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GLOBAL SOLUTIONS FOR SOME OLDROYD MODELS OF NON-NEWTONIAN FLOWS

P. L. LIONS* N. MASMOUDI*

Abstract

The authors consider here some Oldroyd models of non-Newtonian flows consisting of a strong coupling between incompressible Navier-Stokes equations and some transport equations. It is proved that there exist global weak solutions for general initial conditions. The existence proof relies upon showing the propagation in time of the compactness of solutions.

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§1. Introduction

We consider here some Oldroyd models of non-Newtonian flows and prove the existence of global weak solution of the corresponding systems of equations.

More precisely, we study the following system of equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = \mathrm{div}\tau, \quad \mathrm{div}u = 0, \tag{1.1}$$

$$\frac{\partial \tau}{\partial t} + u \cdot \nabla\tau + \tau \cdot \omega - \omega \cdot \tau + a\tau = bD(u), \tag{1.2}$$

where $\nu > 0$, $a, b \geq 0$; u stands for the velocity of the fluid assumed to be incompressible and p for the pressure and τ is a symmetric tensor; we denote by $\omega = \frac{1}{2}(\nabla u - \nabla u^T)$ the vorticity tensor and by $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ the deformation tensor. The parameters ν, a, b correspond respectively to $\frac{\theta}{\mathrm{Re}}$, $\frac{1}{\mathrm{We}}$ and $\frac{2(1-\theta)}{\mathrm{We\ Re}}$, where Re is the Reynolds number, θ is the ratio between the so-called relaxation and retardation times and We is the Weissenberg number which measures the elasticity of the fluid. The above system of equations is set in three dimensions with boundary conditions described below so that u is an N-dimensional vector field and τ is an $N \times N$ symmetric matrix with $N = 3$ and we shall consider as well the case when $N = 2$ in which case the matrix reduces to $\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$, where ω is the usual scalar vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$.

The above system of equations is one of the basic macroscopic models for visco-elastic flows such as polymer flows. We have, however, made a simplifying assumption (taking one

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*CEREMADE - UMR C.N.R.S. 7534, Université Paris 9 - Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France.
E-mail: mazzella@pi.ceremade.dauphine.fr; masoudi@courant.nyu.edu

parameter of the model to be 0, namely the one in front of $[D(u)\cdot\tau+\tau\cdot D(u)]$ in the definition of the so-called objective derivative in Oldroyd models. We shall come back to that issue in a future work where we study other related macroscopic models and also the physically more realistic micro-macro models for polymers that couple the macroscopic scale together with a mesoscopic (or microscopic) scale. For more details on the macroscopic models, we refer the reader to [11, 2] (for instance) while information on the so-called micro-macro models can be found in [3, 10, 19, 20, $\cdots$].

If we complete (1.1)–(1.2) with initial and boundary conditions, it is then natural to study the corresponding evolution (Cauchy) problem for which nothing was known as far as global and general solutions are concerned. The only known results concern either the short-time well-posedness of the problem in appropriate regularity classes or the existence and uniqueness of global small regular solutions and we refer the reader to [4, 7, 8, 9, 12, 18] for such results.

Let us now make precise the initial and boundary conditions. First of all, we consider three examples of boundary conditions (and it should be clear from the arguments below that many other situations can be handled as well, such as exterior problems, non-homogeneous boundary conditions $\cdots$), namely (i) the periodic case where all functions are assumed to be periodic in x_i with a fixed period $T_i>0$ $(1\le i\le n)$—in that case, all the functions spaces we use will be composed of periodic functions and we shall not recall it—or (ii) the whole space case in which (1.1)–(1.2) holds in $\mathbb{R}^N$ and the unknowns u,τ,p vanish at infinity in an appropriate sense described later on, or (iii) the case of (homogeneous) Dirichlet conditions in which (1.1)–(1.2) are set in $\Omega\times(0,\infty)$ where Ω is a bounded, smooth, open domain in $\mathbb{R}^N$ and we impose that u vanishes on $\partial\Omega\times(0,\infty)$. Next, we impose initial conditions

$$u\Big|_{t=0} = u_0, \quad \tau\Big|_{t=0} = \tau_0 \tag{1.3}$$

and we assume that $u_0\in L^2$, $\tau_0\in L^2$ and that $\mathrm{div}u_0=0$ in $\mathcal{D}'$.

