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LINEAR ISOMETRIC NON-ANTICIPATIVE TRANSFORMATIONS OF WIENER PROCESS

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Abstract

The necessary and sufficient conditions are given so that a non-anticipative transformation in Hilbert space is isometric. In terms of second order Wiener process, these conditions assure that a non-anticipative transformation of Wiener process is a Wiener process, too.

Keywords Resolution of identity, Separable Hilbert space, Wide sense Wiener process, Non-anticipative transformation
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This paper is an outcome of numerous discusions and exchange of ideas with late Predrag Peruničić, whose deep and profound mathematical knowledge and human values gave us an inspiration and motivation to finalize our joint ideas.

1. We refer to [3,4] for the notion of spectral multiplicity theory in the separable Hilbert space. Let $\mathcal{H}$ be a cyclic Hilbert space with the resolution of the identity $\{\mathcal{P}(t)\}$ of the maximal spectral type $\|\mathcal{P}(dt)\|^2 = dt$-ordinary Lebesgue measure. A non-anticipative linear transformations is defined by Volterra kernel $g(t, u),\ u \le t$, as

$$\int_0^t g(t, u)\mathcal{P}(du), \quad t > 0. \tag{0}$$

It means that for $\xi, \xi \in \mathcal{H}$, the element $\mathcal{P}(t)\xi$ transforms in $\int_0^t g(t, u)\mathcal{P}(du)\xi$ for each $t > 0$.

We are going to find the necessary and sufficient conditions so that

$$\mathcal{P}_1(t) = \int_0^t g(t, u)\mathcal{P}(du)$$

defines the resolution of identity $\{\mathcal{P}_1(t)\}$ in $\mathcal{H}_1.\mathcal{H}_1 \subset \mathcal{H}$, such that again $\|\mathcal{P}_1(dt)\|^2 = dt$, i.e. that the transformation (0) is isometric. Also, we show that there exists the inverse isometric transformation

$$\mathcal{P}(t) = \int_0^\infty h(t, u)\mathcal{P}_1(du)$$

and determine $h(t, u)$ in terms of $g(t, u)$.

In the sequel we shall use the technique of second-order stochastic processes as the curves in Hilbert space[1].

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2. Let $W = \{W(t),\ t \geq 0\}$ be a standard wide sense Wiener process

$$(\mathbb{E}\{W(t)\} = 0, \quad \mathbb{E}\{W(t)W(s)\} = \langle W(t), W(s)\rangle = \|W(\min(s,t)\|^2 = \min(s,t)).$$

Denote be $\mathcal{H}(W;t)$ the mean-square linear closure of $\{W(u), u \leq t\}, \mathcal{H}(W) = \overline{\vee_t \mathcal{H}(W;t)}$. Each $\xi \in \mathcal{H}(W;t)$ has the representation

$$\xi = \int_0^t f(u)W(du), \quad \|\xi\|^2 = \int_0^t f^2(u)du \tag{1}$$

for some $f \in L_2([0,t];du)$.

A second order process $X = \{X(t), t \leq 0\}$ is a (linear) non-anticipative transformation of W if $X(t) \in \mathcal{H}(W;t)$ for each $t \geq 0$. It follows from (1) that X has the representation

$$X(t) = \int_0^t g(t,u)W(du), \quad \|X(t)\|^2 = \int_0^t g^2(t,u)du \tag{2}$$

for some Volterra kernel $g(t,u) \in L_2([0,t];du)$.

Lévy[5] gave some examples of processes X which are Wiener processes too. Hida[2] treated these examples in the framework of the theory of canonical representation of Gaussian processes.

In this paper we give the necessary and sufficient conditions on the Volterra kernel $g(t,u)$ such that $W_1 = \{W_1(t).t \geq 0\}$ defined by

$$W_1(t) = \int_0^t g(t,u)W(du) \tag{3}$$

is a Wieneer process, i.e. that the linear non-anticipative transformation defined by $g(t,u)$ is isometric. In this case, $g(t,u)$ is called a Wiener kernel.

