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MASLOV-TYPE INDEX THEORY FOR SYMPLECTIC PATHS AND SPECTRAL FLOW (II)**

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Abstract

Based on the spectral flow and the stratification structures of the symplectic group $\mathrm{Sp}(2n,\mathbf{C})$, the Maslov-type index theory and its generalization, the ω-index theory parameterized by all ω on the unit circle, for arbitrary paths in $\mathrm{Sp}(2n,\mathbf{C})$ are established. Then the Bott-type iteration formula of the Maslov-type indices for iterated paths in $\mathrm{Sp}(2n,\mathbf{C})$ is proved, and the mean index for any path in $\mathrm{Sp}(2n,\mathbf{C})$ is defined. Also, the relation among various Maslov-type index theories is studied.

Keywords Maslov-type index theory, Symplectic path, Spectral flow, Relative Morse index, ω-index

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§1. Introduction and Main Results

Starting from the pioneering works [5,6] of H. Amann and E. Zehnder in 1980, C. Conley and E. Zehnder established an index theory in 1984 in their celebrated work [11] for non-degenerate paths in $\mathrm{Sp}(2n)$ started from the identity matrix with $n \geq 2$. This index theory was extended to the nondegenerate case of $n = 1$ by E. Zehnder and the first author in [28] of 1990. This index theory for the degenerate Hamiltonian systems was established by the first author in [21] of 1990 and C. Viterbo in [34] of 1990 via different methods, and then extended to all degenerate symplectic paths by the first author in the recent [26] of 1997. In this paper, we call it the Maslov-type index theory. Note that J. Robbin and D. Salamon[30] defined the Maslov index via the intersection forms, and in [31] they proved that their index coincides with the spectral flow of certain family of Fredholm operators for the nondegenerate case. In [8] and [17] analytic ideas are also used to get some more general index theory for linear operator equations. In [25], the first author introduced the ω-index theory parametrized by all ω on the unit circle in the complex plane, and used it to establish the Bott formula and the iteration theory of the Maslov-type index.

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Our aim in this paper is to give a different approach to the ω-index theory and study the iteration theory of the Maslov-type index via the spectral flow method for complex symplectic paths. Note that in our case the Maslov-type index theory corresponds to the periodic boundary value problems of Hamiltonian systems. This index theory has been applied to the study of periodic solutions of Hamiltonian systems in many papers.

The index theory defined in this paper only differs from those in [11, 21, 26, 28, 34] by a constant (cf. §2.2), but is quite different from those defined in [30] for the degenerate case (cf. §3.2 for details).

We also refer to the pioneering work [1] of V. Arnold and the recent celebrated work [9] of S. Cappell, R. Lee, and E. Miller for related discussions on the Maslov index theory.

For $n \in \mathbf{N}$, we define as usual

$$\mathrm{Sp}(2n,\mathbf{C}) = \{M \in \mathrm{Gl}(2n,\mathbf{C}) \mid M^*JM = J\},$$
$$\mathrm{Sp}(2n) = \{M \in \mathrm{Gl}(2n,\mathbf{R}) \mid M^TJM = J\},$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix on $\mathbf{R}^n$. When there is no confusion, we will omit the subindex of identity matrices.

Let α be the cycle defined in part (a) of Proposition 2.1 below. For any symplectic path γ with nonsingular endpoints, i.e., 1 is not their eigenvalues, the geometric definition of the index $i_{\mathrm{geo}}(\gamma)$ is defined by the intersection number of α and γ. By a method in [9] we can extend the definition to arbitrary symplectic paths.

As in [9], we can also associate to each $M \in \mathrm{Sp}(2n,\mathbf{C})$ a self-adjoint Fredholm operator with discrete point spectrum. Therefore we get our analytic definition $i_{\mathrm{anal}}(\gamma)$ of the index theory for arbitrary symplectic path γ.

Now we can define the ω-indices. The ω-index theory assigns a pair of integers

$$(i_{\tau,\omega}(\gamma), \nu_{\tau,\omega}(\gamma)) \in \mathbf{Z} \times \{0, \cdots, 2n\}$$

to each symplectic path γ such that $\gamma(0) = I$ with $\nu_{\tau,\omega}(\gamma) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(\gamma(\tau) - \omega I)$, where $\omega \in \mathbf{U}$ and $\mathbf{U}$ is the unit circle of $\mathbf{C}$. Therefore we have two natural definitions: $(i_{\mathrm{geo}}(\bar\omega\gamma), \nu_{\tau,\omega}(\gamma))$ and $(i_{\mathrm{anal}}(\bar\omega\gamma), \nu_{\tau,\omega}(\gamma))$. Such definitions coincide with the definition in §11 of [9] when $\omega = 1$ and the paths are real (cf. §3.1), and only differs from those defined in [25] when the path is real (cf. §2.2 for details).

Via the above definitions and some basic properties of the spectral flow we can treat the indices in a simpler way. In particular we give a different proof of the Bott formula established in [25] by the first author for the indices of the iteration of the symplectic paths. The first study of such formula was carried out by R. Bott[7] for the iteration of closed geodesics in 1956.

From this we can define the mean index of γ as in [25] and prove an optimal increasing estimate for the iteration paths due to C. Liu and Y. Long[19].

This paper is organized as follows. In §2, we establish the Maslov-type index theory and the ω-index theory. In §3, we study the relation among various Maslov-type index theory. In §4, we study the iteration theory for the Maslov-type indices. In §A, we study some basic properties of the group $\mathrm{Sp}(2n,\mathbf{C})$.

In this paper we denote the sets of natural, integral, real, complex numbers, the unit circle in the complex plane, the open unit disk in the complex plane, and the set of all self-adjoint $n \times n$ complex matrices by $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{U}$, $\mathbf{D}$, and $\mathrm{D}(n)$ respectively. Without further explaination, the coefficient field is $\mathbf{R}$ in the rest of the paper.

§2. Maslov-Type Index Theory for Symplectic Paths

The Maslov-type index theory for nondegenerate continuous paths starting from the identity matrix I in $\mathrm{Sp}(2n)$ was established by C. Conley and E. Zehnder in [11] for $n \geq 2$. Later on, Y. Long and E. Zehnder[28], Y. Long[21], Viterbo[34], and Long[26] have worked in this area and extended it successively to all the continuous paths starting from the identity matrix in $\mathrm{Sp}(2n)$. In this section, we will define the Maslov-type index theory for arbitrary symplectic paths not necessarily starting from I and study its properties via the spectral flow method. The indices defined here are Maslov indices if we view $\mathrm{Sp}(2n)$ as an open submanifold of the Lagrangian Grassmannian.

2.1. Definition of Maslov-type Indices

Firstly we give the following notation.

Notation 2.1. *For $0 \leq k \leq 2n$, define*

$$\mathrm{Sp}_k(2n, \mathbf{C}) = \{M \in \mathrm{Sp}(2n, \mathbf{C}) \mid \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - I) = k\}\,, \tag{2.1}$$

$$\mathrm{Sp}_k(2n) = \{M \in \mathrm{Sp}(2n) \mid \dim \ker(M - I) = k\}\,. \tag{2.2}$$

It is clear that there are stratifications

$$\mathrm{Sp}(2n, \mathbf{C}) = \bigcup_{0 \leq k \leq 2n} \mathrm{Sp}_k(2n, \mathbf{C}), \tag{2.3}$$

$$\mathrm{Sp}(2n) = \bigcup_{0 \leq k \leq 2n} \mathrm{Sp}_k(2n). \tag{2.4}$$

The following proposition gives the properties of the stratifications. Following [30], we can prove it by viewing the symplectic group as an open submanifold of Lagrangian Grassmannian. The transversal parts of (a) and (b) can be proved via the method of §3 of [7].

Proposition 2.1. (a) *For all $0 \leq k \leq 2n$, $\mathrm{Sp}_k(2n, \mathbf{C})$ is a codimension k^2 smooth submanifold of $\mathrm{Sp}(2n, \mathbf{C})$, and $\frac{d}{dt}\,|_{t=0}\,(Me^{Jt})$, with $M \in \mathrm{Sp}_1(2n, \mathbf{C})$, forms a transverse structure of $\mathrm{Sp}_1(2n, \mathbf{C})$ in $\mathrm{Sp}(2n, \mathbf{C})$. Moreover, we have*

$$\overline{\mathrm{Sp}_k(2n, \mathbf{C})} = \bigcup_{l \geq k} \mathrm{Sp}_l(2n, \mathbf{C}).$$

(b) *For all $0 \leq k \leq 2n$, $\mathrm{Sp}_k(2n)$ is a codimension $\frac{1}{2}k(k+1)$ smooth submanifold of $\mathrm{Sp}(2n)$, and $\frac{d}{dt}\,|_{t=0}\,(Me^{Jt})$, with $M \in \mathrm{Sp}_1(2n)$, forms a transverse structure of $\mathrm{Sp}_1(2n)$ in $\mathrm{Sp}(2n)$. Moreover, we have $\overline{\mathrm{Sp}_k(2n)} = \bigcup_{l \geq k} \mathrm{Sp}_l(2n)$.*

Let α be the oriented codimension 1 cycle $\overline{\mathrm{Sp}_1(2n, \mathbf{C})}$ of $\mathrm{Sp}(2n, \mathbf{C})$. By (a) of Proposition 2.1, the intersection points of the curve $\gamma(t) = Me^{Jt}$ and the cycle α form a discrete subset of $\gamma(\mathbf{R})$. This leads to our geometric definition of the Maslov-type index theory. Recall that a matrix $M \in \mathrm{Sp}(2n, \mathbf{C})$ is called nondegenerate if $\det(M - I) \neq 0$.