In addition, in the case of Dirichlet boundary conditions, u_0 has to satisfy the following compatibility condition: $u_0\cdot n=0$ on $\partial\Omega$, where n denotes the unit normal to $\partial\Omega$.

Our main result, stated in the following section, shows that there exist global weak solutions, satisfying energy identities if $N=2$ or inequalities if $N=3$, of (1.1)–(1.2)–(1.3). It is worth pointing out that if $b=0$ and $\tau_0\equiv 0$, then $\tau\equiv 0$ solves (1.2) and (1.1) reduces to the classical incompressible Navier-Stokes equations: in that case, our result allows to recover the well-known Leray solutions!

The proof of this result is split into two sections where we derive some a priori bounds on u,p and τ (Section 3) and where we state and prove a result concerning the behavior of sequences of solutions. This result is the heart of the matter for the existence proof and it shows that the "possible loss of compactness of τ does not grow too fast along particle paths" and that, in particular, the compactness of τ (in L^2) propagates in t. We also show, by explicit examples, that, in general, there is no compactification mechanism. This type of result presents a striking similitude with those obtained in [15, 14], by one of the authors on compressible Navier-Stokes equations which also consist, like (1.1)–(1.2), of a coupled system of a "parabolic" equation (1.1) with a transport equation (1.2). Although the general approach consisting in establishing the propagation of compactness (through the derivation of a transport equation) is indeed the same, the particular structure of our proof is quite different and involves various elements such as renormalized solutions of transport equations and generalized flows for ordinary differential equations (see [6]), but also renormalization techniques for three-dimensional incompressible Navier-Stokes equations which are, we believe, of independent interest.

§2. Main Results

We begin with a definition of solutions (1.1)–(1.2): we shall consider (u, τ, p) satisfying (1.1)–(1.2) (in the sense of distributions) and (1.3) such that, for all $T \in (0, \infty)$, $u \in L^2(0,T;H^1)$ (H^1_0 in the case of Dirichlet conditions) $\cap L^\infty(0,\infty;L^2)$; $u \in C([0,\infty);L^2)$ if $N = 2$, $u \in C([0,\infty);L^2_w)$ if $N = 3$ where L^2_w means that L^2 is endowed with its weak topology ; $\tau \in C([0,\infty);L^2)$; if $N = 2$, $p \in (L^1(0,T;W^{2,1}) \cap L^2(0,T;W^{1,1}) \cap L^q(0,T;L^r)) + C([0,\infty);L^2)$, where $1 \le q < \infty$ and $r = \frac{q}{q-1}$, except in the case of Dirichlet boundary conditions where we replace $C\big([0,\infty);L^2\big)$ by $C\big([0,\infty);L^2_{\rm loc}\big) + L^2\big(0,T;C^\infty_{\rm loc}\big)$ (for example) and $W^{2,1}$, $W^{1,1}$ by $W^{2,1}_{\rm loc}$, $W^{1,1}_{\rm loc}$ respectively ; and if $N = 3$, $p \in \Big(L^1\big(0,T;W^{2,1}\big) \cap L^2\big(0,T;W^{1,1}\big) \cap L^q\big(0,T;L^r\big)\Big) + C\Big([0,\infty);L^2\Big)$ where $1 \le q < \infty$ and $r = \frac{3q}{3q-2}$, except in the case of Dirichlet boundary conditions where we replace $C\big([0,\infty);L^2\big)$ by $C\big((0,\infty);L^2_{\rm loc}\big) + L^2\big(0,T;C^\infty_{\rm loc}\big)$ (for example) and $W^{2,1}$, $W^{1,1}$ by $W^{2,1}_{\rm loc}$, $W^{1,1}_{\rm loc}$ respectively.

