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THE HEAT FLOW OF HARMONIC MAPS FROM NONCOMPACT MANIFOLDS***

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Abstract

The authors consider the global existence of the heat flow of harmonic maps from noncompact manifolds while imposing restrictions on the initial data.

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§1. Introduction

Let M and N be two Riemannian manifolds of dimension m and n. Suppose their metrics are given by $ds_M^2 = g_{ij}dx^i dx^j$ and $ds_N^2 = h_{\alpha\beta}du^\alpha du^\beta$. Let $u : M \longrightarrow N$ be a smooth map. The energy density function of u is given by

$$e(u) = g^{ij}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}h_{\alpha\beta} = |\nabla u|^2.$$

The total energy is defined by

$$E(u) = \int_M e(u)dx.$$

A mapping $u : M \longrightarrow N$ is called a harmonic map if it is a classical solution of the Euler-Lagrange equation of $E(u)$ which can be written as

$$\tau^\alpha(u(x)) = \triangle u^\alpha(x) + \Gamma^\alpha_{\beta\gamma}(u(x))\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j}g^{ij} = 0,$$

where $\tau(u)$ is called the tension field of u. The corresponding parabolic system with initial data $u_0(x)$ known as the heat equation for harmonic maps is as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = \tau(u), \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

When M and N are compact without boundary and N has nonpositive sectional curvature, Eells-Sampson[7] proved that any C^1 map from M into N can be deformed to a

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harmonic map by solving (1.1). When $M = R^m$ and $N = S^n$, Coron and Ghidaglia[3] exhibited a class of smooth initial data for which the solution of (1.1) blows up in finite time. If M and N are compact, Ding[6] and Chen-Ding[2] showed a blow up of the solution in finite time, provided u_0 belongs to some nontrivial homotopy class and has sufficiently small energy. Therefore, it is necessary to impose restrictions on the initial data to obtain a global smooth solution.

If $M = R^m$, N is compact, Struwe[14] proved that for $u_0 \in H^1_{loc}(R^m, N)$ with $||\bigtriangledown u_0||_\infty \leq C$, one can find $\epsilon(C) > 0$ such that $E(u_0) < \epsilon$ lends to a global smooth solution. If M and N are compact ($m \geq 3$), Chen-Ding[2] proved the same result. Li-Tam[10] generalized the result to the case where M is a noncompact manifold with Ricci curvature bounded from below and the sectional curvature of N is bounded from above, provided $u_0(M)$ is bounded.

Soyeur[13] proved that if $M = R^m$ and N is compact then there exists a constant $\epsilon > 0$ depending only on m and N such that (1.1) has a global smooth solution for initial data u_0 with $|\bigtriangledown u_0| \in L^p(R^m)$ $(p > m)$ and $||\bigtriangledown u_0||_m < \epsilon$.

Since $||\bigtriangledown u_0||_\infty \leq C$ and $E(u_0) = \frac{1}{2}||\bigtriangledown u_0||_2^2 < \epsilon$ yield $|\bigtriangledown u_0| \in L^p(R^m)$ and

$$||\bigtriangledown u_0||_m \leq C^{1-\frac{2}{m}} 2^{\frac{1}{m}} \epsilon^{\frac{1}{m}} \ (m \geq 2),$$

Soyeur's result implies Struwe's result.

We will generalize Soyeur's result to the case where M is a noncompact manifold. We mainly obtain the following two results.

Theorem 1.1. *Suppose M is a noncompact complete Riemannian manifold with non-negative Ricci curvature, N is a compact manifold, and assume that $V_x(r) = \mathrm{Vol}(B_x(r)) \geq C_{m,m_0} r^{m_0}$ $(1 \leq m_0 \leq m)$ for all $x \in M$, $r \geq 1$. There exists a constant $\epsilon > 0$ depending on m, m_0, p and N such that if $|\bigtriangledown u_0| \in L^p(M)$, $p > m$, $||\bigtriangledown u_0||_m + ||\bigtriangledown u_0||_{m_0} < \epsilon$, then* (1.1) *has a global smooth solution which converges to a constant map as $t \longrightarrow \infty$ with the following decay*

$$||\bigtriangledown u(t)||_p \leq \frac{M_p^\infty}{h(t)},$$

where M_p^∞ depends only on m, m_0, p and N, $h(t) = t^{\frac{1}{2}-\frac{m}{2p}}$ if $0 \leq t \leq 1$, and $h(t) = t^{\frac{1}{2}-\frac{m_0}{2p}}$ if $1 \leq t < \infty$.