Definition 2.1. *Let $\gamma\colon [a, b] \to \mathrm{Sp}(2n, \mathbf{C})$ be a continuous path. Then there exists an $\epsilon > 0$ such that for any $t \in (-\epsilon, \epsilon) \setminus \{0\}$ and $s = a$ or b, $\gamma(s)e^{Jt}$ is nondegenerate. The*

Maslov-type index $i_{\rm geo}(\gamma)$ of γ is defined to be the intersection number of $\gamma(\cdot)e^{-Jt}$ and α for all $t \in (0,\epsilon)$.

Now we come to our analytic definition of the Maslov-type index theory. Let $M \in \mathrm{Sp}(2n,\mathbf{C})$. We will associate to M a complex, self-adjoint operator $D(M)$ as follows.

Lemma 2.1. (i) *Let $W^{1,2}([0,1];M)$ denote the Sobolev completion of the set of smooth functions $\phi\colon [0,1] \to \mathbf{C}^{2n}$ satisfying the boundary condition*

$$\phi(1) = M\phi(0). \tag{2.5}$$

Here the Sobolev norm is defined by

$$\|\phi\|^2_{W^{1,2}} = \int_0^1 \Big((\phi,\phi) + \Big(\frac{d\phi}{dt},\frac{d\phi}{dt}\Big)\Big)dt.$$

Let $L^2([0,1];\mathbf{C}^{2n})$ denote the L^2-completion of the set of the smooth functions $\phi\colon [0,1] \to \mathbf{C}^{2n}$. Then $-J\frac{d}{dt}$ defines a complex, self-adjoint operator

$$D(M)\colon W^{1,2}([0,1];M) \to L^2([0,1];\mathbf{C}^{2n}) \tag{2.6}$$

which is unbounded, Fredholm with point spectrum without accumulation points.

(ii) *The kernel of $D(M)$ coincides with the space of constant functions $\phi\colon [0,1] \to \ker_{\mathbf{C}}(M-I)$ and in particular is isomorphic to $\ker_{\mathbf{C}}(M-I)$.*

Proof. The proof is the same as Lemma 3.1 of [9] and therefore is omitted.

Definition 2.2. *Let $\gamma\colon [a,b] \to \mathrm{Sp}(2n,\mathbf{C})$ be a continuous path. We define the Maslov-type index $i_{\rm anal}(\gamma)$ for γ by*

$$i_{\rm anal}(\gamma) = \mathrm{sf}_-\{D(\gamma(t)),[a,b]\}. \tag{2.7}$$

2.2. Definition of the ω-Index Theory

Denote

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau],\mathrm{Sp}(2n,\mathbf{C})) \mid \gamma(0) = I\}, \tag{2.8}$$

$$\mathcal{P}^*_{\tau,\omega}(2n) = \{\gamma \in \mathcal{P}_\tau(2n) \mid \gamma(1) \in \omega\mathrm{Sp}^*(2n,\mathbf{C})\}. \tag{2.9}$$

The ω-index theory of [25] assigns a pair of integers $(i_{\tau,\omega}(\gamma),\nu_{\tau,\omega}(\gamma)) \in \mathbf{Z}\times\{0,\cdots,2n\}$ to each $\gamma \in \mathcal{P}_\tau(2n)$ with

$$\nu_{\tau,\omega}(\gamma) = \dim_{\mathbf{C}}\ker_{\mathbf{C}}(M-\omega I)), \tag{2.10}$$

where $\omega \in \mathbf{U}$. There are two definitions of the ω-index theory available:

$$(i_{\rm geo}(\bar\omega\gamma),\nu_{\tau,\omega}(\gamma)) \quad \text{and} \quad (i_{\rm anal}(\bar\omega\gamma),\nu_{\tau,\omega}(\gamma)).$$

In this subsection we will give the third definition which is only defined for fundamental solutions of linear Hamiltonian systems, show that they are all coincide, and discuss their basic properties.

For $B \in C(\mathbf{R}/\tau\mathbf{Z},\mathrm{gl}(2n,\mathbf{C}))$ with $B(t)$ self-adjoint for all t, consider the linear Hamiltonian system

$$\dot x = JB(t)x, \quad x \in \mathbf{C}^{2n}. \tag{2.11}$$

Denote

$$L_\tau = L^2([0,\tau],\mathbf{C}^{2n}), \quad \text{and} \quad E_{\tau,\omega} = \{x \in W^{1,2}([0,\tau],\mathbf{C}^{2n}) \mid x(\tau) = \omega x(0)\}.$$

There are two self-adjoint operators A, B on L_τ defined by the bilinear form

$$(Ax, y) = \int_0^\tau (-J\dot{x}, y)dt, \qquad (Bx, y) = \int_0^\tau (B(t)x, y)dt, \tag{2.12}$$

for all $x, y \in E_{\tau,\omega}$. Let γ_B be the fundamental solution of (2.11). By Floquet theory we have

$$\nu_{\tau,\omega}(\gamma_B) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(A - B). \tag{2.13}$$

Note that A is an unbounded linear Fredholm operator with compact resolvent.

Definition 2.3. *The ω-index $(i_{\tau,\omega}(\gamma_B), \nu_{\tau,\omega}(\gamma_B))$ of γ_B is defined by*

$$i_{\tau,\omega}(\gamma_B) = I(A, A - B) \tag{2.14}$$

and (2.10) *for all $\omega \in \mathbf{U}$. When $\omega = 1$, we omit the subindex ω.*

There is a list of basic properties of Maslov-type index:

(a)(Path Additivity). Let $\gamma: [a, b] \to \mathrm{Sp}(2n, \mathbf{C})$. If $c \in [a, b]$, there holds

$$i(\gamma) = i(\gamma\mid_{[a,c]}) + i(\gamma\mid_{[c,b]}). \tag{2.15}$$

(b)(Affine Scale Invariance). For all $k > 0$ and $\gamma \in \mathcal{P}_{k\tau}$, we have

$$i(\gamma(kt), 0 \le t \le \tau) = i(\gamma(t), 0 \le t \le k\tau). \tag{2.16}$$

(c)($\diamondsuit$-Additivity). Let $\gamma_1: [a, b] \to \mathrm{Sp}(2k, \mathbf{C})$ and $\gamma_2: [a, b] \to \mathrm{Sp}(2l, \mathbf{C})$ be two symplectic paths. Then we have

$$i(\gamma_1 \diamond \gamma_2) = i(\gamma_1) + i(\gamma_2). \tag{2.17}$$

(d)(Homotopy Invariance). For any two paths γ_1 and γ_2, if $\gamma_1 \sim \gamma_2$ in $\mathrm{Sp}(2n, \mathbf{C})$ with endpoints either fixed or always staying in $\mathrm{Sp}^*(2n, \mathbf{C})$, there holds

$$i(\gamma_1) = i(\gamma_2). \tag{2.18}$$

(e) For all $\gamma \in \mathcal{P}_\tau$, we have

$$i(\gamma) = \inf\left\{ i(\beta) \mid \beta \in \mathcal{P}_\tau^* \text{ is sufficiently } C^0\text{-close to } \gamma \right\}. \tag{2.19}$$

(f)(Normalization). Let $n = 1$. Then

(i) $i(e^{it}I, t \in [0, a]) = \begin{cases} 1, & \text{if } a \in (0, 2\pi), \\ 0, & \text{if } a = 2\pi. \end{cases}$

(ii) For all $(a, b) \in \mathbf{Z} \times \mathbf{Z}$, let $\gamma_{a,b}$ be defined by (A.1). Then we have

$$i(\gamma_{a,b}) = a - b. \tag{2.20}$$

(iii) For all $(a, b) \in (\mathbf{Z} + \frac{1}{2}) \times (\mathbf{Z} + \frac{1}{2})$, let $\gamma_{a,b}$ be defined by (A.1). Then we have

$$i(\gamma_{a,b}) = a - b + 1. \tag{2.21}$$

Lemma 2.2. (1) i_{geo} *satisfies* (a)–(d), (f). (2) i_{anal} *satisfies* (a)–(f).

(3) *The ω-index $i_{\tau,\omega}$ satisfies* (b)–(e)*, and* (ii) *of* (f) *for $\omega \in \mathbf{U} \setminus \{1\}$,* (iii) *of* (f) *for $\omega = 1$ if we change $\mathrm{Sp}^*(2n, \mathbf{C})$ to $\omega\mathrm{Sp}^*(2n, \mathbf{C})$ and restrict it to all C^1 maps only.*

Proof. (1) (a)–(d) follow from the definition.

(f) (i) We claim that for any $\gamma \in C([0, 1], \mathrm{Sp}(2n, \mathbf{C}))$ with endpoints $\gamma(0), \gamma(1) \in \mathrm{Sp}^*(2n, \mathbf{C})$, there holds

$$i_{\mathrm{geo}}(\gamma^{-1}) = -i_{\mathrm{geo}}(\gamma), \tag{2.22}$$

where γ^{-1} is the path $\gamma(t)^{-1}$, $0 \le t \le 1$.

In fact, we can deform γ with end points fixed such that for all intersection points $\gamma(s) \in \alpha$, there hold that $\gamma(s) \in \mathrm{Sp}_1(2n, \mathbf{C})$ and γ has the form $\gamma(t) = \gamma(s)e^{\pm J(t-s)}$ for t near s. Now (2.22) follows from the definition and (d).