Also, in the whole space case, $W^{2,1}$ is to be replaced by the set $\{p \in L^r,\ Dp \in L^1,\ D^2p \in L^1\}$ and, in the periodic case, we normalize p in such a way that $\int p dx = 0$ a.e. for $t > 0$. Let us mention that the slightly complicated spaces to which p belongs and the fact that we have no simple control on p up to the boundary in the case of Dirichlet boundary condition are phenomena that already appear for the classical incompressible Navier-Stokes equations and that are discussed in detail in [13].

We now discuss energy identities which will also be part, in some form, of our definition of solutions. Formally, we multiply (1.1) by u and we obtain

$$\frac{\partial}{\partial t}\Big(\frac{|u|^2}{2}\Big) + \mathrm{div}\Big\{u\Big\{\frac{|u|^2}{2} + p\Big\} - u \cdot \tau\Big\} - \nu\Delta\frac{|u|^2}{2} + \nu|\nabla u|^2 = -\mathrm{Tr}\Big(D(u)\cdot\tau\Big)\ . \tag{2.1}$$

Next, we multiply (1.2) by τ and we obtain taking the trace of the equation

$$\frac{\partial}{\partial t}\Big(\frac{||\tau||^2}{2}\Big) + \mathrm{div}\Big\{u\frac{||\tau||^2}{2}\Big\} + a||\tau||^2 = b\mathrm{Tr}\Big(D(u)\cdot\tau\Big)\ , \tag{2.2}$$

where we denote by $||\tau|| = \mathrm{Tr}(\tau^2)^{1/2} = \Big(\sum\limits_{ij}\tau^2_{ij}\Big)^{1/2}$. Combining (2.1) and (2.2), we obtain

$$\frac{\partial}{\partial t}\Big(\frac{|u|^2}{2}+\frac{1}{2b}||\tau||^2\Big)+\mathrm{div}\Big\{u\Big\{\frac{|u|^2}{2}+\frac{||\tau||^2}{2b}+p\Big\}-u\cdot\tau\Big\}-\nu\Delta\frac{|u|^2}{2}+\nu|\nabla u|^2+\frac{a}{b}||\tau||^2 = 0. \tag{2.3}$$

Of course, we deduce from (2.1), (2.2) and (2.3) at least formally

$$\frac{d}{dt}\int\frac{|u|^2}{2} + \nu\int|\nabla u|^2 \ = \ -\int\mathrm{Tr}(D(u)\cdot\tau), \tag{2.4}$$

$$\frac{d}{dt}\int\frac{||\tau||^2}{2} + a\int||\tau||^2 \ = \ b\int\mathrm{Tr}(D(u)\cdot\tau), \tag{2.5}$$

$$\frac{d}{dt}\int\frac{|u|^2}{2} + \frac{||\tau||^2}{2b} + \frac{a}{b}\int||\tau||^2 + \nu\int|\nabla u|^2 \ = \ 0. \tag{2.6}$$

When $N = 2$, we incorporate in our notion of solution the equalities (2.1) and (2.4)–(2.6), and (2.2)–(2.3) if $\tau_0 \in L^p$ for some $p > 2$; while, if $N = 3$, we request that (2.1) and (2.4)–(2.6) hold replacing equalities by inequalities, and if $\tau_0 \in L^{9/4}$, (2.2)–(2.3) hold replacing again equalities by inequalities.

And all those equalities or inequalities are assumed to hold in distributions sense.