Example 1. Consider a family of second order processes $\{X_\alpha, \alpha > 0\}$ defined by

$$X_\alpha = \int_0^t \Big(\frac{2\alpha+1}{\alpha}\Big(\frac{u}{t}\Big)^\alpha - \frac{\alpha+1}{\alpha}\Big)W(du). \tag{4}$$

Then $X_\alpha = \{X_\alpha(t), t \geq 0\}$ is a Wiener process for every $\alpha > 0$.

For $\alpha = 1$ we have the well-known Lévy's example

$$X_1(t) = \int_0^t \Big(-2 + 3\frac{u}{t}\Big)W(du). \tag{5}$$

Treating linear combinations of Wiener kernels from (4), one can prove that for every $n \geq 1$ and $\alpha_1, \cdots, \alpha_n > 0$ we can determine the constants $c_k (0 \leq k \leq n)$ so that the process

$$\int_0^t \Big(c_0 + \sum_{k=1}^n c_k \Big(\frac{u}{t}\Big)^{\alpha_k}\Big)W(du)$$

is a Wiener process. For $n = 2$ and $\alpha \neq \beta$ we get

$$X(t) = \int_0^t \Big(\frac{\alpha+\beta+1}{\alpha-\beta}(g_\alpha(t,u) - g_\beta(t,u)) + 1\Big)W(du),$$

where $g_\alpha(t,u)$ is the Wiener kernel defined by (4).

Note that the spaces $\mathcal{H}(W_0;t)$ and $\mathcal{H}(W_1;t)$ coincide for each $t > 0$ if and only if

$$W_1(t) = \int_0^t h(u)W_0(du), \tag{6}$$

where $h^2(u) = 1$, a.s. We say that W_0 and W_1 are equivalent and in fact we deal with classes of equivalences Wiener processes (or Wiener kernels).

Theorem 1. *Let $g(t,u)$ be an analytic function on $\{0, u \le t\}$. Then $g(t,u)$ is a Wiener kernel if and only if*
(i) $g^2(t,t) = 1$;
(ii) *$g(t,u)$ is a homogeneous function, i.e. $g(t,u) = g(u/t)$;*
(iii) *random variable $\xi = \int_0^t ug'(u/t)dW(u)$ belongs to $\mathcal{H}(W;t) \ominus \mathcal{H}(W_1;t)$, i.e.*

$$\int_0^s ug'\Big(\frac{u}{t}\Big)g\Big(\frac{u}{s}\Big)du = 0 \quad \textit{for all} \quad s \le t.$$

Proof. ($\Longrightarrow$) Differentiating in s and t the relation $\int_o^s g(t,u)g(s,u)du = s$ one gets

$$\int_0^s g_1'(t,u)g(s,u)du = 0, \quad g(t,s)g(s,s) + \int_0^s g(t,u)g_1'(s,u)du = 1$$

respectively. By continuity, letting $s \uparrow t$ we get

$$\int_0^t g_1'(t,u)g(t,u)du = 0, \quad g^2(t,t) + \int_0^s g(t,u)g_1'(t,u)du = 1,$$

or $g^2(t,t) = 1$. In the sequel we shall suppose that $g(t,t) = 1$, due to equivalence classes of Wiener kernels.

Consider the kernel $g(xt, xu), x > 0$ and the processes

$$W_x(t) = \int_0^t g(xt, xu)W(du). \tag{7}$$

It is easy to verify that

$$\langle W_x(s), W_x(t)\rangle = \int_0^s g(xt, xu)g(xs, xu)du = s, \quad s \le t.$$

It means that $W_x(t)$ is a Wiener process too. Denote

$$f(x,y,s,t) = \langle W_x(t), W_y(s)\rangle = \int_0^s g(xt,xu)g(ys,yu)du, \quad s \le t, \quad x,y > 0. \tag{8}$$