Now we prove our assertion. Choosing an $M \in \mathrm{Sp}(2n, \mathbf{C})$ with no eigenvalue on the unit circle, by definition and (d) we have

$$i_{\mathrm{geo}}(e^{it}I, t \in [0, 2\pi]) = i_{\mathrm{geo}}(Me^{it}, t \in [0, 2\pi]) = 0. \tag{2.23}$$

By (a) and (2.23) we have

$$i_{\mathrm{geo}}(e^{it}I, t \in [0, a]) = i_{\mathrm{geo}}(e^{-it}I, t \in [0, a]) \tag{2.24}$$

for all $a \in (0, 2\pi)$. By (a), (d) and (2.22) we have

$$\begin{aligned} & i_{\mathrm{geo}}(e^{it}I, t \in [0, a]) + i_{\mathrm{geo}}(e^{-it}I, t \in [0, a]) \\ &= i_{\mathrm{geo}}(e^{it}e^{-2J\epsilon}, t \in [0, a]) + i_{\mathrm{geo}}(e^{-it}e^{-2J\epsilon}, t \in [0, a]) \\ &= i_{\mathrm{geo}}(e^{it}e^{-2J\epsilon}, t \in [0, a]) - i_{\mathrm{geo}}(e^{it}e^{2J\epsilon}, t \in [0, a]) \\ &= i_{\mathrm{geo}}(e^{Js}, s \in [-2\epsilon, 2\epsilon]) - i_{\mathrm{geo}}(e^{ia}e^{Js}, s \in [-2\epsilon, 2\epsilon]) \\ &= i_{\mathrm{geo}}(e^{Js}e^{i\epsilon}, s \in [-2\epsilon, 2\epsilon]) = 2, \end{aligned}$$

where $\epsilon > 0$ is small. By (2.24), we get $i_{\mathrm{geo}}(e^{it}I, t \in [0, a]) = 1$. Note that we have proved that for $\epsilon > 0$ small, $i_{\mathrm{geo}}(e^{Js}, s \in [-\epsilon, \epsilon]) = 2$.

(ii) By (i) we have

$$\begin{aligned} i_{\mathrm{geo}}(\gamma_{1,0}) &= i_{\mathrm{geo}}(e^{it}I, t \in [0, \pi]) + i_{\mathrm{geo}}(e^{i\pi}e^{Jt}, t \in [0, \pi]) \\ &= i_{\mathrm{geo}}(e^{it}I, t \in [0, \pi]) + i_{\mathrm{geo}}(e^{-J(\pi-t)}, t \in [0, \pi]) = 1, \end{aligned}$$

and similarly $i_{\mathrm{geo}}(\gamma_{0,1}) = -1$. Since $[\gamma_{a,b}] = a[\gamma_{1,0}] + b[\gamma_{0,1}]$ in $\pi_1(\mathrm{Sp}(2n, \mathbf{C}))$,

$$i_{\mathrm{geo}}(\gamma_{a,b}) = a i_{\mathrm{geo}}(\gamma_{1,0}) + b i_{\mathrm{geo}}(\gamma_{0,1}) = a - b.$$

(iii) follows from (i), (ii) and the fact

$$i_{\mathrm{geo}}(\gamma_{a,b}) = i_{\mathrm{geo}}(\gamma_{a-\frac{1}{2}, b-\frac{1}{2}}) + i_{\mathrm{geo}}(\gamma_{\frac{1}{2},\frac{1}{2}}).$$

(2) (a)–(d) and (f) follow from the definition and Proposition 2.2 in [35].

(e) Define the path γ_s by $\gamma_s(t) = \gamma(t)e^{-Jst}, \quad t \in [0, \tau], \quad s > 0$. Then $\gamma_s \sim \gamma$, $\gamma_s \in \mathcal{P}_\tau^*$ and

$$i_{\mathrm{anal}}(\gamma_s) - i_{\mathrm{anal}}(\gamma) = i_{\mathrm{anal}}(\gamma(1)e^{-Ja\tau}, \quad a \in [0, s]) = 0 \tag{2.25}$$

for small s. By the definition of the spectral flow (e) is true.

(3) (a)–(d) and (f) follow from the definition and Proposition 2.2 in [35].

(e) Let the paths γ_s be the fundamental solutions of (2.11) with $B_s = B - sI$. By Proposition 2.2 in [35] we have

$$\begin{aligned} i_{\tau,\omega}(\gamma_s) &= I(A, A - B + sI) \\ &= I(A + sI, A - B + sI) + I(A, A + sI) \\ &= I(A, A - B) = i_{\tau,\omega}(\gamma) \end{aligned}$$

for $s > 0$ small. Then by Proposition 2.2 in [35] (e) is true.

Corollary 2.1. *Let $\gamma \in \mathcal{P}_\tau$ be a real symplectic path. Let $\tilde{i}_{\tau,\omega}(\gamma)$ be the ω-index of γ defined in* [25]. *Then we have*

$$\tilde{i}_{\tau,\omega}(\gamma) - i_{\tau,\omega}(\gamma) = \begin{cases} -n, & \text{if } \omega = 1, \\ 0, & \text{if } \omega \in \mathbf{U} \setminus \{1\}. \end{cases} \tag{2.26}$$

We call $i_{CZ}(\gamma) \equiv \tilde{i}_{\tau,1}(\gamma)$ *the Conley-Zehnder index of* γ.

Proof. By Lemma 2.2 the ω-index for any path in $\mathcal{P}^*_{\tau,\omega}(2n,k)$ defined in [25, p.7] is $k+n$ for $\omega=1$ and k for $\omega\in \mathbf{U}\setminus\{1\}$. By Theorem 2.19 of [25] and Lemma 2.2 our corollary is proved.

Our main results of this subsection are the following.

Theorem 2.1. *There exists a unique collection of functions* $i_{\tau,\omega}$ *satisfying* (3) *of Lemma* 2.2.

Proof. The existence follows from Lemma 2.2, and we will prove the uniqueness by giving a formula for the calculation of the index.

Let $\gamma\in\mathcal{P}_\tau$ be C^1. By (e) we can assume $\gamma\in\mathcal{P}^*_{\tau,\omega}$. By Proposition A.2, for $M=\gamma(\tau)$ there are smooth curves $\beta_\omega\colon[0,1]\to\omega\mathrm{Sp}^*(2n,\mathbf{C})$ such that $\beta_\omega(1)=M$, and

$$\beta_\omega(0)=\begin{cases}-I, & \text{if } \omega=1,\\ I, & \text{if } \omega\in\mathbf{U}\setminus\{1\}.\end{cases}$$

Moreover, we can assume that $\gamma_1=\beta_\omega^{-1}*\gamma$ is a smooth path, where β_ω^{-1} is the reverse of β_ω. Then there is a smooth homotopy $\gamma\overset{H}{\sim}\gamma_1$, where $H\colon[0,\tau]\times[0,1]$ is a smooth homotopy such that $H(0,s)=I$ and $H(\tau,s)\in\mathrm{Sp}^*(2n,\mathbf{C})$ for all $0\le s\le 1$. By Corollary A.1 there is a unique $(a,b)\in\mathbf{Z}\times\mathbf{Z}$ for $\omega\in\mathbf{U}\setminus\{1\}$ and $(a,b)\in(\mathbf{Z}+\frac12)\times(\mathbf{Z}+\frac12)$ for $\omega=1$ such that

$$\gamma_1\sim\gamma_{a,b}\ \text{rel. } 0,1.$$

By (b)–(d) and (f) there holds

$$\begin{aligned} i_{\tau,\omega}(\gamma)&=i_{\tau,\omega}(\gamma_1)=i_{\tau,\omega}(\gamma_{a,b})\\ &=\begin{cases}a-b, & \text{if } \omega\in\mathbf{U}\setminus\{1\},\\ a-b+1, & \text{if } \omega=1.\end{cases}\end{aligned}$$

Corollary 2.2. (i) *For any* C^1 *path* $\gamma\in\mathcal{P}_\tau$, *there holds*

$$i_{\tau,\omega}(\gamma)=i_{\mathrm{anal}}(\bar\omega\gamma). \tag{2.27}$$

(ii) *For any continuous symplectic path* γ, *there holds*

$$i_{\mathrm{geo}}(\gamma)=i_{\mathrm{anal}}(\gamma). \tag{2.28}$$

Proof. (i) follows from Theorem 2.1. (ii) From the proof of Theorem 2.1, (2.28) holds for all $\gamma\in\mathcal{P}^*_\tau$. By Lemma 2.2, (2.25) and the definition of i_{geo}, (2.28) holds.

§3. The Relation Among Various Maslov-Type Index Theory

3.1. The Relation with Maslov Index Theory for Lagrangian Intersections

In [9] there are several definitions of Maslov indices. We will show that our definitions coincide with those in [9].

Let $V=\mathbf{C}^{2n}\oplus\mathbf{C}^{2n}$, and $(\cdot,\cdot)$ be the standard inner product of V. We define

$$\{v,w\}=(\mathcal{J}v,w),\ \forall v,w\in V,\ \text{where } \mathcal{J}=\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}.$$

A complex subspace X of V is called Lagrangian iff

(a) X is isotropic, i.e. $\{v,w\}=0,\ \forall v,w\in X$, and (b) $\dim_{\mathbf{C}} X=2n$.

We denote by $\mathrm{Lag}(V)$ the set of Lagrangian subspaces of V and topologize it as a subspace of $\mathrm{G}_{2n}(V)$, where $\mathrm{G}_k(V)$ is the Grassmannian of all k-dimensional complex subspaces of V.

Lemma 3.1. *Let* $g : \mathrm{Sp}(2n, \mathbf{C}) \to \mathrm{Lag}(V)$ *be the embedding*

$$g(M) = \mathrm{Gr}(M) \equiv \left\{ \begin{pmatrix} x \\ Mx \end{pmatrix} \Big| x \in \mathbf{C}^{2n} \right\}, \quad \forall M \in \mathrm{Sp}(2n, \mathbf{C}). \tag{3.1}$$

Then we have

$$e^{\mathcal{J}t} g(M) = g(e^{Jt} M e^{Jt}) \tag{3.2}$$

for all $M \in \mathrm{Sp}(2n, \mathbf{C})$.

Proof. Direct computation.

Note that we can extend the definitions of Maslov indices in [9] to the complex case by the study of $\mathrm{Lag}(V)$ in [7], and for any path $\gamma \subset \mathrm{Lag}(V)$, the Maslov index of γ is defined as that of $(g(I), \gamma)$ in [9], where g is defined by (3.1).