Finally, we introduce a class of functions, denoted by $I_{p,q}$ for $1 < p < \infty$, defined by the set of vector fields u in $W^{-1,q}$ such that $\mathrm{div}u = 0$ and $\nabla w \in L^p(0,T;L^q)$ (for all $T \in (0,\infty)$)

where w solves the Stokes equation (with the corresponding boundary conditions)

$$\frac{\partial w}{\partial t} - \nu \Delta w + \nabla p = 0, \quad \mathrm{div} w = 0$$

and $w\big|_{t=0} = u$. This space is quite technical to describe explicitly and, in terms of regularity, is very close to $W^{1-2/p,q}$ (in the case of Dirichlet boundary conditions, the condition on the normal trace $u \cdot n = 0$ on $\partial\Omega$ has to be imposed as soon as $\frac{2}{p'} = 2(1-\frac{1}{p}) > \frac{1}{q}$) and, in the case $p = q \geq 2$, $W^{1-2/p,q}$ (with the vanishing normal trace condition in the case of Dirichlet boundary conditions and with the divergence free condition in all cases) is contained in $I_{p,q}$.

We may now state our main results:

Theorem 2.1. (i) *There exists a global solution of* (1.1)–(1.2)–(1.3).

(ii) *If* $N = 2$, $\tau_0 \in L^q$, $u_0 \in I_{p,q}$, *for some* $1 < p < \infty$, $2 < q < \infty$, *then* $\nabla u \in L^p(0, T; L^q)$ *for all* $T \in (0, \infty)$ *and* $\tau \in C([0, \infty); L^q)$.

(iii) *If* $N = 3$, $\tau_0 \in L^q$, $u_0 \in I_{p,q}$ *for some* $2 < q \leq 3$, $1 < p \leq q/(2q-3)$, *then* $\nabla u \in L^p(0, T; L^q)$ *for all* $T \in (0, \infty)$ *and* $\tau \in C([0, \infty); L^q)$.

Remarks. (i) Since the above result applies, as a very special case, to the classical incompressible Navier-Stokes equations, it is worth comparing it with the known results in that case. First of all, when $N = 3$, the above result is strictly analogous to Leray celebrated results[13,14] on global weak solutions. And, since the regularity of solutions (say, for smooth u_0) and their uniqueness are still fundamental open problems, we cannot expect these issues to be any easier in a more general framework! Also, like in the classical newtonian case, it is not known whether (2.3) holds and the proof below shows that (2.3) holds with a right-hand side given by a non-positive bounded measure which is, roughly speaking, "concentrated on large values of u".

In the case when $N = 2$, the above result also yields classical results in the Newtonian case ($b = 0$, $\tau \equiv 0$) and in view of what is known in that case, we might expect uniqueness results and further regularity results. These are, however, not known in our general setting. The main reason is that we are not able to obtain an L^∞ bound on τ: if it were known, then uniqueness and regularity would follow in a straightforward manner (see [4] for more general results of that sort).

(ii) Of course, we may add to the equation (1.1) force terms, i.e. a right-hand side f. The only assumption we need for the existence part of the theorem is: $f \in L^2(0, T; H^{-1})$ ($T > 0$ is fixed). And the L^q regularity results hold as soon as $f \in L^p(0, T; W^{-1,q})$.

(iii) As we mentioned in the Introduction, our arguments may be modified to other non-Newtonian models and we shall come back to that issue in a future work. Roughly speaking, the main feature we are using for models involving transport equations for τ is the possibility of deriving energy bounds. Let us, however, mention immediately a straightforward adaptation of our results and proofs to the case when $\tau = \tau_1 + \cdots + \tau_m$ ($m \geq 1$) and each symmetric tensor solves a transport equation like (1.2) with different constants b.

(iv) Contrarily to what is known on classical (Newtonian) incompressible Navier-Stokes equations, we do not know how to extend the above results to dimensions $N \geq 4$. The main reason is that we need to be able to obtain bounds on ∇u in $L^1(0, T; L^s)$ for some $s > 2$ or bounds on $(u.\nabla)u$ in $L^1(0, T; L^r)$ for some $r > \frac{2N}{N+2}$. And since the "best " bounds on u and ∇u we can derive come from the energy, we find only a bound on $(u.\nabla)u$ in $L^1(0, T; L^{r_0})$ where $\frac{1}{r_0} = \frac{N-2}{2N} + \frac{1}{2}$ (by Sobolev embeddings) and $\frac{1}{r_0} < \frac{N+2}{2N}$ if and only if $N < 4$.