Theorem 1.2. *Let M be a Cartan-Hadamard manifold with bounded curvature tensor and its first and second covariant derivatives. Let N be a compact manifold. Let λ be the bottom of the spectrum of $(-\triangle)$. Suppose $\lambda > 0$. Then there exists a positive constant $\epsilon > 0$ depending on M, N, p and m_1 $(m_1 \geq m)$ such that if $|\bigtriangledown u_0| \in L^p(M)$ $(p > m_1)$ and $||\bigtriangledown u_0||_{m_1} < \epsilon$, then* (1.1) *has a global smooth solution which converges to a constant map as $t \longrightarrow \infty$ with the following decay*

$$||\bigtriangledown u(t)||_p \leq \frac{M_p^\infty}{t^{\frac{1}{2}-\frac{m_1}{2p}}},$$

where M_p^∞ depends on M, N, p, and m_1.

§2. Local Existence Theorem

In the following of this paper, we always assume that M is complete noncompact. For simplicity we assume N is compact and suppose N is isometrically embedded into R^Q. Then the Equation (1.1) is equivalent to the following system of differential equations

$$\begin{cases} \frac{\partial u}{\partial t} - \triangle u = A(u)(du, du), \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where $A(u)(du, du) = g^{ij} B(u)(\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j})$, $B(u)$ is the second fundamental form of N.

Since N is compact, we have

$$|A(u)(du, du)| \leq C|\bigtriangledown u|^2, \tag{2.2}$$

where C depends only on N.

Let $S(t)$ denote the semi-group associated to the linear heat equation from M to R^Q:

$$(S(t)u_0)^\alpha(x) = \int_M H(x,y,t)(u_0(y))^\alpha dy,$$

where $1 \leq \alpha \leq Q$, $H(x,y,t)$ is the heat kernel of M. We will show that the operator $S(t)$ maps $[L^q(M)]^Q$ into $[L^p(M)]^Q$ for $1 \leq q \leq p \leq \infty$.

Let $\Gamma_p = \{u : M \longrightarrow R^Q \mid u \in L^\infty(M, R^Q), \ |\bigtriangledown u| \in L^p(M, R^Q)\}$, $p > m$. Given $u_0 \in \Gamma_p$, let $D = 2(||u_0||_\infty + ||\bigtriangledown u_0||_p)$, and

$$\Gamma_p^T = \{u \in C([0,T], \Gamma_p) \mid \sup_{0 \leq t \leq T} (||u(t) - S(t)u_0||_\infty + ||\bigtriangledown u(t) - \bigtriangledown S(t)u_0||_p) \leq D\},$$

which is a complete metric space for the distance

$$d(u,v) = \sup_{0 \leq t \leq T} \{||u(t) - v(t)||_\infty + ||\bigtriangledown u(t) - \bigtriangledown v(t)||_p\}.$$

In this section we will prove

Theorem 2.1. *Suppose M is a complete noncompact manifold with Ricci tensor $R_{ij} \geq -(m-1)b^2 g_{ij}$ $(b \geq 0)$. And assume that $\inf_{x \in M} V_x(1) > 0$, where $V_x(r) = \mathrm{Vol}(B_x(r))$, $B_x(r) = \{y \in M \mid \mathrm{dist}(y,x) < r\}$. Let $u_0 \in \Gamma_p$, $p > m$. Then there exists $T > 0$ and $u \in \Gamma_p^T$ is smooth on $M \times (0,T)$ which is a solution of* (2.1).

To prove Theorem 2.1, we first derive the following lemmas.

Lemma 2.1. *Suppose M is a complete noncompact manifold with Ricci tensor $R_{ij} \geq -(m-1)b^2 g_{ij}$ $(b \geq 0)$. H(x,y,t) is the heat kernel of M. Then*

$$\int_M |\bigtriangledown H(x,y,t)| dy \leq \Big(2mb + \frac{2m}{t}\Big)^{\frac{1}{2}}, \tag{2.3}$$

$$|\bigtriangledown H(x,y,t)| \leq H^{\frac{1}{2}}(y,y,t)\Big(\Big(2mb + \frac{2m}{t}\Big)H(x,x,t) + C_m \frac{1}{t} H\Big(x,x,\frac{t}{2}\Big)\Big)^{\frac{1}{2}}. \tag{2.4}$$

Proof. By Hölder's inequality one has

$$\int_M |\bigtriangledown H| dy \leq \Big(\int_M \frac{|\bigtriangledown H|^2}{H} dy\Big)^{\frac{1}{2}} \Big(\int_M H dy\Big)^{\frac{1}{2}}.$$