Lemma 3.2. *Let* $M \in \mathrm{Sp}(2n, \mathbf{C})$ *and* $\epsilon > 0$ *be such that* $Me^{Jt} \in \mathrm{Sp}^*(2n, \mathbf{C})$ *for all* $t \in (-2\epsilon, 2\epsilon) \setminus \{0\}$. *Then for any* $a \in (0, \epsilon)$, *we have*

$$i_{\mathrm{geo}}(Me^{-Jt}, 0 \le t \le a) = i_{\mathrm{geo}}(e^{-Jt} M e^{-Jt}, 0 \le t \le a) = 0. \tag{3.3}$$

Proof. For any $s \in [0, 1]$ and $t \in (0, a]$, we have

$$\det(e^{-Jst} M e^{-Jt} - I) = \det(M e^{-Jt(s+1)} - I),$$

and hence $e^{-Jst} M e^{-Jt} \in \mathrm{Sp}^*(2n, \mathbf{C})$. By definition and the deformation invariance rel. endpoints of i_{geo}, we have $i_{\mathrm{geo}}(Me^{-Jt}, 0 \le t \le a) = i_{\mathrm{geo}}(Me^{-J(t+t_0)},\ \ 0 \le t \le a) = 0$, where $t_0 \in (0, \epsilon)$ and

$$\begin{aligned} & i_{\mathrm{geo}}(e^{-Jt} M e^{-Jt}, 0 \le t \le a) \\ = {} & i_{\mathrm{geo}}(Me^{-Jt}, 0 \le t \le a) + i_{\mathrm{geo}}(e^{-Jsa} M e^{-Ja}, 0 \le s \le 1) = 0. \end{aligned}$$

Proposition 3.1. *There hold*

$$i_{\mathrm{geo}}(\gamma) = \mu_{\mathrm{geo}_1}(g \circ \gamma), \tag{3.4}$$

$$i_{\mathrm{anal}}(\gamma) = \mu_{\mathrm{anal}_2}(g \circ \gamma) \tag{3.5}$$

for all path $\gamma : [a, b] \to \mathrm{Sp}(2n, \mathbf{C})$, *where* μ_{geo1} *and* μ_{anal2} *are defined in* [9]. *In particular, we have* $i_{\mathrm{geo}} = i_{\mathrm{anal}}$ *by* [9].

Proof. First, we prove (3.4). Assume that γ is a proper path, i.e., $\gamma(a)$ and $\gamma(b)$ are nondegenerate symplectic matrices. Since the map $g : \mathrm{Sp}(2n, \mathbf{C}) \to \mathrm{Lag}(V)$ is an embedded open submanifold, (3.4) is true in this case. The general case follows from Lemmata 3.1, 3.2, and the definition of i_{geo} and μ_{geo1}.

Now we consider the Equation (3.5). Let $M \in \mathrm{Sp}(2n, \mathbf{C})$. It is clear that λ is an eigenvalue of $D(M)$ iff $\det(M - e^{J\lambda}) = 0$. Note that for $D(g(I), g(M))$ defined in §3 of [9], λ is an eigenvalue of $D(g(I), g(M))$ iff there exists $v, w \in \mathbf{C}^{2n} \setminus \{0\}$ such that $(v, Mv) = (e^{-J\lambda} w, e^{J\lambda} w)$, iff $\det(M - e^{2J\lambda}) = 0$. By definition (3.5) is true.

3.2. Calculation of the Maslov Index

In this subsection, we will caculate the Maslov indices for Lagrangian intersections defined by [9] via the intersection forms defined by [30] and study the relation between the CLM index theory and the RS index theory. Firstly we recall the definitions.

Let $\mathbf{C}^{2n}$ be a (complex) symplectic space with

(a) Hermitian structure: $(v, w) = w^* v$, $\forall v, w \in \mathbf{C}^{2n}$, and

(b) complex symplectic structure: $\{v, w\} = (Jv, w)$, $\forall v, w \in \mathbf{C}^{2n}$.

Denote by $\Sigma(n, \mathbf{C})$ the space of all Lagrangian subspaces in $\mathbf{C}^{2n}$. For $a < b$, we denote by $P([a, b]; \mathbf{C}^{2n})$ the space of continuous maps

$$f : [a, b] \to \{\text{pairs of Lagrangian subspaces in } \mathbf{C}^{2n}\}.$$

The topology on $P([a, b]; \mathbf{C}^{2n})$ is given by the usual compact open topology.

Definition 3.1.[9] *The CLM index is a uneque interger-valued function*

$$i_{\mathrm{CLM}} : P([a, b]; C^{2n}) \to \mathbf{Z}$$

which satisfies Properties I–V *in* [9] *and the following complex version of Property* VI *on* p. 128 *in* [9]

Property VI' (Normalization). *Define the path $f(t)$ in $P([-\frac{\pi}{4}, \frac{\pi}{4}]; \mathbf{C}^2)$ by the formula*

$$f(t) = (\mathbf{C}, e^{Jt}\mathbf{C}), \qquad -\frac{\pi}{4} \le t \le \frac{\pi}{4}.$$

Then

(i) $i_{\mathrm{CLM}}(f|[-\frac{\pi}{4}, \frac{\pi}{4}]) = 1$; (ii) $i_{\mathrm{CLM}}(f|[-\frac{\pi}{4}, 0) = 0$; (iii) $i_{\mathrm{CLM}}(f|[0, \frac{\pi}{4}]) = 1$.

By Theorem 1.1 in [30], for any curve $\Lambda(t) \in \Sigma(n, \mathbf{C})$ of Lagrangian subspaces with $\Lambda(0) = \Lambda$ and $\dot{\Lambda}(0) = \hat{\Lambda}$, there is a quadradic form $Q(\Lambda, \hat{\Lambda})$ defined on Λ defined as follows. Let W be a fixed Lagrangian complement of $\Lambda(t)$. For $v \in \Lambda$ and small t, define $w(t) \in W$ by $v + w(t) \in \Lambda(t)$. The form

$$Q(\Lambda, \hat{\Lambda})(v) = \frac{d}{dt}\Big|_{t=0} \{v, w(t)\}$$

is independent of the choice of W.

Definition 3.2.[30] *Let $(\Lambda_1(t), \Lambda_2(t))$ be in $P([a, b]; \mathbf{C}^{2n})$. For $t \in [a, b]$, the crossing form is a quadratic form defined by*

$$\Gamma(\Lambda_1, \Lambda_2, t) = (Q(\Lambda_1(t), \dot{\Lambda}_1(t)) - Q(\Lambda_2(t), \dot{\Lambda}_2(t)))\ |_{\Lambda_1(t)\cap\Lambda_2(t)}\ .$$

A crossing is a time $t \in [a, b]$ such that $\Lambda_1(t) \cap \Lambda_2(t) \ne \{0\}$. A crossing is called regular if $\Gamma(\Lambda_1, \Lambda_2, t)$ is nondegenerate. For a pair with only regular crossings, the RS index is given by

$$i_{\mathrm{RS}}(\Lambda_1, \Lambda_2) = \frac{1}{2}\mathrm{sign}\ \Gamma(\Lambda_1, \Lambda_2, a) + \sum_{a<t<b} \mathrm{sign}\ \Gamma(\Lambda_1, \Lambda_2, t) + \frac{1}{2}\mathrm{sign}\ \Gamma(\Lambda_1, \Lambda_2, b).$$

Theorem 3.1. *Let $f(t) = (\Lambda_1(t), \Lambda_2(t))$ be in $P([a, b]; \mathbf{C}^{2n})$.*

(i) *We have*

$$i_{\mathrm{CLM}}(\Lambda_1, \Lambda_2) = i_{\mathrm{RS}}(\Lambda_2, \Lambda_1) - \frac{1}{2}(h_{12}(b) - h_{12}(a)), \tag{3.7}$$

where $h_{12}(t) = \dim_{\mathbf{C}} \Lambda_1(t) \cap \Lambda_2(t)$.

(ii) *Assume that $f(t)$ has only regular crossings. Then we have*

$$i_{\mathrm{CLM}}(\Lambda_1, \Lambda_2) = m^+(\Gamma(\Lambda_2, \Lambda_1, a)) + \sum_{a<t<b} \mathrm{sign}\ \Gamma(\Lambda_2, \Lambda_1, t) - m^-(\Gamma(\Lambda_2, \Lambda_1, b)), \tag{3.8}$$

where m^+ and m^- denote the Morse positive index and the Morse negative index respectively.

Proof. (i) It follows from [30] that the function in the right hand of Equation (3.7) satisfies Properties I-V and the complex version of Property VI in [9]. Note that $\pi_1(\Sigma(n, \mathbf{C})) = \mathbf{Z}$. By the proof of Theorem 1.1 in [9] such a function is unique and hence (3.7) holds.

(ii) follows from (i) and the definition of RS index.

Corollary 3.1. *Let* $\gamma\colon[a,b]\to \mathrm{Sp}(2n,\mathbf{C})$ *be a symplectic path.*

(i) *Let* $g\colon \mathrm{Sp}(2n,\mathbf{C})\to \mathrm{lg}(V)$ *be the map defined by* (3.1). *Then we have*

$$i_{\rm RS}(g(\gamma),g(I))=i_{\rm anal}(\gamma)+\frac{1}{2}(\nu(b)-\nu(a)). \tag{3.9}$$

(ii) *Assume that* γ *is a* C^1 *curve and* $\Gamma(t)\equiv B(t)|\ker_{\mathbf{C}}(\gamma(t)-I)$ *is nondegenerate, where* $B(t)=-J\dot\gamma(t)\gamma(t)^{-1}$. *Then we have*

$$i_{\rm anal}(\gamma)=m^+(\Gamma(a))+\sum_{a<t<b}\mathrm{sign}\,\Gamma(t)-m^-(\Gamma(b)). \tag{3.10}$$

Proof. Note that (i) follows from Proposition 3.1, and (ii) follows from Lemma 3.1 in [13] and Proposition 3.1

3.3 The Relation Between the Ekeland Indices and the ω-Indices

In this subsection we will study the relation between Ekeland indices defined by I. Ekeland[16], generalized Ekeland indices defined by C. Viterbo[33], and ω-index theory defined in current paper. The idea comes from Lemma 5.2 of [30]. Firstly we recall the definitions.

