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# GENUS MINIMIZING IN SYMPLECTIC 4-MANIFOLDS**

CHO Yongseung*     CHO Misung*

**Abstract**

The authors show that a symplectically embedded surface in a symplectic 4-manifold with $b_2^+$ greater than one minimizes genus in its homology class.

**Keywords** Generalized Thom Conjecture, Seiberg-Witten invariant, Symplectic 4-manifolds
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## §1. Introduction

**1.** In [9] a generalized Thom conjecture for symplectic manifolds is stated as follows: In a symplectic 4-manifold $X$, does a symplectic surface $\Sigma$ minimize genus in its homology class ? In [10] Kronheimer and Mrowka show that an algebraic curve of degree $d$ in $\mathbb{CP}^2$ minimizes genus in its homology class. In [13] Morgan, Szabó and Taubes give positive answer when $X$ is a compact Kähler surface, $\Sigma$ a smooth holomorphic curve and $\Sigma \cdot \Sigma \geq 0$, using Seiberg-Witten gauge theory.

A 2-form $\omega$ on a closed, oriented, smooth 4-manifold $X$ is symplectic if $\omega$ is a closed, non-degenerated 2-form. In this case the pair $(X, \omega)$ is called a symplectic 4-manifold. The symplectic structure determines a compatible almost complex structure on $X$, and the canonical complex line bundle $K_X$ over $X$. In [14] Taubes shows that the canonical Spin$^c$ structure $K_X$ is a basic class of $X$, that is, the Seiberg-Witten invariant of $K_X$ is non-zero (in fact, $\pm 1$). The other Spin$^c$ structure on $X$ is given by

$$W^+ = E \oplus (K^{-1} \otimes E)$$

for some complex line bundle $E$ over $X$. In this case, the Seiberg-Witten invariant for $W^+$ is equal to the Gromov invariant for $E$ (see [15]).

There are non-symplectic 4-manifolds which have non-zero Seiberg-Witten invariants[2]. In [3] we study $G$-invariant Seiberg-Witten invariants and Seiberg-Witten invariants on the quotient setting when a finite group $G$ acts in the question. In [7] we show that there

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*Department of Mathematics, Ewha Women's University, Seoul 120-750, Korea.
**E-mail:** yescho@mm.ewha.ac.kr

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are simply-connected symplectic 4-manifolds corresponding to integral lattice points of the Noether line below.

In the symplectic 4-manifolds, the adjunction formula still holds. That is, if $\Sigma$ is a symplectically embedded surface in $X$, then

$$2g(\Sigma) - 2 = \Sigma \cdot \Sigma + c_1(K) \cdot \Sigma,$$

where $K$ is the canonical complex line bundle over $X$. Thus the Generalized Thom Conjecture for symplectic manifolds is still open when $b_2^+ = 1$ or the self-intersection is negative. In this paper we would like to prove that for a generic almost complex structure, a symplectically embedded 2-manifold minimizes genus in its homology class in a symplectic 4-manifold. In [4] we have a similar result for almost complex 4-manifolds.

**2.** A blow-up $(\bar{X}, \bar{\omega}_\psi)$ of $(X, \omega)$ of weight $\lambda$ is obtained from a symplectic embedding $\psi$ of a closed ball $B^4(\lambda)$ into $X$ by extending $\psi$ to a symplectic embedding $\psi_0$ of the ball $B^4(\lambda+\delta)$ for some small $\delta > 0$, and then replacing the image $\psi_0(B^4(\lambda+\delta))$ by the standard neighbourhood $L(\delta)$, where $L(\delta) = \{z \in L \mid |z| < \delta\}$, $L$ is the tautological bundle over $\mathbb{CP}^1$.

Let $\pi_1 : L \to \mathbb{CP}^1$ and $\pi_2 : L \to \mathbb{C}^2$ be the canonical projections of the tautological bundle $L$. Then $L(\delta) = \pi_2^{-1}(B^4(\delta))$. The symplectic form

$$\rho(\lambda) = \pi_2^* \omega_0 + \lambda^2 \pi_1^* \tau_0,$$

where $\tau_0$ is the standard symplectic form on $\mathbb{CP}^1$. Thus the manifold $\bar{X}$ is defined to be

$$\bar{X} = [X \backslash \psi_0(\text{int} B^4(\lambda + \delta)] \cup L(\delta).$$

The form $\bar{\omega}_\psi$ equals $\omega$ on $X \backslash \psi_0(\text{int} B^4(\lambda+\delta))$ and equals $\rho(\lambda)$ on $L(\delta)$. Conversely, the pair $(X, \omega)$ is called the blow-down of $(\bar{X}, \bar{\omega}_\psi)$.