Since

$$\int_M H(x,y,t) dy = 1, \tag{2.5}$$

we have

$$\int_M |\bigtriangledown H| dy \leq \Big(\int_M \frac{|\bigtriangledown H|^2}{H} dy\Big)^{\frac{1}{2}}.$$

The gradient estimates[12] imply

$$\frac{|\bigtriangledown H|^2}{H} \le 2\frac{\partial H}{\partial t} + \Big(2mb + \frac{2m}{t}\Big)H. \tag{2.6}$$

By (2.5) one has $\int_M \frac{\partial H}{\partial t} dy = 0$, so $\int_M |\bigtriangledown H| dy \le (2mb + \frac{2m}{t})^{\frac{1}{2}}$.

Clearly,

$$\bigtriangledown H(x,y,t) = \int_M \bigtriangledown H\Big(x,z,\frac{t}{2}\Big) H\Big(z,y,\frac{t}{2}\Big) dz,$$

$$|\bigtriangledown H(x,y,t)| \le \Big(\int_M \Big|\bigtriangledown H\Big(x,z,\frac{t}{2}\Big)\Big|^2 dz\Big)^{\frac{1}{2}} \Big(\int_M H^2\Big(z,y,\frac{t}{2}\Big)dz\Big)^{\frac{1}{2}}.$$

Substituting (2.6) into the last inequality we obtain

$$\begin{aligned}|\bigtriangledown H(x,y,t)| \le & \int_M 2\frac{\partial H}{\partial t}\Big(x,z,\frac{t}{2}\Big) H\Big(x,z,\frac{t}{2}\Big) dz \\ & + \Big(2mb + \frac{2m}{t}\Big)\Big(\int_M H^2\Big(x,z,\frac{t}{2}\Big)dz\Big)^{\frac{1}{2}} \Big(\int_M H^2\Big(z,y,\frac{t}{2}\Big)dz\Big)^{\frac{1}{2}}.\end{aligned}$$

By semi-group property we have

$$\int_M H^2\Big(z,y,\frac{t}{2}\Big)dz = H(y,y,t)$$

and since $(\triangle - \frac{\partial}{\partial t})H(x,z,\frac{t}{2}) = 0$, we have

$$\int_M \frac{\partial H}{\partial t}\Big(x,z,\frac{t}{2}\Big) H\Big(x,z,\frac{t}{2}\Big) dz = \int_M \triangle H\Big(x,z,\frac{t}{2}\Big) H\Big(x,z,\frac{t}{2}\Big) dz.$$

Therefore

$$\begin{aligned}|\bigtriangledown H(x,y,t)| \le & 2H^{\frac{1}{2}}(y,y,t) \int_M \triangle H\Big(x,z,\frac{t}{2}\Big) H\Big(x,z,\frac{t}{2}\Big) dz \\ & + \Big(mb + \frac{m}{t}\Big)\Big(\int_M H^2\Big(x,z,\frac{t}{2}\Big)dz\Big)^{\frac{1}{2}}.\end{aligned}$$

We know that (see [4, Lemma 7])

$$\Big|\int_M \triangle H\Big(x,z,\frac{t}{2}\Big) H\Big(x,z,\frac{t}{2}\Big) dz\Big| \le C_m \frac{1}{t} H\Big(x,x,\frac{t}{2}\Big).$$

So, we have (2.4).

Lemma 2.2. *Suppose M is a complete noncompact manifold with Ricci tensor $R_{ij} \ge -(m-1)b^2 g_{ij}$ $(b \ge 0)$. Assume that $\delta = \inf_{x\in M} V_x(1) > 0$. Then for all $x, y \in M$, $0 < t \le 1$,*

$$H(x,y,t) \le C_{m,b}\delta^{-1} t^{-\frac{m}{2}} e^{-\frac{\rho^2(x,y)}{5t}}, \tag{2.7}$$

$$\int_M |\bigtriangledown H(x,y,t)| dy \le C_{m,b}\frac{1}{\sqrt{t}}, \tag{2.8}$$

$$|\bigtriangledown H(x,y,t)| \le C_{m,b}\frac{\delta}{\sqrt{t}} t^{-\frac{m}{2}}. \tag{2.9}$$

Proof. By the estimate of the heat kernel derived by Li-Yau[12] we have

$$H(x,y,t) \le C_m V_x^{-\frac{1}{2}}(\sqrt{t}) V_y^{-\frac{1}{2}}(\sqrt{t}) e^{(-\frac{\rho^2(x,y))}{5t} + Cbt)},$$

where $\rho(x,y) = \mathrm{dist}(x,y)$. Li-Schoen[9] proved that $t^{-m}e^{-\frac{\sqrt{m-1}}{2}bt}V_x(t)$ is a decreasing function. So, for $0 < t \le 1$

$$V_x(\sqrt{t}) \ge V_x(1) t^{\frac{m}{2}} e^{\frac{\sqrt{m-1}}{2}b(t-1)} \ge C_{m,b}\delta t^{\frac{m}{2}},$$

we therefore have (2.7).