Let A, B be two self-adjoint operators defined by (2.12).

Definition 3.3.[16] Define the Hilbert space $L_{0,\tau}$ by

$$L_{0,\tau}=\Big\{x\in L_\tau\Big|\int_0^\tau x=0\Big\}.$$

Let Q be the orthogonal projection from L_τ onto $L_{0,\tau}$. If $-B(t)$ is positive definite for all $t\in[0,\tau]$, the Ekeland index of γ_B for $\omega=1$ is defined by

$$\tilde{i}_\tau(\gamma_B)=m^-(-QB^{-1}Q+(QAQ)^{-1}). \tag{3.11}$$

Remark 3.1. Note that the quadratic form $q_s(u,u)$ defined by I.4.(11) of [16] coincides with the quadratic form $((QAQ)^{-1}(-Ju),-Ju)-(QB^{-1}Q(-Ju),-Ju)$. Thus (3.11) coincides with Definition I.4.3 of [16].

Definition 3.4.[16,33] Let $K\ge 0$ be an integer and $\omega\in\mathbf{U}$ such that $A-KI$ is invertible. If $-B(t)+KI$ is positive definite for all $t\in[0,\tau]$, the generalized Ekeland index of γ_B is defined by

$$\tilde{i}_{k,\tau,\omega}(\gamma_B)=m^-((-B+KI)^{-1}+(A-KI)^{-1}). \tag{3.12}$$

For the relations among such index theories and ω-index theory, we have the following theorem.

Theorem 3.2. (i) Under the condition of Definition 3.3, we have

$$\tilde{i}_\tau(\gamma_B)+i_{\tau,1}(\gamma_B)+\nu_\tau(\gamma_B)=0. \tag{3.13}$$

(ii) Under the condition of Definition 3.4, we have

$$\tilde{i}_{k,\tau,\omega}(\gamma_B)+i_{\tau,\omega}(\gamma_B)+\nu_{\tau,\omega}(\gamma_B)=\dim\ker A-I(-A,-A+KI). \tag{3.14}$$

Proof. (i) By Theorem I.4.4 of [16], we have

$$\dim\ker(-QB^{-1}Q+s(QAQ){-1})=\begin{cases}0, & \text{if } s=0,\\ \dim\ker(A-sB), & \text{if } s>0.\end{cases}$$

Denote by P_s the orthogonal projection from $L_{0,\tau}$ onto $\ker(-QB^{-1}Q + s(QAQ)-1)$ for $s \in [0,1]$. Since B is negative definite, $-QB^{-1}Q$ is positive definite. By (c) of Proposition 2.2 and Corollary 4.1 in [35] we have

$$\begin{aligned}\tilde{i}_\tau(\gamma_B) &= I(-QB^{-1}Q, -QB^{-1}Q + (QAQ)^{-1}) \\ &= -\sum_{s\in(0,1)} \text{sign } (P_s(QAQ)^{-1}P_s) - m^+(P_1(QAQ)^{-1}P_1) \\ &= -\sum_{s\in(0,1)} \text{sign } (s^{-1}P_sQB^{-1}QP_s) - m^+(P_1QB^{-1}QP_1) \\ &= \sum_{s\in(0,1)} \dim\ker(-QB^{-1}Q + s(QAQ)^{-1}) \\ &= \sum_{s\in(0,1)} \dim\ker(A - sB).\end{aligned}$$

Denote by $\tilde{P}_s$ the orthogonal projection from $L\tau(2n)$ onto $\ker(A - sB)$ for $s \in [0,1]$. Since B is negative definite, by Corollary 4.1 in [35] we have

$$\begin{aligned}I(A, A-B) &= m^-(\tilde{P}_0(-B)\tilde{P}_0) - \sum_{s\in(0,1)} \text{sign } (\tilde{P}_s(-B)\tilde{P}_s) - m^+(\tilde{P}_1(-B)\tilde{P}_1) \\ &= -\sum_{s\in(0,1]} \dim\ker(A - sB) = -\tilde{i}_\tau(\gamma_B) - \nu_{\tau,1}(\gamma_B).\end{aligned}$$

So (3.13) is proved.

(ii) By (c) of Proposition 2.2 in [35] and Corollary 4.2 in [35] we have

$$\begin{aligned}\tilde{i}_{k,\tau,\omega}(\gamma_B) &= I((-B+KI)^{-1}, (-B+KI)^{-1} + (A-KI)^{-1}) \\ &= I(-A+KI, -A+B) \\ &= I(-A, -A+B) - I(-A, -A+KI) \\ &= -I(A, A-B) + \dim\ker A - \dim\ker(A-B) - I(-A, -A+KI) \\ &= -i_{\tau,\omega}(\gamma_B) + \dim\ker A - \nu_{\tau,\omega}(\gamma_B) - I(-A, -A+KI).\end{aligned}$$

So (3.14) is proved.

3.4. The Relation Between the Bott Functions and the ω-Indices

In this subsection we will use the idea in [13] and give a geometric proof of Corollary 3.2 below.

In his pioneering work [7], R. Bott studied the periodic Hermitian systems in $\mathbf{C}^n$,

$$Lx \equiv -\frac{d}{dt}\Big(p\frac{d}{dt}x + qx\Big) + q^*\frac{d}{dt}x + rx = 0, \tag{3.15}$$

where $p, q \in C^1(S_\tau, \text{gl}(n,\mathbf{C}))$, $r \in C^1(S_\tau, \text{gl}(n,\mathbf{C}))$, $p = p^*$, $r = r^*$, and $p(t)$ is positive definite for all t. Here $q^*(t)$ denotes the complex conjugate of the transpose of $q(t)$ (cf. (1.1) of [7]). Bott defined his function $\Lambda(\cdot)$ and $N(\cdot)$ in §1 of [7] for such systems. In the rest of this subsection we assume that $p, q \in C^1([0,\tau], \text{gl}(n,\mathbf{C}))$, $r \in C([0,\tau], \text{gl}(n,\mathbf{C}))$, $p = p^*$, $r = r^*$, and $p(t)$ is positive definite for all t in (3.15).

Let

$$b_\lambda(t) \equiv \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}+\lambda \end{pmatrix} \equiv \begin{pmatrix} p^{-1}(t) & -p^{-1}(t)q(t) \\ -q^*(t)p^{-1}(t) & q^*(t)p^{-1}(t)q(t) - r(t) + \lambda \end{pmatrix}. \tag{3.16}$$

Let $u_x = ((p\frac{d}{dt}x + qx)^T, x^T)^T$. It is easy to see that the equation $Lx = \lambda x$ is equivalent to

$$\dot{u}_x = Jb_\lambda u_x. \tag{3.17}$$

Let $W \in \mathrm{Lag}(V)$ with V defined in §3.1. Let $g\colon \mathrm{Sp}(2n, \mathbf{C}) \to \mathrm{Lag}(V)$ be defined by (3.1) For any symplectic path $\gamma \in C([0,\tau], \mathrm{Sp}(2n, \mathbf{C}))$, we define $i_{\tau,W}(\gamma) = i_{\mathrm{CLM}}(g(\gamma), W)$.

Define L_W on $L_\tau(n) = L^2([0,\tau], \mathbf{C}^n)$ by $L_W x = Lx$ with its domain

$$\mathbf{D}(L_W) = \{x \in W^{2,2}([0,\tau], \mathbf{C}^n) \mid ((u_x(0))^T, (u_x(\tau))^T)^T \in W\}.$$

Then L_W is a self-adjoint operator.

Let $X_\lambda(t)$ be the fundamental matrix solution of (3.17). The matrix $X_0(1)$ is called the Poincaré matrix of L.

Definition 3.5.[7] *The Bott functions of the curve $X_0(t)$, $0 \le t \le \tau$ is defined by*

$$N(\omega) = \dim_{\mathbf{C}} \ker_{\mathbf{C}} L_{g(\omega)}, \tag{3.18}$$

$$\lambda(\omega) = m^-(L_{g(\omega)}), \tag{3.19}$$

where $\omega \in \mathbf{U}$.

Lemma 3.3. *Let V be the symplectic space defined in* §3.1 *and W be a $2n$-dimensional subspace of V. Define $\pi_k\colon V_k \to \mathbf{C}^{2n}$, $k = 1,2$ by*

$$\pi_1((x^T, 0^T, y^T, 0^T)^T) = (x^T, y^T)^T, \quad \pi_2((0^T, x^T, 0^T, y^T)^T) = (-x^T, y^T)^T$$

for all $x, y \in \mathbf{C}^n$, where $V_1 = \mathbf{C}^n \times \{0\} \times \mathbf{C}^n \times \{0\}$ and $V_2 = \{0\} \times \mathbf{C}^n \times \{0\} \times \mathbf{C}^n$. Set $N_k = W \cap V_k$, $k = 1,2$. Then the following three conditions are equivalent.

(i) *$W \in \mathrm{Lag}(V)$ and $(x, y) = (z, w)$ for all $(x^T, y^T, z^T, w^T)^T \in W$,*

(ii) *$W \in \mathrm{Lag}(V)$ and $W = N_1 \oplus N_2$, and*

(iii) *we have the following orthogonal decomposition*

$$\mathbf{C}^{2n} = \pi_1(N_1) \oplus \pi_2(N_2). \tag{3.20}$$

Proof. (i)⇒(ii). Let $(x_k^T, y_k^T, z_k^T, w_k^T)^T \in W$, $k = 1,2$. Then we have $(x_1 \pm x_2, y_1 \pm y_2) = (z_1 \pm z_2, w_1 \pm w_2)$. So we have

$$(x_1, y_2) + (x_2, y_1) = (z_1, w_2) + (z_2, w_1).$$

Since $W \in \mathrm{Lag}(V)$, we have $-(x_1, y_2) + (x_2, y_1) + (z_1, w_2) - (z_2, w_1) = 0$. So

$$\{(x_1^T, 0^T, y_1^T, 0^T)^T, (x_2^T, y_2^T, z_2^T, w_2^T)^T\} = -(x_1, y_2) + (z_1, w_2) = 0.$$

Since $W \in \mathrm{Lag}(V)$, we have $(x_1^T, 0^T, y_1^T, 0^T)^T \in W$. Hence (ii).