**Lemma 1.**[12] *For small $\lambda, \delta > 0$ the space $(L(\delta) - L(0), \rho(\lambda))$ is symplectomorphic to the spherical shell $(B^4(\lambda+\delta) - B^4(\lambda), \omega_0)$ in $\mathbb{C}^2$.*

A symplectic 4-manifold is minimal if it contains no symplectically embedded 2-spheres of self-intersection number $-1$, which are called exceptional spheres.

**Theorem 1.**[12] *Every symplectic 4-manifold $(\bar{X}, \bar{\omega})$ covers a minimal symplectic manifold $(X, \omega)$ which may be obtained from $\bar{X}$ by blowing down a finite collection of disjoint exceptional spheres. Moreover $(X, \omega)$ is determined up to symplectomorphism by the homology classes of these exceptional spheres.*

The blow-up $\bar{X}$ of a symplectic 4-manifold $X$ is diffeomorphic to $X \sharp \overline{\mathbb{CP}^2}$.

**Theorem 2.**[1] *If $K$ is a basic class on a symplectic 4-manifold $X$, then $K \pm E$ are basic classes on the blow-up space $\bar{X} = X \sharp \overline{\mathbb{CP}^2}$, where $E$ is the exceptional sphere in $\overline{\mathbb{CP}^2}$.*

**Corollary.** *Let $X$ be a symplectic 4-manifold with canonical class $K_X$, and let $\bar{X}$ be the n-times blow-up space of $X$. Then the classes $K_X \pm E_1 \pm \cdots \pm E_n$ are basic classes of $\bar{X}$, where the $E_i$ are exceptional spheres in $\bar{X}$, obtained by the blow-up's.*

**3.** Let $X$ be a symplectic 4-manifold and let $\Sigma$ be an embedded 2-dimensional submanifold of $X$ with self-intersection number $\Sigma \cdot \Sigma = 0$. Let

$$W^+ = E \oplus (K^{-1} \otimes E) \to X$$

be a Spin$^c$ structure on $X$, and $E$ is a complex line bundle over $X$. Let

$$L = \det W^+ = E \otimes K^{-1} \otimes E.$$

For a connection $A$ of $L$ and a section $\phi \in \Gamma(W^+)$ of $W^+$ the Seiberg-Witten equations are

$$\begin{cases} D_A\phi = 0, \\ F_A^+ = -\tau(\phi, \bar{\phi}). \end{cases}$$

Since $\Sigma \cdot \Sigma = 0$, there is a tubular neighbourhood $N(\Sigma)$ of $\Sigma$ in $X$ such that its boundary

$$\partial N(\Sigma) \cong S^1 \times \Sigma \equiv Y.$$

Let $(X_R, g_R)$ be the Riemannian manifold obtained from $X$ by cutting along $Y$ and inserting a cylinder $[-R, R] \times Y$, where $g_R$ is a product metric on $[-R, R] \times Y$. In [10] Kroheimer and Mrowka shows the following:

**Proposition.**[10] *Suppose the moduli space $\mathcal{M}(L, g_R)$ is non-empty for all sufficiently large $R$. Then there is a solution of the equations on the cylinder $\mathbb{R} \times Y$ which is translation-invariant in a temporal gauge.*

In the temporal gauge a connection $A$ and a section $\phi$ on the cylinder $\mathbb{R} \times Y$ can be thought as a path $A(t)$ of connections and a path $\phi(t)$ in the restricted Spin$^c$ structure $L \to Y$ over the 3-manifold $Y$. In this case, the Seiberg-Witten equations become

$$(A) \begin{cases} \frac{d\phi}{dt} = -\bar{D}_A\phi, \\ \frac{dA}{dt} = - * F_A - \tau(\phi, \bar{\phi}), \end{cases}$$

where $\bar{D}_A$ is the Dirac operator in 3-dimensional Spin$^c$ structure $W$ and $\tau$ is a pairing obtained from Clifford multiplication by using the hermitian metric on $W$.

By the uniformization theorem there is a Riemannian metric on $S^1 \times \Sigma = Y$ such that $\Sigma$ has constant scalar curvature. Then using the Gauss-Bonnet Theorem Kronheimer and Mrowka get the following theorem.