(2.8) follows from (2.3) and (2.9) follows from (2.4) and (2.7).

Proof of Theorem 2.1. If $p=\infty$, $1\le q\le\infty$,

$$||S(t)u||_\infty \le ||H(x,\cdot,t)||_{q'}||u||_q,$$

where $\frac{1}{q}+\frac{1}{q'}=1$. By (2.5) and (2.7) we have

$$||S(t)u||_\infty \le C_{m,b,\delta}t^{-\frac{m}{2q}}||u||_q$$

for all $0<t\le 1$.

If $1=q\le p<\infty$,

$$\begin{aligned}|s(t)u|^p &\le \Big(\int_M H(x,y,t)|u(y)|dy\Big)^p\\ &= \Big(\int_M H(x,y,t)|u(y)|dy\Big)\Big(\int_M H(x,y,t)|u(y)|dy\Big)^{p-1}.\end{aligned}$$

(2.5) and (2.7) implies that

$$|S(t)u|^p \le C_{m,b,\delta}t^{-\frac{m(p-1)}{2}}\int_M H(x,y,t)|u(y)|dy||u||_1^{p-1}.$$

So

$$||S(t)u||_p \le C_{m,b,\delta}t^{-\frac{m}{2}(1-\frac{1}{p})}||u||_1$$

for all $0<t\le 1$.

If $1<q\le p<\infty$,

$$|S(t)u|^p \le \Big(\int_M H(x,y,t)|u(y)|dy\Big)^p.$$

By Hölder's inequality we have

$$\begin{aligned}|S(t)u|^p &\le \Big(\int_M H(x,y,t)|u(y)|^q dy\Big)\Big(\int_M H(x,y,t)|u(y)|^{\frac{q-1}{p-1}}dy\Big)^{p-1}\\ &\le \Big(\int_M H(x,y,t)|u(y)|^q dy\Big)\Big(\int_M H(x,y,t)^{\frac{q(p-1)}{p(q-1)}}dy\Big)^{\frac{p(q-1)}{q}}\Big(\int_M |u(y)|^q dy\Big)^{p-1}.\end{aligned}$$

By (2.5) and (2.7) again we have

$$|S(t)u|^p \le C_{m,b,\delta}t^{-\frac{mp}{2}(\frac{1}{q}-\frac{1}{p})}\Big(\int_M H(x,y,t)|u(y)|^q dy\Big)\Big(\int_M |u(y)|^q dy\Big)^{p-1}.$$

So

$$||S(t)u||_p \le C_{m,b,\delta}t^{-\frac{m}{2}(\frac{1}{q}-\frac{1}{p})}||u||_q \tag{2.10}$$

for all $0<t\le 1$, $1\le q\le p\le\infty$.

Similarly we have

$$||\bigtriangledown S(t)u||_p \le C_{m,b,\delta}t^{-(\frac{1}{2}+\frac{m}{2}(\frac{1}{q}-\frac{1}{p}))}||u||_q \tag{2.11}$$

for all $0<t\le 1$, $1\le q\le p\le\infty$, by Hölder's inequality, (2.8) and (2.9).

We consider the integral equation associated to (2.1). Let

$$Fu(t)=S(t)u_0+\int_0^t S(t-\tau)A(u)(du,du)(\tau)d\tau.$$

By an argument similar to the one used in the proof of Theorem 1 in [13], we can show that F maps Γ_p^T into itself and has a unique fixed point in Γ_p^T for T small. This completes the proof.

§3. Heat Flow from a Nonnegatively Curved Manifold

In this section we suppose M is a noncompact complete manifold with nonnegative Ricci curvature. Let u_0 be a bounded C^1 function, we define $u(x,t) = \int_M H(x,y,t)u_0(y)dy$. Bochner's formula implies that $(\triangle - \frac{\partial}{\partial t})| \bigtriangledown u| \geq 0$, we therefore have

$$| \bigtriangledown u(x,t)| \leq \int_M H(x,y,t)| \bigtriangledown u_0(y)|dy, \tag{3.1}$$

$$|| \bigtriangledown u(x,t)||_p \leq || \int_M H(x,y,t)| \bigtriangledown u_0|dy||_p \tag{3.2}$$

for $1 \leq p \leq \infty$.