(ii)⇒(i). Since $(x^T, y^T, z^T, w^T)^T \in W$, $(x^T, 0^T, z^T, 0^T)^T \in W$, and $(0^T, y^T, 0^T, w^T)^T \in W$. Now (i) follows from $W \in \mathrm{Lag}(V)$.

(ii)⇒(iii). Note that the right hand side of (3.20) is an orthogonal decomposition. Now (3.20) follows from counting of the dimension.

(iii)⇒(ii). By (3.20) $N_1 \oplus N_2 \in \mathrm{Lag}(V)$. Since $W \supset N_1 \oplus N_2$ and $\dim_{\mathbf{C}} W = 2n$, $W \in \mathrm{Lag}(V)$. Therefore (ii) holds.

Lemma 3.4. *Suppose that $W \in \mathrm{Lag}(V)$ and $W = N_1 \oplus \mathbf{N}_2$. Then there exists a $\mu < 0$ such that for all $\lambda \le \mu$ and $t \in (0,\tau]$,*

$$g(X_\lambda(t)) \cap W = \{0\}. \tag{3.21}$$

Proof. Since p is positive definite, we can choose $\mu < 0$ such that

$$K \equiv \begin{pmatrix} p & q \\ q^* & r-\mu \end{pmatrix}$$

is positive definite. Then for all $x \in \mathbf{D}(L_W)$, $x \neq 0$, there holds

$$\int_{[0,t]} ((L_W - \mu)x, x) = \int_{[0,t]} (Kv_x, v_x) > 0,$$

where $v_x = (\frac{d}{dt}x^T, x^T)^T$. Therefore (3.17) has no non-trivial solution in $\mathbf{D}(L_W)$, and (3.21) is proved.

The following lemma follows from Lemma 3.1 in [13], Theorem 3.1 and Lemma 3.4.

Lemma 3.5. *Suppose* $W \in \mathrm{lg}(V)$, $W = N_1 \oplus N_2$ *and* $\mu \ll 0$. *Then* $i_{\tau,W}(X_\mu(t), 0 \leq t \leq \tau) = \dim_{\mathbf{C}} N_1 \cap g(I)$.

Note that Lemmata 3.4 and 3.5 were mentioned in [13].

In our setting, Theorem 4.3 of [13] can be read as:

Theorem 3.3. *Let* $W \in \mathrm{Lag}(V)$ *with* V *defined in* §3.1. *Suppose that* $W = N_1 \oplus N_2$. *Then for any Hermitian system* (3.15), $m^-(L_W)$ *is finite, and we have*

$$\dim_{\mathbf{C}} \ker_{\mathbf{C}} L_W = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(g(X_0(\tau)) \cap W), \tag{3.22}$$

$$m^-(L_W) = i_{\tau,W}(X_0(t), 0 \leq t \leq \tau) - \dim_{\mathbf{C}} N_1 \cap g(I). \tag{3.23}$$

Proof. Let χ be the composition of the curves

$$\chi_1 = X_\mu(t),\ t \text{ running from } \tau \text{ to } 0, \quad \chi_2 = X_\lambda(0),\ \lambda \text{ running from } \mu \text{ to } 0,$$

$$\chi_3 = X_0(t),\ t \text{ running from } 0 \text{ to } \tau, \quad \chi_4 = X_\lambda(\tau),\ \lambda \text{ running from } 0 \text{ to } \mu.$$

Then χ is homotopy to zero, $\chi_2 = I$, and $\chi_3 = \gamma_B$. Therefore (3.22) is trivial. Now we prove (3.23). By Lemma 3.1 in [13], Propositon 4.1 in [13], Lemma 3.5 and Theorem 3.1, there holds

$$\begin{aligned} m^-(L_W) &= -i_{\tau,W}(\chi_4) = i_{\tau,W}(\chi_3) + i_{\tau,W}(\chi_1) \\ &= i_{\tau,W}(X_0(t), 0 \leq t \leq \tau) - \dim_{\mathbf{C}} N_1 \cap g(I), \end{aligned}$$

and (3.23) is proved.

Taking $W = g(\omega)$ in Theorem 3.3, we have

Corollary 3.2. *For any Hermitian system* (3.15), *the Bott functions* $\Lambda(\cdot)$ *and* $N(\cdot)$ *of* (3.15) *and our* ω*-index theory of* $X_0(t)$, $0 \leq t \leq \tau$ *satisfy*

$$N(\omega) = \nu_{\tau,\omega}(X_0(t), 0 \leq t \leq \tau), \qquad \forall \omega \in \mathbf{U}, \tag{3.24}$$

$$\lambda(\omega) - i_{\tau,\omega}(X_0(t), 0 \leq t \leq \tau) = \begin{cases} -n & \text{if } \omega = 1, \\ 0 & \text{if } \omega \in \mathbf{U} \setminus \{1\}. \end{cases} \tag{3.25}$$

§4. Iteration Theory for the Maslov-Type Indices

4.1. Properties of ω-Indices

Now fix $n \in \mathbf{N}$, $\tau > 0$ and a path $\gamma \in \mathcal{P}_\tau$. Set $M = \gamma(\tau)$. We will study the properties of the ω-indices of γ. Set

$$i_\tau(\omega) \equiv i_{\tau,\omega}(\gamma), \quad \nu_\tau(\omega) \equiv \nu_{\tau,\omega}(\gamma), \tag{4.1}$$

for all $\omega \in \mathbf{U}$. By Lemma 2.2 we have

$$i_\tau(\omega_2) - i_\tau(\omega_1) = i_{\mathrm{anal}}(Me^{it}, -t_1 \le t \le -t_2) - f(t_2) + f(t_1), \tag{4.2}$$

where $\omega_j = e^{it_j}$, $t_j \in \mathbf{R}$, $j = 0, 1$ and

$$f(t) = i_{\mathrm{anal}}(e^{is}, 0 \le s \le -t) = \begin{cases} n, & \text{if } t \not\equiv 0 \bmod 2\pi, \\ 0, & \text{if } t \equiv 0 \bmod 2\pi. \end{cases}$$

Therefore our problem is reduced to studying the properties of the functions

$$i_M(t) \equiv i_{\mathrm{anal}}(Me^{is}, 0 \le s \le t). \tag{4.3}$$

Clearly we have

Lemma 4.1. *The function i_M defined by (4.3) is locally constant on $\mathbf{U} \setminus \sigma(M)$ and $\nu_\tau = 0$ on $\mathbf{U} \setminus \sigma(M)$.*

The following theorem is Theorem 4.5 of [25]. Here we give a different proof. Recall that the Krein form G on $\mathbf{C}^{2n}$ is defined by

$$(Gx, y) := (iJx, y) = i\Big\{\sum_{k=1}^{n}(x_k \bar{y}_{n+k} - x_{n+k}\bar{y}_k)\Big\}, \tag{4.4}$$

for all $x = (x_1, \cdots, x_{2n}) \in \mathbf{C}^{2n}$, $y = (y_1, \cdots, y_{2n}) \in \mathbf{C}^{2n}$. Let $\lambda \in \sigma(M) \cap \mathbf{U}$. The restriction of G to the root vector space E_λ must be nondegenerate. The signature (p, q) of G on E_λ is called the Krein type of λ.

Theorem 4.1. *Let $\omega \in \mathbf{U} \cap \sigma(M)$ be of Krein type (p, q). Then there holds*

$$\lim_{\epsilon \to 0^+} \big(i_\tau(e^{i\epsilon}\omega) - i_\tau(e^{-i\epsilon}\omega)\big) = p - q. \tag{4.5}$$

Proof. By (4.2) and (4.3) we need only to prove (4.5) for the function $i_M(t)$. Without lost of generality, we can assume $\omega = 1$ and all the curves concerned are in a contractible neighborhood of M in $\mathrm{Sp}(2n, \mathbf{C})$ by dividing ω.

By Corollary 3.1 we get the required result in the case that 1 is a simple eigenvalue of M. In the general case, by Lemma A.1 there is a smooth curve $\gamma(s)$ with $s \in \mathbf{R}$ sufficiently small such that $\gamma(0) = M$ and $\gamma(s)$ has only simple eigenvalues for $s \ne 0$. Choose a sufficiently small neighborhood N of 1 in $\mathbf{C}$ so that $N \cap \sigma(M) = \{1\}$ and N is symmetric about the unit circle and the real line. Let $\epsilon > 0$ be such that $e^{i\epsilon} \in \partial N$. Let $s > 0$ be sufficiently small such that $\sigma(\gamma(s)) \cap \partial N = \emptyset$. By Lemma 2.2 we have

$$i_M(-\epsilon) - i_M(\epsilon) = i_{\gamma(s)}(-\epsilon) - i_{\gamma(s)}(\epsilon) = \sum_{\lambda \in N \cap \sigma(\gamma(s))} (p(s, \lambda) - q(s, \lambda)) = p - q,$$

where $(p(s, \lambda), q(s, \lambda))$ is the Krein type of λ for $\gamma(s)$. Now our theorem is proved.

As in [7] and [25], we study next the splitting numbers.