**Theorem 3.**[10] *If there is a solution to the Seiberg-Witten equations on $\mathbb{R} \times Y$ which is translation-invariant in a temporal gauge, then*

$$|c_1(L)[\Sigma]| \le 2g(\Sigma) - 2.$$

**4.** We assume that the line bundle $L$ over $\Sigma \times S^1$ is pulled back from a line bundle over $\Sigma$. This may be justified when the cohomology class of $L$ over $\Sigma \times S^1$ has no component in $H^1(\Sigma) \otimes H^1(S^1)$. The equations of $S^1$-invariant solutions of $(A)$ reduce to the following vortex equations $(B)$ over $\Sigma$. By the symmetry between $L$ and $L^{-1}$, we may suppose that the degree $d = c_1(L^{-1}) \cdot \Sigma > 0$ is non-negative.

$$(B) \begin{cases} \bar{\partial}_A\psi = 0, \\ F_A = -|\psi|^2, \end{cases}$$

where $A$ is a connection of $L^{-1} \to \Sigma$ and $\psi$ is a section of

$$K_\Sigma^{\frac{1}{2}} \otimes L^{-\frac{1}{2}} \to \Sigma.$$

**Theorem 4.** [4,16] *Under the above assumption,*

(i) *If a Spin$^c$ structure $E \oplus (K^{-1} \otimes E) \to X$ has a solution of the Seiberg-Witten equations, then the reduced vortex equations over $\Sigma$ has a solution. In this case $2g(\Sigma) - 2 \ge c_1(L) \cdot \Sigma (\equiv d)$ if*

$$E = K_\Sigma^{\frac{1}{2}} \otimes L^{-\frac{1}{2}}.$$

(ii) *If $r = (2g - 2) - d \ge 0$, then the space of solutions of the vortex equations is identified with the symmetric product $s^r(\Sigma)$ of $\Sigma$.*

**5.** We introduce some results about $J$-holomorphic curves on symplectic 4-manifold which will be needed later. Let $(X, J)$ be an almost complex manifold and $(\Sigma, j)$ be a Riemann surface. A smooth map $u : \Sigma \to M$ is called $J$-holomorphic (or, pseudo-holomorphic) if the differential $du$ is a complex linear map with respect to $j$ and $J$, i.e.,

$$J \circ du = du \circ j.$$

**Lemma 2.** *Let $(X, \omega_X)$ be a symplectic 4-manifold. If $F$ is a symplectically embedded surface of $X$, then $F$ is a pseudo-holomorphic curve in $X$.*

**Proof.** Since $F$ is a symplectic surface in $X$, the restriction $\omega_F = \omega_X|_F$ of the symplectic form $\omega_X$ to the submanifold $F$ is a symplectic form on $F$. Let $i : (F, \omega_F) \to (X, \omega_X)$ be an inclusion. Then $i$ is symplectic.

Let $g_X$ be a metric on $X$ and let $g_F = i^* g_X$ be a pull-back metric on $F$. Let $J_X$ be an $\omega_X$-compatible almost complex structure on $X$ and let $J_F$ be an $\omega_F$-compatible almost complex structure on $F$. Then

$$\omega_F(u, v) = i^*\omega_X(u, v) = \omega_X(i_* u, i_* v) = g_X(J_X(i_* u), i_* v)$$

for all tangent vectors $u, v$ in $TF$. Also,

$$\omega_F(u, v) = g_F(J_F(u), v) = i^* g_X(J_F(u), v) = g_X(i_*(J_F(u)), i_* v).$$

Therefore

$$g_X(J_X(i_* u), i_* v) = g_X(i_*(J_F(u)), i_* v) \quad \text{for all } v \text{ in } TF.$$

Since $i^* g_X = g_F$ is a metric on $F$,

$$J_X(i_* u) = i_*(J_F(u)) \quad \text{for all } u \ \text{ in } TF.$$

Therefore, $i : (F, J_F) \to (X, J_X)$ is $J_X$-holomorphic.

**Theorem 5.**[11] *Let $X$ be a closed symplectic 4-manifold and let $C$ be the $J$-holomorphic image of a Riemann surface $\Sigma_g$ of genus $g$. Then for generic $J$, $C$ must satisfy the inequality*

$$c_1(C) \geq 1 - g.$$

**Remark.** If $J$ is not a generic almost complex structure, then $J$-holomorphically embedded surfaces may not satisfy the above inequality. From now we will assume that almost complex structures are generic and compatible with given symplectic structure.

**Theorem 6.** *Let $(X, \omega)$ be a closed minimal symplectic 4-manifold and let $F$ be a symplectically embedded surface in $X$. Then the self-intersection number of $F$, $F \cdot F \geq 0$, is non-negative.*

**Proof.** Since $F$ is a symplectically embedded surface in $X$, let $u : (\Sigma, i) \to (X, J)$ be a $J$-holomorphic map with its image $u(\Sigma) = F$. In general the geuns $g(F)$ of $F$ is not less than that $g(\Sigma)$ of $\Sigma$, i.e.,

$$g(F) \geq g(\Sigma)$$

with equality if and only if $C$ is embedded. The adjunction formula says

$$c_1(F) = 2 - 2g(F) + F \cdot F.$$

Theorem 6 above says

$$c_1(F) \geq 1 - g(\Sigma).$$

Thus we have an inequality,

$$F \cdot F \geq g(F) - 1.$$

Since $X$ is minimal, there is no symplectically embedded 2-sphere with self-intersection number $-1$. Therefore $F \cdot F \geq 0$.