If in addition we assume $V_x(r) = \mathrm{Vol}(B_x(r)) \geq C_{m,m_0} r^{m_0}$ $(1 \leq m_0 \leq m)$ for all $x \in M$, $r \geq 1$, by the estimate of the heat kernel[12], Bishop[1] comparison theorem, (2.3) and (2.4) we have

$$H(x,y,t) \leq C_{m,m_0} \frac{1}{g(\sqrt{t})}, \tag{3.3}$$

$$| \bigtriangledown H(x,y,t)| \leq \frac{C_{m,m_0}}{\sqrt{t}} \frac{1}{g(\sqrt{t})}, \tag{3.4}$$

$$\int_M | \bigtriangledown H(x,y,t)|dy \leq \frac{C_m}{\sqrt{t}}, \tag{3.5}$$

where $g(t) = t^{m_0}$ if $t > 1$, $g(t) = t^m$ if $0 \leq t \leq 1$.

Now we prove the following theorem.

Theorem 3.1. *Suppose M is a noncompact complete Riemannian manifold with nonnegative Ricci curvature, N is a compact manifold, and assume that $V_x(r) \geq C_{m,m_0} r^{m_0}$ $(1 \leq m_0 \leq m)$ for all $x \in M$, $r \geq 1$. There exists a constant $\epsilon > 0$ depending on m, m_0, p and N such that if $| \bigtriangledown u_0| \in L^p(M)$, $p > m$, $|| \bigtriangledown u_0||_m + || \bigtriangledown u_0||_{m_0} < \epsilon$, then (1.1) has a global smooth solution which converges to a constant map as $t \longrightarrow \infty$ with the following decay:*

$$|| \bigtriangledown u(t)||_p \leq \frac{M_p^\infty}{h(t)},$$

where M_p^∞ depends only on m, m_0, p and N, $h(t) = t^{\frac{1}{2} - \frac{m}{2p}}$ if $0 \leq t \leq 1$, and $h(t) = t^{\frac{1}{2} - \frac{m_0}{2p}}$ if $1 \leq t < \infty$.

Proof. Using the inequalities (2.5), (3.3), (3.4) and (3.5), by an argument similar to the one used in obtaining (2.10) we can get

$$||S(t)u||_p \leq \frac{C_{m,m_0,p,q}}{g(\sqrt{t})^{(\frac{1}{q} - \frac{1}{p})}} ||u||_q, \tag{3.6}$$

$$|| \bigtriangledown S(t)u||_p \leq \frac{C_{m,m_0,p,q}}{\sqrt{t} g(\sqrt{t})^{(\frac{1}{q} - \frac{1}{p})}} ||u||_q \tag{3.7}$$

for all $0 < t \leq \infty$, $1 \leq q \leq p \leq \infty$.

Suppose $u(t)$ is a solution of (2.1). Then

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)A(u)(du,du)(\tau)d\tau.$$

Using (3.2), (3.6), (3.7) and (2.2) we have

$$|| \bigtriangledown u(t)||_p \leq \frac{C_{m,m_0,p}}{g(\sqrt{t})^{\frac{1}{m_0} - \frac{1}{p}}} || \bigtriangledown u_0||_{m_0} + C_{m,m_0,p} \int_0^t \frac{|| \bigtriangledown u(\tau)||_p^2 d\tau}{(t-\tau)^{\frac{1}{2}} g(\sqrt{t-\tau})^{\frac{1}{p}}}$$

and

$$|| \bigtriangledown u(t)||_p \le \frac{C_{m,m_0,p}}{g(\sqrt{t})^{\frac{1}{m}-\frac{1}{p}}} || \bigtriangledown u_0||_m + C_{m,m_0,p} \int_0^t \frac{|| \bigtriangledown u(\tau)||_p^2 d\tau}{(t-\tau)^{\frac{1}{2}} g(\sqrt{t-\tau})^{\frac{1}{p}}}.$$

We set

$$M_p(T) = \sup_{0 \le t \le T} \{h(t)|| \bigtriangledown u(t)||_p\}.$$

When $0 \le t \le 1$,

$$\begin{aligned} h(t)|| \bigtriangledown u(t)||_p &\le C_{m,m_0,p}|| \bigtriangledown u_0||_m + C_{m,m_0,p} M_p^2(t) \int_0^t \frac{h(t) d\tau}{(t-\tau)^{\frac{1}{2}} g(\sqrt{t-\tau})^{\frac{1}{p}} \tau^{1-\frac{1}{p}}} \\ &\le C_{m,m_0,p}|| \bigtriangledown u_0||_m + C_{m,m_0,p} M_p^2(t). \end{aligned}$$