Definition 4.1. *For $M \in \mathrm{Sp}(2n, \mathbf{C})$ and $\omega = e^{it} \in \mathbf{U}$, define*

$$\begin{aligned} S^+_{-,M}(\omega) &= \mathrm{sf}_-\{D(Me^{is}), t \le s \le t + \epsilon\}, \\ S^-_{-,M}(\omega) &= -\mathrm{sf}_-\{D(Me^{is}), t - \epsilon \le s \le t\}, \\ S^+_{+,M}(\omega) &= -\mathrm{sf}\{D(Me^{is}), t \le s \le t + \epsilon\}, \\ S^-_{+,M}(\omega) &= \mathrm{sf}\{D(Me^{is}), t - \epsilon \le s \le t\}, \end{aligned} \tag{4.6}$$

for $\epsilon > 0$ sufficiently small. We call them the splitting numbers of M at ω.

Suppose $\omega \in \sigma(M)$ has Krein type (p, q). As in I.3 of [16] and §4 of [25], we choose a sufficiently small neighborhood N of ω in $\mathbf{C}$ so that $N \cap \sigma(M) = \{\omega\}$. Define

$$E_M(s) = E_M^+(s) \oplus E_M^0(s) \oplus E_M^-(s), \quad \forall s \in (-\epsilon, \epsilon), \tag{4.7}$$

where $\epsilon > 0$ is sufficiently small, $E_M^+(s)$ (or $E_M^-(s)$) is the direct sum of invariant subspaces in $\mathbf{C}^{2n}$ associated with the Krein positive (or negative) eigenvalues of Me^{Js} in $N \cap \mathbf{U}$, $E_M^0(s)$ is the direct sum of invariant subspaces in $\mathbf{C}^{2n}$ associated with the eigenvalues of Me^{Js} in $N \setminus \mathbf{U}$. Denote $p_s = \dim_{\mathbf{C}} E_M^+(s)$, $q_s = \dim_{\mathbf{C}} E_M^-(s)$, and $2r_s = \dim_{\mathbf{C}} E_M^0(s)$ for $-\epsilon < s < \epsilon$. By definition there hold

$$p = p_s + r_s, \quad q = q_s + r_s. \tag{4.8}$$

Lemma 4.2. *With the notation above, there hold*

(a) *The non-negative integers p_s, q_s, and r_s are constants for $-\epsilon < s < 0$ and $0 < s < \epsilon$. Define*

$$\begin{aligned} &p_- = p_s, \quad q_- = q_s, \quad r_- = r_s, \qquad \forall -\epsilon < s < 0, \\ &p_+ = p_s, \quad q_+ = q_s, \quad r_+ = r_s, \qquad \forall 0 < s < \epsilon. \end{aligned} \tag{4.9}$$

As in [16] *we call $2r_+$ and $2r_-$ the eigenvalue arriving number of M at ω.*

(b) *We have*

$$\begin{aligned} &S_{-,M}^+(\omega) = p - r_- = p_- \geq 0, \quad S_{-,M}^-(\omega) = q - r_- = q_- \geq 0, \\ &S_{+,M}^+(\omega) = q - r_+ = q_+ \geq 0, \quad S_{+,M}^-(\omega) = p - r_+ = p_+ \geq 0. \end{aligned} \tag{4.10}$$

(c) *We have*

$$S_{-,M}^+(\omega) + S_{+,M}^+(\omega) = S_{-,M}^-(\omega) + S_{+,M}^-(\omega) = \nu_\tau(\omega). \tag{4.11}$$

Proof. By definition, (4.8), and Theorem 4.1, we see that

$$\begin{aligned} S_{-,M}^+(\omega) &= \mathrm{sf}_-\{D(Me^{is}), t \leq s \leq t + \epsilon\} \\ &= \mathrm{sf}_-\{D(Me^{is}e^{-J\epsilon}), t \leq s \leq t + \epsilon\} = p_s = p - r_s \end{aligned}$$

is independent of $-\epsilon < s < 0$. Similarly we get other equations in (4.10). Therefore (a) and (b) are proved. By Proposition 2.2 in [35] (c) is true.

4.2. The Bott Formula of the Maslov-Type Index Theory for Iterated Symplectic Paths

Let $\tilde{\gamma} \in C([0, +\infty), \mathrm{Sp}(2n, \mathbf{C}))$ be the iteration paths of γ defined by (4.13). Our purpose of this subsection is to prove the following Theorem 4.2. We need a lemma.

Fix $\tau > 0$ and $B \in C(S_\tau, D(2n))$, where $S_\tau = \mathbf{R}/\tau\mathbf{Z}$. Let $\gamma\colon [0, +\infty) \to \mathrm{Sp}(2n, \mathbf{C})$ be the fundamental solution of (2.11). Fix $k \in \mathbf{N}$. For $\omega, z \in \mathbf{U}$ satisfying $\omega^k = z$, recall that

$$L_{k\tau,z} = L^2([0, k\tau], \mathbf{C}^{2n}).$$

Define

$$L_{k\tau,z}(\tau, \omega) = \{y \in L_{k\tau,z} \mid y(t + \tau) = \omega y(t), \forall t \in [0, k\tau]\}.$$

In the following for notation simplicity we identify $L_{k\tau,z}(\tau, \omega)$ with $L_{\tau,\omega}$.

Lemma 4.3. *There is an L^2 orthogonal decomposition*

$$L_{k\tau,z} = \bigoplus_{\omega^k = z} L_{k\tau,z}(\tau, \omega). \tag{4.12}$$

Moreover, these subspaces are invariant subspaces of $(A-B)\mid L_{k\tau,z}$.

Proof. Let $\omega,\eta\in\mathbf{U}$ be such that $\omega^k=\eta^k=z$ and $\omega\neq\eta$. For any $x\in L_{k\tau,z}(\tau,\omega)$ and $y\in L_{k\tau,z}(\tau,\eta)$, we have

$$(x,y)_{k\tau}=(x,y)_\tau\sum_{m=0}^{k-1}(\omega\bar{\eta})^m=0,$$

where $(x,y)_\tau$ is the L^2_τ inner product. So (4.12) follows. The second part of the lemma follows from the τ-periodicity of B.

Let $\gamma\in C([0,\tau],\mathrm{Sp}(2n,\mathbf{C}))$ with $\gamma(0)=I$. As in [25], the iteration path of γ is defined by

$$\tilde{\gamma}(t)=\gamma(t-j\tau)\gamma(\tau)^j\quad\text{for } j\in\{0\}\cup\mathbf{N}\text{ and } j\tau\le t\le(j+1)\tau.\tag{4.13}$$

Then we have

Theorem 4.2. *For any* $\tau>0$, $z\in\mathbf{U}$, $\gamma\in\partial_\tau$, *and* $k\in\mathbf{N}$, *there hold*

$$i_{k\tau,z}(\tilde{\gamma})=\sum_{\omega^k=z}i_{\tau,\omega}(\gamma),\tag{4.14}$$

$$\nu_{k\tau,z}(\tilde{\gamma})=\sum_{\omega^k=z}\nu_{\tau,\omega}(\gamma).\tag{4.15}$$

Proof. (4.15) is clear. Now we prove (4.14). By Proposition 2.2 in [35], Lemma 4.3 and the definition of the ω-indices, (4.14) is true for fundamental solutions of periodic Hamiltonian systems. In the general case, let $\gamma_1\colon[0,+\infty)\to\mathrm{Sp}(2n,\mathbf{C})$ be one of such solutions satisfying $\gamma_1(\tau)=\gamma(\tau)$. Set $M=\gamma(\tau)$. Since $\tilde{\gamma}_1\mid_{[j,j+1]}*\tilde{\gamma}\mid_{[j-1,j]}$ is homotopic to $\tilde{\gamma}\mid_{[j,j+1]}*\tilde{\gamma}_1\mid_{[j-1,j]}$ rel. endpoints, by Lemma 2.2, it holds that

$$i_z(\tilde{\gamma}\mid[j\tau,(j+1)\tau])-i_\omega(\gamma)$$

depends only on j, M, z and ω, where $j\in\{0\}\cup\mathbf{N}$, $z,\omega\in\mathbf{U}$. Hence (4.14) follows.

4.3. The Mean Indices for Symplectic Paths

Let $\tilde{\gamma}\in C([0,+\infty),\mathrm{Sp}(2n,\mathbf{C}))$ be the iteration path of γ defined by (4.13) and set $i_{k\tau}=i_{k\tau}(\tilde{\gamma})$. In this subsection we prove that the limit $\lim\limits_{k\to\infty}i_{k\tau}/k$ for every path $\gamma\in\mathcal{P}_\tau$ always exists and is finite, and study some properties of this limit.

Theorem 4.3. *For any* $\tau>0$ *and* $\gamma\in\mathcal{P}_\tau$, *there holds*

$$\hat{i}_\tau(\gamma)\equiv\lim_{k\to\infty}\frac{i_{k\tau}}{k}=\frac{1}{2\pi}\int_{\mathbf{U}}i_{\tau,\omega}(\gamma)d\omega.\tag{4.16}$$

In particular, $\hat{i}_\tau(\gamma)$ *is always a finite real number.*

Proof. It follows from Theorem 4.2, Lemma 4.1 and the definition of Riemannian intergrals.

Definition 4.2. *The mean index per period* τ *of* $\gamma\in\mathrm{P}_\tau$ *is defined by*

$$\hat{i}_\tau\equiv\hat{i}_\tau(\gamma)=\lim_{k\to\infty}\frac{i_{k\tau}}{k}\in\mathbf{R}.\tag{4.17}$$

Remark 4.1. By Corollary 2.1 our definition of mean indices coincides with that of Y. Long[25] previously.

