1/t}\Big) \leq CP\Big(\|S_n\| \geq \frac{\epsilon}{2} \cdot (n\varphi(n))^{1/t}\Big) \tag{2.1}$$

for sufficiently large n. $\forall \epsilon > 0$, when $2^m < n \leq 2^{m+1}$ for sufficiently large m, by the Ottaviani inequality we can obtain

$$P(\|S_{2^m}\| \geq \epsilon \cdot (2^m\varphi(2^m))^{1/t}) \leq CP(\|S_n\| \geq c\epsilon \cdot (n\varphi(n))^{1/t}). \tag{2.2}$$

By the property of $l(x)$, the definition of $\varphi(x)$ and using Lemma 2.1 of [2], (2.2) and noting that $r > 1$, we have

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P\Big(\sup_{k\geq n} \|S_k/(k\varphi(k))^{1/t}\| \geq \epsilon\Big) \leq C \sum_{n=1}^{\infty} n^{r-2} l(n) P(\|S_n\| \geq c\epsilon \cdot (n\varphi(n))^{1/t}) + C. \tag{2.3}$$

Hence, by (2.1) and (2.3), we obtain that (i), (ii) and (iii) are equivalent. Thus, we need only to prove that (i) holds. By the symmetrization inequality[8, p.114], we may suppose that $\{X_n\}$ is symmetric.

Let $Y_{ni} = X_i I(\|X_i\| \leq (n\varphi(n))^{1/t})$, $S'_n = \sum_{i=1}^{n} Y_{ni}$, $S''_n = S_n - S'_n$. Obviously

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} l(n) P(\|S_n\| \geq \epsilon \cdot (n\varphi(n))^{1/t}) \\
&\leq C \sum_{i=0}^{\infty} 2^{i(r-1)} l(2^i) \max_{2^i\leq n<2^{i+1}} P\Big(\|S''_n\| \geq \frac{\epsilon}{2} \cdot (n\varphi(n))^{1/t}\Big) \\
&\quad + C \sum_{i=0}^{\infty} 2^{i(r-1)} l(2^i) \max_{2^i\leq n<2^{i+1}} P\Big(\|S'_n\| \geq \frac{\epsilon}{2} \cdot (n\varphi(n))^{1/t}\Big) =: I_1 + I_2 + C.
\end{aligned}$$

By the properties of $l(x)$ and $\varphi(x)$ and using Lemma 2.1 we get

$$I_1 \leq C \sum_{i=0}^{\infty} 2^{-\delta i} l(2^i) + C < +\infty. \tag{2.4}$$

Since $r > 1$, by the property (3) of $l(x)$ we get $S''_n/(n\varphi(n))^{1/t} \xrightarrow{P} 0$ from (2.4), further we obtain $S'_n/(n\varphi(n))^{1/t} \xrightarrow{P} 0$ from assumption, and using Lemma 3.1 of [3] we have

$$E\|S'_n/(n\varphi(n))^{1/t}\| \to 0 \quad \text{as } n \to \infty. \tag{2.5}$$

Therefore, to prove $I_2 < \infty$, by (2.5) it suffices to show that

$$I_2^* =: C \sum_{i=0}^{\infty} 2^{i(r-1)} l(2^i) \max_{2^i\leq n<2^{i+1}} P(|\|S'_n\| - E\|S'_n\|| \geq \epsilon \cdot (n\varphi(n))^{1/t}) < +\infty, \quad \forall \epsilon > 0.$$

In fact, choosing $q > \max\Big\{2, \frac{2t(r-1)}{2-t}, \frac{2(r-1)}{r+\delta-1}, rt\Big\}$, and using Theorem 2.1 of [3], we get

$$\begin{aligned}
I_2^* &\leq C \sum_{i=0}^{\infty} 2^{i(r-1-q/t)} l(2^i) (\varphi(2^i))^{-q/t} \max_{2^i\leq n<2^{i+1}} \Big[\sum_{k=1}^{n} E\|Y_{nk}\|^2\Big]^{q/2} \\
&\quad + C \sum_{i=0}^{\infty} 2^{i(r-1-q/t)} l(2^i) (\varphi(2^i))^{-q/t} \max_{2^i\leq n<2^{i+1}} \sum_{k=1}^{n} E\|Y_{nk}\|^q =: I_3 + I_4.
\end{aligned}$$

Using the definition of $\varphi(x)$ and Lemma 2.1 we may obtain that for every $\alpha, \beta > 0$,

$$\varphi(x^\alpha) \le Cx^\beta \tag{2.6}$$

for sufficiently large x. By the monotonicity of $\varphi(x)$, $x/\varphi(x)$, using the property (3) of $l(x)$ and (2.6) we have

$$\begin{aligned}
I_3 &\le C\sum_{i=0}^{\infty} 2^{i(r-1-q/t)} l(2^i)(\varphi(2^i))^{-q/t} \max_{2^i \le n < 2^{i+1}} \Big[\sum_{k=1}^{n} \int_0^{n\varphi(n)} x^{2/t-1} P(\|X_k\|^t > x)dx\Big]^{q/p} \\
&\le C\sum_{i=0}^{\infty} 2^{i(r-1-q/t+q/2)} l(2^i)(\varphi(2^i))^{-q/t} \\
&\quad + \begin{cases} C\sum\limits_{i=0}^{\infty} 2^{i[r-1-q(r+\delta)/2+q/2)} l(2^i) & \text{if } \frac{2}{t}-r-\delta > 0 \\ C\sum\limits_{i=0}^{\infty} 2^{i(r-1-q/t+q/2)} l(2^i)(\varphi(2^i))^{q(r+\delta)/2-q/t} [\log(2^i\varphi(2^i)]^{q/2} & \text{if } \frac{2}{t}-r-\delta = 0 \\ C\sum\limits_{i=0}^{\infty} 2^{i(r-1-q/t+q/2)} l(2^i)(\varphi(2^i))^{q(r+\delta)/2-q/t} & \text{if } \frac{2}{t}-r-\delta < 0 \end{cases} \\
&< +\infty.
\end{aligned}$$

Similarly, we can get $I_4 < \infty$.