So

$$C_{m,m_0,p} M_p^2(t) - M_p(t) + C_{m,m_0,p}|| \bigtriangledown u_0||_m \ge 0. \tag{3.9}$$

When $1 \le t \le 2$,

$$\begin{aligned} h(t)|| \bigtriangledown u(t)||_p &\le C_{m,m_0,p}|| \bigtriangledown u_0||_{m_0} + C_{m,m_0,p} M_p^2(t) \int_0^t \frac{h(t) d\tau}{(t-\tau)^{\frac{1}{2}} g(\sqrt{t-\tau})^{\frac{1}{p}} h^2(\tau)} \\ &\le C_{m,m_0,p}|| \bigtriangledown u_0||_{m_0} + C_{m,m_0,p} M_p^2(t). \end{aligned}$$

So

$$C_{m,m_0,p} M_p^2(t) - M_p(t) + C_{m,m_0,p}|| \bigtriangledown u_0||_{m_0} \ge 0. \tag{3.10}$$

When $t \ge 2$,

$$\begin{aligned} h(t)|| \bigtriangledown u(t)||_p \le\; & C_{m,m_0,p}|| \bigtriangledown u_0||_{m_0} \\ & + C_{m,m_0,p} M_p^2(t) \int_0^t \frac{h(t) d\tau}{(t-\tau)^{\frac{1}{2}} g(\sqrt{t-\tau})^{\frac{1}{p}} h^2(\tau)}, \end{aligned}$$

$$\int_0^t \frac{h(t) d\tau}{(t-\tau)^{\frac{1}{2}} g(\sqrt{t-\tau})^{\frac{1}{p}} h^2(\tau)} = \int_0^1 + \int_1^{t-1} + \int_{t-1}^t \frac{h(t) d\tau}{(t-\tau)^{\frac{1}{2}} g(\sqrt{t-\tau})^{\frac{1}{p}} h^2(\tau)},$$

$$\begin{aligned} \int_0^1 \frac{h(t) d\tau}{(t-\tau)^{\frac{1}{2}} g(\sqrt{t-\tau})^{\frac{1}{p}} h^2(\tau)} &= \int_0^1 \frac{t^{\frac{1}{2}-\frac{m_0}{2p}} d\tau}{(t-\tau)^{\frac{1}{2}+\frac{m_0}{2p}} \tau^{1-\frac{m}{p}}} \\ &\le C_{m,p} \frac{t^{\frac{1}{2}-\frac{m_0}{2p}}}{(t-1)^{\frac{1}{2}+\frac{m_0}{2p}}} \le C_{m,m_0,p}, \end{aligned}$$

$$\begin{aligned} \int_{t-1}^t \frac{h(t) d\tau}{(t-\tau)^{\frac{1}{2}} g(\sqrt{t-\tau})^{\frac{1}{p}} h^2(\tau)} &= \int_{t-1}^t \frac{t^{\frac{1}{2}-\frac{m_0}{2p}} d\tau}{(t-\tau)^{\frac{1}{2}+\frac{m}{2p}} \tau^{1-\frac{m_0}{p}}} \\ &\le C_{m,p} \frac{t^{\frac{1}{2}-\frac{m_0}{2p}}}{(t-1)^{1-\frac{m_0}{p}}} \le C_{m,m_0,p}, \end{aligned}$$

$$\begin{aligned} \int_1^{t-1} \frac{h(t) d\tau}{(t-\tau)^{\frac{1}{2}} g(\sqrt{t-\tau})^{\frac{1}{p}} h^2(\tau)} &= \int_1^{t-1} \frac{t^{\frac{1}{2}-\frac{m_0}{2p}} d\tau}{(t-\tau)^{\frac{1}{2}+\frac{m_0}{2p}} \tau^{1-\frac{m_0}{p}}} \\ &\le C_{m,m_0,p} t^{\frac{1}{2}-\frac{m_0}{2p}-\frac{1}{2}+\frac{m_0}{2p}-1+1} = C_{m,m_0,p}. \end{aligned}$$

So

$$C_{m,m_0,p} M_p^2(t) - M_p(t) + C_{m,m_0,p}|| \bigtriangledown u_0||_{m_0} \ge 0. \tag{3.11}$$