A direct consequence of this definition is

Corollary 4.1. *There holds*

$$\hat{i}_{k\tau}=k\hat{i}_\tau,\qquad\forall k\in\mathbf{N}.\tag{4.18}$$

Lemma 4.4. *Let $\gamma \in \mathcal{P}_\tau$ and $\omega \in \mathbf{U} \setminus \{1\}$. Then there holds*

$$i_\tau(\gamma) - 2n + \nu_\tau(\gamma) \le i_{\tau,\omega}(\gamma) \le i_\tau(\gamma) - \nu_{\tau,\omega}(\gamma). \tag{4.19}$$

Proof. Assume that $\omega = e^{it}$ where $t \in (0, 2\pi)$. Let $\gamma(\tau) = M$ and

$$(\sigma(M) \cap \mathbf{U}) \cup \{1, \omega\} = \left\{ e^{it_j} \mid j = 0, 1, \cdots, m \right\},$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 2\pi$ and $t_k = t$. Let $S_j^\pm \equiv S_{-,M}^\pm(e^{it_j})$, $j = 0, \cdots, m$, $S_{+,0}^+ \equiv S_{+,M}^+(1)$ and $S_{+,k}^- \equiv S_{+,M}^-(\omega)$. By Lemma 2.2 we have

$$i_{\tau,\omega}(\gamma) - i_\tau(\gamma) = S_0^+ + \sum_{0<j<k} (S_j^+ - S_j^-) - S_k^- - n. \tag{4.20}$$

Since the Krein form has signature 0, we have

$$\sum_{\omega \in \mathbf{U}} p_\omega = \sum_{\omega \in \mathbf{U}} q_\omega \le n, \tag{4.21}$$

where (p_ω, q_ω) are the Krein type numbers of ω for M. By Lemma 4.2 we have

$$\sum_{j=0}^m S_j^+ \le n, \tag{4.22}$$

$$S_{+,0}^+ + \sum_{j=1}^m S_j^- \le n, y \tag{4.23}$$

$$S_{+,0}^+ + S_0^+ = \nu_\tau(\gamma), \tag{4.24}$$

$$S_{+,k}^- + \sum_{j=0}^{k-1} S_j^+ \le n, \tag{4.25}$$

$$S_{+,k}^- + S_k^- = \nu_{\tau,\omega}(\gamma), \tag{4.26}$$

$$S_j^\pm \ge 0. \tag{4.27}$$

Combiing the Equations from (4.20) to (4.27) we obtain (4.19).

Corollary 4.2. *There holds*

$$k i_\tau - (k-1)(2n - \nu_\tau) \le i_{k\tau} \le k i_\tau + \nu_\tau - \nu_{k\tau}, \qquad \forall k \in \mathbf{N}. \tag{4.28}$$

Proof. By Theorem 4.2 and Lemma 4.4.

The following estimate belongs to [19]. Here we give a different proof.

Corollary 4.3. *Let $\tilde{\gamma}(k\tau) = M$. Then we have*

(i) *There holds*

$$k\hat{i}_\tau \le i_{k\tau} \le k\hat{i}_\tau + 2n - \nu_{k\tau}, \quad \forall k \in \mathbf{N}. \tag{4.29}$$

(ii) *The left equality in* (4.29) *holds iff $S_{-,M}^\pm(1) = n$. In this case, we have $\sigma(M) = \{1\}$.*

(iii) *The right equality in* (4.29) *holds iff $S_{+,M}^\pm(1) = n$. In this case, we have $\sigma(M) = \{1\}$.*

(iv) *Both of the equalities in* (4.29) *hold iff $M = I$.*

Proof. (i) By Corollary 4.1 the proof is reduced to the case $k = 1$ as in [19]. Let $\gamma(\tau) = M$. By Definition 4.2 and Lemma 4.4 we obtain (4.29) in the case $k = 1$.

(ii) By Lemma 4.2 and the proof of Lemma 4.4, the left equality of (4.29) holds iff $S_{-,M}^+(1) = n$ and $S_{-,M}^\pm(\omega) = 0$ for all $\omega \in \mathbf{U} \setminus \{1\}$, iff $S_{-,M}^\pm(1) = n$. In this case the dimension of the root vector space E_1 of M is $2n$ and hence $\sigma(M) = \{1\}$.

(iii) can be proved similarly. (iv) is obvious.

§A. Some Properties of Sp($2n$,C)

A.1. Basic Properties

As in [24], it is convenient to give the following notations.

Notation A.1. *We define the $\diamond$-product* $\diamond\colon \mathrm{Sp}(2k,\mathbf{C})\times\mathrm{Sp}(2l,\mathbf{C})\to\mathrm{Sp}(2k+2l,\mathbf{C})$ *by*

$$M_1\diamond M_2=\begin{pmatrix}A_{11}&0&A_{12}&0\\0&B_{11}&0&B_{12}\\A_{21}&0&A_{22}&0\\0&B_{21}&0&B_{22}\end{pmatrix},$$

where $M_1\in\mathrm{Sp}(2k,\mathbf{C})$ *and* $M_2\in\mathrm{Sp}(2l,\mathbf{C})$ *are in square block forms*

$$M_1=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix},\quad M_2=\begin{pmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}.$$

Moreover, we define $M^{\diamond k}=M\diamond\cdots\diamond M$, *the* k*-fold* $\diamond$*- product of* $M\in\mathrm{Sp}(2n,\mathbf{C})$.

Define

$$\mathrm{P}(n)=\{M\in\mathrm{gl}(n,\mathbf{C})\mid M=M^*\text{ and }M\text{ is positive define}\}.$$

The proof of the following proposition is the same as that of Theorem 4.1 in Chapter 1 of [22] and therefore is omitted.

Proposition A.1. (i) $\mathrm{Sp}(2n,\mathbf{C})=PK$ *where* $P=\mathrm{Sp}(2n,\mathbf{C})\cap\mathrm{P}(2n)$ *and* $K=\mathrm{H}(2n)\equiv\mathrm{Sp}(2n,\mathbf{C})\cap\mathrm{U}(2n)$.

(ii) P *is diffeomorphic to* $\mathbf{R}^{2n^2}$ *and* K *is diffeomorphic to* $\mathrm{U}(n)\times\mathrm{U}(n)$.

(iii) $\mathrm{Sp}(2n,\mathbf{C})$ *is path connected and* $\pi_1(\mathrm{Sp}(2n,\mathbf{C}))=\mathbf{Z}\oplus\mathbf{Z}$.

Corollary A.1. *For any* $(a,b)\in\frac12\mathbf{Z}\times\frac12\mathbf{Z}$, *we define the path* $\gamma_{a,b}$ *by*

$$\gamma_{a,b}(t)=(e^{\pi i(a+b)t}e^{\pi J_1(a-b)t})\diamond I_{2n-2},\tag{A.1}$$

where $i=\sqrt{-1}$, $t\in[0,1]$ *and* $J_1=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. *Then we have*

(a) *For any loop* $\gamma\colon[0,1]\to\mathrm{Sp}(2n,\mathbf{C})$ *such that* $\gamma(0)=\gamma(1)=I$, *there is a unique* $(a,b)\in\mathbf{Z}\times\mathbf{Z}$ *such that* $\gamma\sim\gamma_{a,b}$ rel. $0,1$.

(b) *For any path* $\gamma\colon[0,1]\to\mathrm{Sp}(2n,\mathbf{C})$ *such that* $\gamma(0)=I$ *and* $\gamma(1)=-I$, *there is a unique* $(a,b)\in(\mathbf{Z}+\frac12)\times(\mathbf{Z}+\frac12)$ *such that* $\gamma\sim\gamma_{a,b}$ rel. $0,1$.

A.2. The Topological Structure of Sp*($2n$,C)

In this subsection we want to show that the nonsingular part of $\mathrm{Sp}(2n,\mathbf{C})$ defined by

$$\mathrm{Sp}^*(2n,\mathbf{C})=\{M\in\mathrm{Sp}(2n,\mathbf{C}))\mid\det(M-I)\neq0\}$$

is path connected. Note that this is rather different from the real case studied in [11] and [28].

Lemma A.1. *Let* $M\in\mathrm{Sp}(2n,\mathbf{C})$. *Then there is a smooth curve*

$$\gamma\colon(-\epsilon,\epsilon)\to\mathrm{Sp}(2n,\mathbf{C}),\quad\epsilon>0$$

such that $\gamma(0)=M$ *and* $\gamma(t)$ *have only simple eigenvalues for all* $(-\epsilon,\epsilon)\setminus\{0\}$.

Proof. The proof is the same as that of Theorem 5.1 in Chapter 1 of [22] and therefore is omitted.

Proposition A.2. $\mathrm{Sp}^*(2n,\mathbf{C})$ *is path connected.*

Proof. Let $M\in\mathrm{Sp}^*(2n,\mathbf{C})$. By Lemma A.1 there is a curve $\gamma_1\subset\mathrm{Sp}^*(2n,\mathbf{C})$ with endpoints M and M_1 such that M_1 has only simple eigenvalues. It is clear that there are smooth curves

$$f_\lambda\colon [0,1] \to \mathbf{U} \setminus \{1\},\ \forall \lambda \in \sigma(M_1) \cap \mathbf{U},$$
$$f_\lambda\colon [0,1] \to \mathbf{D} \cup \{-1\},\ \forall \lambda \in \sigma(M_1) \cap \mathbf{D}$$

such that $f_\lambda(0) = \lambda$ and $f_\lambda(1) = -1$. Let $f_\mu(t) = (\overline{f_\lambda(t)})^{-1}$ for $\lambda \in \sigma(M_1) \cap \mathbf{D}$, $\mu = \bar{\lambda}^{-1}$, and $t \in [0,1]$. For all $\lambda \in \sigma(M_1)$, choose $v_\lambda \neq 0$ such that $M_1 v_\lambda = \lambda v_\lambda$. Define a smooth curve $\gamma_2\colon [0,1] \to \mathrm{Sp}^*(2n, \mathbf{C})$ by

$$\gamma_2(t) v_\lambda = f_\lambda(t) v_\lambda,\ \forall \lambda \in \sigma(M_1),\ t \in [0,1].$$

Clearly $\gamma_2(0) = M_1$ and $\gamma_2(1) = -I$. So M can be connected to $-I$ via continuous curve in $\mathrm{Sp}^*(2n, \mathbf{C})$ and our proposition is proved.

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