The rest of this paper is devoted to the proof of the above result. Let us immediately mension that we shall only present in detail the proof in the periodic case and mention the necessary modifications for the case of Dirichlet boundary conditions. The case of the whole space is a straightforward adaptation of the periodic case and we skip it.

As usual for evolution nonlinear partial differential equations, the existence of global (weak) solutions follows upon approximating conveniently the equations and proving that the approximated solutions yield in the limit the desired solutions. The main step is then the passage to the limit since we only have L^2 estimates (see the definition of solutions) on ∇u and τ and thus the approximating sequences are only known to converge weakly in L^2. But then, we cannot a priori pass to the limit in Equation (1.2) because of the quadratic terms $\tau\cdot\omega-\omega\cdot\tau$. In order to solve that difficulty, we shall investigate and prove in Section 4 the strong compactness in L^2 of τ (when the corresponding initial condition τ_0 belongs to a compact set of $L^2\cdots$) which is thus the heart of the matter. The a priori bounds we need in order to prove that compactness and the above result are derived in the next section.

Although we shall consider only sequences of solutions of (1.1)–(1.2), it is easy to check (and we leave it to the reader) that the compactness result of Section 4 and its proof immediately adapt to appropriate approximated systems of equations like, for instance,

$$\frac{\partial u}{\partial t}+(\tilde u\cdot\nabla)u-\nu\Delta u+\nabla p=\mathrm{div}\tau,\quad \mathrm{div}u=0, \tag{2.7}$$

$$\frac{\partial \tau}{\partial t}+\mathrm{div}(\tilde u\cdot\tau)+\tau\cdot\tilde\omega-\tilde\omega\tau+a\tau=b\ D(u), \tag{2.8}$$

where $\tilde\omega=\frac12(\nabla\tilde u-\nabla\tilde u^T)$ and $\tilde u$ solves (for example),

$$-\epsilon\Delta\tilde u+\tilde u+\nabla\pi=u,\quad \mathrm{div}\tilde u=0 \tag{2.9}$$

(with the same boundary condition as for u). Let us also observe that in the periodic case or in the case of the whole space, we might choose for $\tilde u$ a simple regularization of u by convolution$\cdots$. Finally, we may also regularize the initial conditions (u_0,τ_0) if needed, in which case we obtain smooth approximated solutions for which all the computations that follow in the next sections are justified.

§3. A Priori Bounds

3.1. Energy Bounds

We begin with the straightforward bound which can be derived from the formal energy identities. All the a priori bounds we obtain, unless explicit mention, are uniform in (u_0,τ_0) provided (u_0,τ_0) are uniformly bounded accordingly. Obviously, (2.6) immediately yields a bound on u in $L^2(0,T;H^1)\cap L^\infty(0,\infty;L^2)$ and on τ in $L^\infty(0,\infty;L^2)$ (for all $t\in(0,\infty)$). From now on, in order to simplify notation, T will denote an arbitrary and fixed positive number and we shall simply write $L^p(X)$ in place of $L^p(0,T;X)$ for any function space X. If $N=2$, we deduce from these bounds the following (classical) bounds

$$\|u\|_{L^\alpha(L^\beta)}<\infty\quad,\quad \|(u\cdot\nabla)u\|_{L^\gamma(L^\delta)}<\infty \tag{3.1}$$

for $2<\alpha\le\infty$, $\beta=\frac{2\alpha}{\alpha-2}$, $1<\gamma\le2$, $\delta=\frac{2\gamma}{3\gamma-2}$, while, if $N=3$, we obtain (3.1) for $2\le\alpha\le\infty$, $\beta=\frac{6\alpha}{3\alpha-4}$, $1\le\gamma\le2$, $\delta=\frac{3\gamma}{4\gamma-2}$ and we observe that, if $1\le\gamma<2$, we may replace L^δ by the Lorentz space $L^{\delta,1}$: in particular, $(u\cdot\nabla)u$ is bounded in $L^1\big(L^{\frac32,1}\big)$.