We are going to show that $f(x,y,s,t)$ satisfies the following linear partial differential equation

$$xf_x'(x,y,s,t) + yf_y'(x,y,s,t) - sf_s'(x,y,s,t) - tf_t'(x,y,s,t) + f(x,y,s,t) = 0. \tag{9}$$

Indeed

$$f_x'(x,y,s,t) = \int_0^s (tg_1'(xt,xu) + ug_2'(xt,xu))g(ys,yu)du, \tag{10}$$

$$f_t'(x,y,s,t) = \int_0^s xg_1'(xt,xu)g(ys,yu)du, \tag{11}$$

$$f_s'(x,y,s,t) = g(xt,xs) + \int_0^s g(xt,xu)yg_1'(ys,yu)du. \tag{12}$$

We have from (10) and (11)

$$\frac{tf_t'(x,y,s,t)}{x} = f_x'(x,y,s,t) - \int_0^s ug_2'(xt,xu)g(ys,yu)du,$$

and by partial integration

$$y\int_0^s g(xt,xu)ug_2'(ys,yu)du = tf_t'(x,y,s,t) - xf_x'(x,y,s,t) + sf_s'(x,y,s,t) - f(x,y,s,t). \tag{13}$$

So, adding (12) and (13) we have

$$uf_y'(x,y,s,t) = tf_t'(x,y,s,t) - xf_x'(x,y,s,t) - f(x,y,s,t),$$

which is equivalent to (9).

The general solution of Equation (9) is $f(x,y,s,t) = \frac{\Phi(tx,sx,y/x)}{x}$, where $\Phi(\cdot,\cdot,\cdot)$ is an arbitrary function. If $s \uparrow t$, by continuity we have

$$\langle W_x(t), W_y(t)\rangle = \frac{\Psi\big(tx, \frac{y}{x}\big)}{x},$$

where $\Psi(\cdot,\cdot)$ is an arbitrary function. From symmetry in x and y it follows that

$$\frac{\Psi\big(tx, \frac{y}{x}\big)}{x} = \frac{\Psi\big(ty, \frac{x}{y}\big)}{y}, \quad \text{or} \quad \Psi\Big(tx, \frac{y}{x}\Big)\frac{y}{x} = \Psi\Big(ty, \frac{x}{y}\Big).$$

Putting $tx = \alpha$ and $y/x = \beta$, we see that the pervious relation becomes

$$\Psi(\alpha,\beta)\beta = \Psi\Big(\alpha\beta, \frac{1}{\beta}\Big). \tag{14}$$

As $\Psi(\alpha,\beta)$ is an analytic function, we have $\Psi(\alpha,\beta) = \sum\limits_{m,n=0}^{\infty} a_{m,n}\alpha^m\beta^n$. Condition (14) yields

$$\sum_{m,n=0}^{\infty} a_{m,n}\alpha^m\beta^{n+1} = \sum_{m,n=0}^{\infty} a_{m,n}\alpha^m\beta^{m-n}.$$

This equality could be satisfied only if, for $n+1 \neq m-n$, i.e. $m \neq 2n+1$, coefficients $a_{m,n}$ are equal to zero. Hence, the function $\Psi(\alpha,\beta)$ must be of form $\Psi(\alpha,\beta) = \sum\limits_{n=0}^{\infty} b_n\alpha^{2n+1}\beta^n$. Recalling the definition of the function Ψ we have

$$\langle W_x(t), W_y(t)\rangle = \sum_{n=0}^{\infty} b_n t^{2n+1} x^n y^n \tag{15}$$

from $\Psi(tx, y/x) = \sum\limits_{n=0}^{\infty} b_n t^{2n+1} x^{n+1} y^n$.

If $x = y$, then $\langle W_x(t), W_x(t)\rangle = t$, so $t = \sum\limits_{n=0}^{\infty} b_n t^{2n+1} x^{2n}$, from which it is obvious that $b_0 = 1$ and $b_n = 0,\ n \geq 1$. Then (15) becomes $\langle W_x(t), W_y(t)\rangle = t$, and $\|W_x(t) - W_y(t)\| = 0$. So, $g(xt, xu)$ does not depend on x, i.e. $g(t,u)$ is a homogeneous function of the form $g(u/t)$.