By (3.9), (3.10), (3.11) we know that for all $T > 0$,

$$C_{m,m_0,p}M_p^2(T) - M_p(T) + C_{m,m_0,p}(||\bigtriangledown u_0||_{m_0} + ||\bigtriangledown u_0||_m) \geq 0. \tag{3.12}$$

Choose $\epsilon > 0$ so small that

$$4C_{m,m_0,p}C_{m,m_0,p}\epsilon < 1.$$

It is clear that ϵ depends on m, m_0, p and N. Note that $M_p(0) = 0$ and $M_p(T)$ is continuous with respect to T. Assuming $||\bigtriangledown u_0||_m + ||\bigtriangledown u_0||_{m_0} < \epsilon$ and solving the inequality (3.12) one gets

$$\begin{aligned} M_p(T) &\leq \frac{1 + \sqrt{1 - 4C_{m,m_0,p}C_{m,m_0,p}(||\bigtriangledown u_0||_m + ||\bigtriangledown u_0||_{m_0})}}{2C_{m,m_0,p}} \\ &\leq \frac{1 + \sqrt{1 - 4C_{m,m_0,p}C_{m,m_0,p}\epsilon}}{2C_{m,m_0,p}} \\ &:= M_p^\infty. \end{aligned}$$

So we obtain that $M_p(T) \leq M_p^\infty$ for all $T \geq 0$. We therefore have $||\bigtriangledown u(t)||_p \leq \frac{M_p^\infty}{h(t)}$, where M_p^∞ is the constant depending on ϵ defined as above.

(3.8) also implies

$$||u(t)||_\infty \leq ||u_0||_\infty + C_{m,m_0,p}(M_p^\infty)^2 \int_0^t \frac{d\tau}{g(\sqrt{t-\tau})^{\frac{2}{p}} h^2(\tau)}.$$

Similarly one can show that

$$\int_0^t \frac{d\tau}{g(\sqrt{t-\tau})^{\frac{2}{p}} h^2(\tau)} \leq C_{m,m_0,p},$$

which implies

$$||u(t)||_\infty \leq ||u_0||_\infty + C_{m,m_0,p}(M_p^\infty)^2.$$

Therefore $u(t)$ can not blow up in finite time and we get a global smooth solution of (2.1).

§4. Heat Flow from a Cartan-Hadamard Manifold

In this section we assume M is a Cartan-Hadamard manifold, that is, M is a simply connected complete manifold with nonpositive sectional curvature. We[11] know that the inequality (3.1) does not hold on constant negative curvature space form. However we may have

Lemma 4.1. *Let M be a Cartan-Hadamard manifold with bounded curvature tensor and its first and second covariant derivatives. Let N be a compact manifold. Let λ be the bottom of the spectrum of $(-\triangle)$. Suppose $\lambda > 0$. If u_0 is a bounded C^1 function and $|\bigtriangledown u_0| \in L^p(M)$, $u(x,t) = \int_M H(x,y,t)u_0(y)dy$, then*

$$||\bigtriangledown u(\cdot,t)||_p \leq \Big\| \int_M H(\cdot,y,t)(-\triangle)^{\frac{1}{2}}u_0(y)dy \Big\|_p, \tag{4.1}$$

$$||(-\triangle)^{\frac{1}{2}}u_0(y)||_p \leq C_p||\bigtriangledown u_0(y)||_p \tag{4.2}$$

for $1 < p < \infty$, where C_p depends on p and M.

Proof. Lohoué[8] showed that if M satisfies the hypotheses of this lemma then

$$||\bigtriangledown f||_p \leq ||(-\triangle)^{\frac{1}{2}}f||_p \leq C_p||\bigtriangledown f||_p$$

for $|\bigtriangledown f| \in L^p(M)$. So

$$||\bigtriangledown u(x,t)||_p \leq ||(-\triangle)^{\frac{1}{2}} u(x,t)||_p = \Big\| \int_M H(x,y,t)(-\triangle)^{\frac{1}{2}} u_0(y)dy \Big\|_p.$$

We therefore have (4.1) and (4.2).

Remark 4.1. If $p=2$, one has (4.1) and (4.2) on any complete manifolds[15].

We also need the following estimate of the heat kernel, which was proved by Davies[5].