(b) $\Rightarrow$ (a). Note that (ii) and (iii) imply (i). We first consider that $\{X_n\}$ is symmetric. By the properties of $l(x)$ and $\varphi(x)$ and noting that $r > 1$, we have

$$\sum_{m=1}^{\infty} 2^{m(r-1)} l(2^m) P(\|S_{2^m}\| \ge \epsilon \cdot (2^{m+1}\varphi(2^{m+1})^{1/t}) < \infty. \tag{2.7}$$

Therefore

$$S_n/(n\varphi(n))^{1/t} \xrightarrow{P} 0. \tag{2.8}$$

For general random element sequence $\{X_n\}$, let $X_n^s = X_n - X_n'$ where X_n' denotes the independent copy of X_n and put $S_n^s = \sum\limits_{i=1}^{n} X_i^s$. Then $\{X_n^s\}$ is a sequence of independent symmetric B-valued random elements. As (2.8), we have

$$S_n^s/(n\varphi(n))^{1/t} \xrightarrow{P} 0. \tag{2.9}$$

So, there exists an n_0, such that when $n \ge n_0$, we have

$$\sup_{n \ge n_0} P(\|S_n^s\| > (n\varphi(n))^{1/t}) < \frac{1}{8 \cdot 3^{rt}}. \tag{2.10}$$

By Lemma 2.7 of [4] and (2.10), when $n \ge n_0$, we have

$$E\|S_n^s\|^{rt} \le 3^{rt} E\Big(\sup_{1\le k\le n} \|X_k^s\|\Big)^{rt} + 4\cdot 3^{rt}(n\varphi(n))^r + \frac{1}{2}\int_{(n\varphi(n))^r}^{\infty} P(\|S_n^s\| > z^{1/rt})dz.$$

So

$$E\|S_n^s/(n\varphi(n))^{1/t}\|^{rt} \le 8\cdot 3^{rt} + \frac{2\cdot 3^{rt}}{(n\varphi(n))^r} E\Big(\sup_{1\le k\le n}\|X_k^s\|\Big)^{rt} < \infty,$$

from which follows the fact that $\{S_n^s/(n\varphi(n))^{1/t}\}$ is uniformly integrable, and hence from (2.9),

$$E\|S_n^s/(n\varphi(n))^{1/t}\| \to 0, \quad \text{as } n \to \infty. \tag{2.11}$$

By $EX_n = 0$ and the Fubini theorem, it is easy to verify

$$E\|S_n/(n\varphi(n))^{1/t}\| \le E\|S_n^s/(n\varphi(n))^{1/t}\|, \quad n \ge 1.$$

Then it follows from (2.11) that $S_n/(n\varphi(n))^{1/t} \xrightarrow{P} 0$.

Proof of Theorem 1.2. Noting the properties of $l(x)$ and $\varphi(x)$, by $S_n/(n(\varphi(n))^d)^{1/t} \xrightarrow{P} 0$ and using the Ottaviani inequality we get

$$\sum_{n=1}^{\infty}\frac{l(n)}{n}P\Big(\sup_{k\geq n}\|S_k/(k(\varphi(k))^d)^{1/t}\|\geq\epsilon\Big)\leq C\sum_{n=1}^{\infty}\frac{l(n)\log n}{n}P(\|S_n\|\geq c\epsilon\cdot(n(\varphi(n))^d)^{1/t}).$$

So, as the proof (a) $\Rightarrow$ (b) in Theorem 1.1 we obtain (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c).

(b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a). If (b) or (c) is statisfied, then

$$\sum_{n=1}^{\infty}\frac{1}{n}P(\|S_n\|\geq\epsilon\cdot(n(\varphi(n))^d)^{1/t})<\infty \quad \forall\epsilon>0. \tag{2.12}$$

We first consider that $\{X_n\}$ is symmetric. By (2.12), as the proof of Proposition 1.1 in [5], we get $S_n/(n(\varphi(n))^d)^{1/t} \xrightarrow{P} 0$.

The rest is as the proof (b) $\Rightarrow$ (a) in Theorem 1.1.

Lemma 2.2. *Let $0<t<2, \varphi(x)\in S, \delta>0, d=1$ or -1, and let $\{X_n\}$ be any sequence of B-valued random elements; when $1\leq t<2$, further let $\{X_n\}$ be a sequence of zero mean independent random elements in a space of type p for some $t<p\leq 2$. If*

$$\sum_{i=1}^{n}P(\|X_i\|^t(\varphi(\|X_i\|))^{-d}>x)\leq Cnx^{-(1+\delta)} \tag{2.13}$$

for sufficiently large x, n, then $S_n/(n(\varphi(n))^d)^{1/t} \xrightarrow{P} 0$.

Proof. By C_r-inequality and the p-type property of Banach space, the proof of this lemma is easy, so is omitted here.

Proof of Corollary 1.1. Obviously, (1.3) implies (2.13), so, by Theorems 1.1 and 1.2 and Lemma 2.2, Corollary 1.1 is proved.

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