The condition (iii) follows from $\int_0^s g(u/t)g(u/s)du = s,\ s \leq t$, via differentation in t:

$$\int_0^s \Big(-\frac{u}{t^2}\Big)g'\Big(\frac{u}{t}\Big)g\Big(\frac{u}{s}\Big)du = 0, \quad s \leq t.$$

($\Longleftarrow$) From (iii) we have $\int_0^s g(u/t)g(u/s)du = \Phi(s)$ and, as $t \downarrow s$, $\int_0^s g^2(u/s)du = \Phi(s)$. Differentation in s yields $\Phi'(s) = 1$, because of conditions (i) and (iii). Now, $\Phi(s) = s + C$, where C is an arbitrary constant. But, since $s\int_0^s g^2(z)dz = \Phi(s)$, we have $C = 0$ and $\Phi(s) = s$, which means that $g(u/t)$ is Wiener kernel. The proof is complete.

Now we shall consider the existence of the inverse transformation of (3). As $\mathcal{H}(W_1;t)$ is a proper subspace of $\mathcal{H}(W;t)$, such a transformation is anticipative.

Theorem 2. *Let g be an analytic Wiener kernel. Then $\mathcal{H}(W_1) = \mathcal{H}(W)$, and the inverse isometric transformation of* (3) *is*

$$W(t) = \int_0^t \Big(\int_0^u \frac{\partial}{\partial u}g\Big(\frac{v}{u}\Big)dv + 1\Big)W_1(du) + \int_t^\infty \Big(\int_0^t \frac{\partial}{\partial u}g\Big(\frac{v}{u}\Big)dv\Big)W_1(du), \tag{16}$$

or

$$\mathcal{P}(t) = \int_0^t \Big(\int_0^u \frac{\partial}{\partial u} g\Big(\frac{v}{u}\Big)dv + 1\Big)\mathcal{P}_1(du) + \int_t^\infty \Big(\int_0^t \frac{\partial}{\partial u} g\Big(\frac{v}{u}\Big)dv\Big)\mathcal{P}_1(du).$$

Proof. Since

$$\langle W_1(s), W(t)\rangle = \int_0^{t\wedge s} g\Big(\frac{u}{s}\Big)du,$$

$$\frac{\partial}{\partial s}\langle W_1(s), W(t)\rangle = \int_0^{t\wedge s} \frac{\partial}{\partial s} g\Big(\frac{u}{s}\Big)du + \mathbb{I}\{s \le t\},$$

it is evident that $\mathcal{P}^{W_1}(W(t))$ (the projection of $W(t)$ on $\mathcal{H}(W_1)$) is equal to the right-hand side in (16). So we must prove that

$$\mathcal{P}^{W_1}(W(t)) = W(t). \tag{17}$$

First of all, as

$$\int_0^t \Big(\frac{\partial}{\partial t} g\Big(\frac{u}{t}\Big)\Big)^2 du = \int_t^\infty \Big(\frac{\partial}{\partial v} g\Big(\frac{t}{v}\Big)\Big)^2 dv$$

and $\frac{\partial}{\partial t} g\Big(\frac{u}{t}\Big) \in L_2([0,t]; du)$, it follows that $\frac{\partial}{\partial v} g\Big(\frac{t}{v}\Big) \in L_2([t,\infty), du)$, so

$$\int_t^\infty \frac{\partial}{\partial v} g\Big(\frac{t}{v}\Big) W_1(dv) \in \mathcal{H}(W_1). \tag{18}$$

Rewriting (3) in the form $W_1(t) = \int_0^t g\Big(\frac{u}{t}\Big)W(du)$ we have

$$W_1(dv) = \Big(\int_0^v \frac{\partial}{\partial v} g\Big(\frac{u}{v}\Big)W(du)\Big)dv + W(dv), \tag{19}$$

$$\int_t^\infty \frac{\partial}{\partial v} g\Big(\frac{t}{v}\Big)W_1(dv) = \int_0^\infty \Big(\frac{\partial}{\partial v} g\Big(\frac{t}{v}\Big)\frac{\partial}{\partial v} g\Big(\frac{u}{v}\Big)dv\Big)W(du) + \int_t^\infty \frac{\partial}{\partial v} g\Big(\frac{t}{v}\Big)W(dv).$$