Lemma 4.2. *Suppose M is a complete manifold with Ricci tensor $R_{ij} \geq -(m-1)b^2 g_{ij}$ ($b \geq 0$). Let $\lambda \geq 0$ be the bottom of the spectrum of $-\triangle$. Then*

$$H(x,y,t) \leq C_\delta (V_x(\sqrt{t}))^{-\frac{1}{2}} (V_y(\sqrt{t}))^{-\frac{1}{2}} e^{(\delta-\lambda)t} e^{-\frac{\rho^2(x,y)}{(4+\delta)t}} \tag{4.3}$$

for $x, y \in M$, $0 < t < \infty$, $\delta > 0$, where $\rho(x,y) = \mathrm{dist}(x,y)$.

Now we prove the main result of this section.

Theorem 4.1. *Let M be a Cartan-Hadamard manifold with bounded curvature tensor and its first and second covariant derivatives. Let N be a compact manifold. Let λ be the bottom of the spectrum of $(-\triangle)$. Suppose $\lambda > 0$. Then there exists a positive constant $\epsilon > 0$ depending on M, N, p and m_1 ($m_1 \geq m$) such that if $|\bigtriangledown u_0| \in L^p(M)$ ($p > m_1$) and $||\bigtriangledown u_0||_{m_1} < \epsilon$ then (1.1) has a global smooth solution which converges to a constant map as $t \longrightarrow \infty$ with the following decay:*

$$||\bigtriangledown u(t)||_p \leq \frac{M_p^\infty}{t^{\frac{1}{2}-\frac{m_1}{2p}}},$$

where M_p^∞ depends on M, N, p, and m_1.

Proof. Suppose the Ricci curvature of M satisfies Ric $M \geq -(m-1)b^2$. We first show that

$$||S(t)u||_p \leq \frac{C_{m,m_1,b,\lambda,p,q}}{(4\pi t)^{\frac{m_1}{2}(\frac{1}{q}-\frac{1}{p})}} ||u||_q \tag{4.4}$$

for $1 \leq q \leq p \leq \infty$, $m_1 \geq m$.

$$||\bigtriangledown S(t)u||_p \leq \frac{C_{m,m_1,b,\lambda,p,q}}{(4\pi t)^{\frac{1}{2}+\frac{m_1}{2}(\frac{1}{q}-\frac{1}{p})}} ||u||_q \tag{4.5}$$

for $1 \leq q < p \leq \infty$, $m_1 \geq m$.

We only prove (4.5), the proof of (4.4) is similar. From

$$|\bigtriangledown S(t)u|^p \leq \Big(\int_M |\bigtriangledown H(x,y,t)||u(y)|dy \Big)^p,$$

by Hölder's inequality we have

$$|\bigtriangledown S(t)u|^p \leq \Big(\int_M |\bigtriangledown H(x,y,t)|\ |u(y)|^q dy \Big) \Big(\int_M |\bigtriangledown H(x,y,t)|\ |u(y)|^{\frac{q-1}{p-1}} dy \Big)^{p-1}.$$

So

$$||\bigtriangledown S(t)u||_p \leq (I_1 I_2)^{\frac{1}{p}} ||u||_q,$$

where

$$I_1 = \sup_{x\in M} \int_M |\bigtriangledown H(x,y,t)| dy,$$

$$I_2 = \sup_{x\in M} \Big(\int_M |\bigtriangledown H(x,y,t)|^{\frac{q(p-1)}{p(q-1)}} dy \Big)^{\frac{p(q-1)}{q}}.$$

By Lemma 2.1 we have $I_1 \leq (2mb + \frac{2m}{t})^{\frac{1}{2}}$. By Lemma 2.1 and Lemma 4.2 we have

$$| \bigtriangledown H(x,y,t)| \leq C_{m,b,\lambda} t^{-\frac{m}{2}-\frac{1}{2}} e^{-\frac{1}{4}\lambda t}.$$

So

$$I_2 \leq C_{m,b,\lambda,p,q} t^{-(\frac{1}{2}+\frac{m}{2})\frac{p-q}{q}} \Big(1+\frac{1}{\sqrt{t}}\Big)^{\frac{p(q-1)}{q}} e^{-\frac{\lambda}{4}\frac{p-q}{q}t}$$

and

$$I_1 \cdot I_2 \leq C_{m,m_1,b,\lambda,p,q} \Big(\frac{1}{4\pi t}\Big)^{\frac{p}{2}+\frac{m_1}{2}\frac{p-q}{q}}.$$

Clearly, (4.5) follows.

Using (4.1), (4.2), (4.4) and (4.5), by an argument similar to that in the proof of Theorem 2 in [13], we can prove this theorem.

Remark 4.2. Theorem 2.1, Theorem 3.1 and Theorem 4.1 also hold when N is noncompact and $u_0(M)$ is bounded in N. In this case, ϵ and M_p^∞ depend on $u_0(M)$ too.

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