Now we are ready to prove our main theorem by using the previous results.

**Theorem 7.** *Let $X$ be a closed symplectic 4-manifold with $b_2^+(X)$ greater than one. Then for generic almost complex structure, symplectically embedded surfaces in $X$ minimize genus in their homology classes.*

**Proof.** Let $\Sigma$ be a symplectically embedded surface in $X$. Then the symplectic form $\omega$ on $X$ descends to a symplectic structure on $\Sigma$ and so is the almost complex structure which is compatible with $\omega$. Then by the proof of Corollary above, $\Sigma \cdot \Sigma \geq 0$ or $\Sigma$ is an exceptional sphere. If $\Sigma$ is an exceptional sphere, then we have nothing to prove it. We may assume that the self-intersection number $\Sigma \cdot \Sigma$ of $\Sigma$ is non-negative.

Let $\Sigma'$ be a $C^\infty$-embedding of a Riemann surface representing the same homology class as $\Sigma$ in $X$. Since

$$\Sigma' \cdot \Sigma' = \Sigma \cdot \Sigma,$$

the self-intersection $\Sigma' \cdot \Sigma'$ of $\Sigma'$ is non-negative.

First, if $\Sigma' \cdot \Sigma' = 0$, then the boundary of a tubular neighbourhood of $\Sigma'$ is the 3-manifold $\Sigma' \times S^1$. Embed $\Sigma' \times S^1 \times [-r, r]$ into $X$ as an isometry. The restriction

$$K_X \to \Sigma' \times S^1 \times [-r, r]$$

of the canonical line bundle $K_X \to X$ has a solution of the Seiberg-Witten equations which is invariant under translation.

Thus we have the solution of the three dimensional Seiberg-Witten equations of $K_X \to \Sigma' \times S^1$. These reduce to the vortex equations over $\Sigma'$ with a line bundle of degree

$$d = K_X \cdot \Sigma' = K_X \cdot \Sigma = K_X \cdot (\omega|_\Sigma).$$

To get a solution for the vortex equation the degree $d$ is less than or equal to $2g(\Sigma') - 2$. Therefore we have

$$\begin{aligned} 2g(\Sigma') - 2 \geq K_X \cdot \Sigma' &= K_X \cdot \Sigma' + \Sigma' \cdot \Sigma' \\ &= K_X \cdot \Sigma + \Sigma \cdot \Sigma = 2g(\Sigma) - 2. \end{aligned}$$

Thus we prove the theorem when $\Sigma' \cdot \Sigma' = 0$.

Secondly, if $\Sigma' \cdot \Sigma' = n > 0$, then we can reduce it to the case of self-intersection number zero by the blow-up's of $n$ points on $\Sigma'$ in $X$. Let $\bar{X}$ be the connected sum of $X$ with $n$ copies of $\overline{\mathbb{CP}^2}$, where one can think of the connected sums as being made at $n$ points of $\Sigma'$. Let $\bar{\Sigma} \subset \bar{X}$ be the surface obtained by taking an internal connected sum with $n$ copies of the projective lines in the $\overline{\mathbb{CP}^2}$'s. In this case the proper transform of $\Sigma'$ should be performed. Then $\bar{\Sigma}$ has the form $\Sigma' - E_1 \cdots - E_n$ and self-intersection number zero. The class $\bar{K}_{\bar{X}} = K_X + E_1 + \cdots E_n$ is a basic class in the symplectic manifold $\bar{X}$, by the blow-up formula for basic classes. The degree of the line bundle $\bar{K}_{\bar{X}} \to \bar{\Sigma}$ is

$$\bar{K}_{\bar{X}} \cdot \bar{\Sigma} = K_X \cdot \Sigma' + n = K_X \cdot \Sigma' + \Sigma' \cdot \Sigma'.$$

Since $\bar{\Sigma} \cdot \bar{\Sigma} = 0$, we have

$$\bar{K}_{\bar{X}} \cdot \bar{\Sigma} \leq 2g(\bar{\Sigma}) - 2 = 2g(\Sigma') - 2.$$

**Remark.** If $b_2^+(X) = 1$, then the Seiberg-Witten invariants on $X$ depend on the metrics on $X$. In [13] they define and use the $x$-negative Seiberg-Witten invariant of $X$ for a class $x \in H^2(X; \mathbb{R} \setminus \{0\})$ of non-negative square. In [10] they show that if $g$ is a metric on $\mathbb{CP}^2 \sharp n \overline{\mathbb{CP}^2}$ with $c_1(L) \cdot [\omega_g] < 0$, then the moduli space $\mathcal{M}_X(L, g)$ of $L$ with $c_1(L) = c_1(TX)$ is non-empty. Perhaps we may use this method to prove our theorem for the case $b_2^+(X) = 1$.

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