Putting $x = \frac{tu}{v}$, we can easily see that

$$\int_{t\vee u}^\infty \frac{\partial}{\partial v} g\Big(\frac{t}{v}\Big)dv = \int_0^{t\wedge u} \frac{\partial}{\partial u} g\Big(\frac{x}{u}\Big)\frac{\partial}{\partial t} g\Big(\frac{x}{t}\Big)dx = -\frac{\partial}{\partial t\vee u} g\Big(\frac{t\wedge u}{t\vee u}\Big).$$

Therefore

$$\begin{aligned}\int_t^\infty \frac{\partial}{\partial v} g\Big(\frac{t}{v}\Big)W_1(dv) &= -\int_0^\infty \frac{\partial}{\partial t\vee u} g\Big(\frac{t\wedge u}{t\vee u}\Big)W(du) + \int_0^\infty \frac{\partial}{\partial u} g\Big(\frac{t}{u}\Big)W(du)\\ &= -\int_0^t \frac{\partial}{\partial t} g\Big(\frac{u}{t}\Big)W(du). \end{aligned}\tag{20}$$

From the relations (19) and (20) follows

$$W_1(dt) = -\Big(\int_t^\infty \frac{\partial}{\partial v} g\Big(\frac{t}{v}\Big)W_1(dv)\Big)dt + W(dt).$$

Hence

$$\begin{aligned}W(t) &= W_1(t) + \int_0^t \Big(\int_u^\infty \frac{\partial}{\partial v} g\Big(\frac{u}{v}\Big)W_1(dv)\Big)du\\ &= W_1(t) + \int_0^\infty \Big(\int_0^{t\wedge v} \frac{\partial}{\partial v} g\Big(\frac{u}{v}\Big)du\Big)W_1(dv)\\ &= \int_0^t \Big(\int_0^u \frac{\partial}{\partial u} g\Big(\frac{v}{u}\Big)dv + 1\Big)W_1(du) + \int_t^\infty \Big(\int_0^t \frac{\partial}{\partial u} g\Big(\frac{v}{u}\Big)dv\Big)W_1(du)\\ &= \mathcal{P}^{W_1}(W(t))\end{aligned}$$

and the statement is proved.

Remark. Note that if we consider Wiener process $\{W(t), 0 \le t \le T\}$ on the finite interval, the inverse transformation does exist.

Denote

$$\sigma^2(s,t) = \|W(t) - \mathcal{P}_s^{W_1}(t))\|^2,$$

where $\mathcal{P}_s^{W_1}$ is the projection operator on $\mathcal{H}(W_1; s)$. For $s \ge t$ we have $\sigma^2(s,t) = t\phi\left(\frac{t}{s}\right)$, where $\phi(x)$ could be determined from the equation

$$\phi'(x) = \left(\frac{1}{x}\int_0^x sg'(s)ds\right)^2, \quad x > 0.$$

Example 3. Consider Lévy's kernel (5)

$$g\left(\frac{u}{t}\right) = -2 + 3\frac{u}{t}, \quad 0 < u \le t.$$

In that case

$$W(t) = -\frac{1}{2}W_1(t) - \frac{3}{2}t^2\int_t^\infty \frac{1}{u^2}W_1(du).$$

If we project $W(t)$ on $\mathcal{H}(W_1; s)$, then for $s \ge t$,

$$\mathcal{P}_s^{W_1}(W(t)) = -\frac{1}{2}W_1(t) - \frac{3}{2}t^2\int_t^s \frac{1}{u^2}W_1(du), \quad \sigma^2(s,t) = \frac{3t^2}{4s^2}.$$

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