We now deduce some bounds on the pressure p. Recalling that we deal with the periodic case, we observe that p solves the folloging equation

$$-\Delta p=\mathrm{div}\big((u\cdot\nabla)u\big)-\partial_{ij}\tau_{ij}=\partial_{ij}(u_iu_j-\tau_{ij})=\partial_iu_j\partial_ju_i-\partial_{ij}(\tau_{ij}),\qquad \int p dx=0,$$

where we use the implicit summation convention. We may thus write $p=p_1+p_2$ where

p_1, p_2 solve the following equations

$$-\Delta p_1 = \partial_{ij}(u_i u_j) = \operatorname{div}\Big((u.\nabla)u\Big) = \partial_i u_j\ \partial_j u_i, \quad \int p_1 dx = 0, \tag{3.2}$$

$$-\Delta p_2 = -\partial_{ij}(\tau_{ij}), \quad \int p_2 dx = 0. \tag{3.3}$$

And we obtain a bound on p_2 in $L^\infty(L^2)$ and on p_1 in $L^1(W^{2,1}) \cap L^2(W^{1,1}) \cap L^q(L^r)$ with $1 \le q < \infty$, $r = \frac{q}{q-1}$ if $N = 2$ and $r = \frac{3q}{3q-2}$ if $N = 3$. Let us notice that the bounds in $L^1(W^{2,1})$ and in $L^2(W^{1,1})$ use the fact that $\partial_i u_j \partial_j u_i$ is bound in $L^1(\mathcal{H}')$, where $\mathcal{H}^1$ denotes the usual Hardy space (see [13] and [5] for more details). In fact, we also obtain a bound on ∇p_1 in L^2 $(\mathcal{H}^1)$ since $(u \cdot \nabla)u$ is bounded in $L^2(\mathcal{H}^1)$. In the case of Dirichlet boundary conditions, the bounds on the pressure are more delicate: we first split u into $u_1 + u_2$ where u_1, u_2 solve respectively

$$\frac{\partial u_1}{\partial t} - \nu\Delta u_1 + \nabla p_1 = -(u \cdot \nabla)u, \quad \operatorname{div} u_1 = 0\ , \tag{3.4}$$

$$\frac{\partial u_2}{\partial t} - \nu\Delta u_2 + \nabla p_2 = \operatorname{div}\tau, \quad \operatorname{div} u_2 = 0 \tag{3.5}$$

with the following initial conditions $u_1|_{t=0} \equiv 0$, $u_2|_{t=0} \equiv u_0$ and such that u_1 and u_2 vanish on the boundary.

Then, exactly as in [5], one obtains bounds on p_1 in $L^1(W^{2,1}_{\text{loc}}) \cap L^2(W^{1,1}_{\text{loc}}) \cap L^q(L^r)$ (for the range of (q, r) described in Section 2). Bounds on p_2 may be obtained in various ways. For instance, one may write $u_2 = u_3 + v + w$, $p_2 = p_3 + p_4 + \pi$, where u_3, v, w solve respectively

$$\frac{\partial u_3}{\partial t} - \nu\Delta u_3 + \nabla p_3 = 0,\ \operatorname{div} u_3 = 0\ ,\ u_3\big|_{t=0} = u_0, \tag{3.6}$$

$$\frac{\partial v}{\partial t} - \nu\Delta v + \nabla p_4 = -\frac{\partial w}{\partial t},\ \operatorname{div} v = 0,\ v\big|_{t=0} = 0\ , \tag{3.7}$$

$$-\nu\Delta w + \nabla\pi = \operatorname{div}\tau, \quad \operatorname{div} w = 0. \tag{3.8}$$

And we have obviously a bound on w in $C(H^1_0)$, on π in $C(L^2)$ and, using (1.2), on $\frac{\partial w}{\partial t}$ in $L^2(L^r)$ for all $1 < r < \frac{N}{N-1}$. Then, classical regularity results for Stokes' equations yield a bound on p_4 in $L^2(W^{1,r})$ and in fact in $L^2(C^\infty_{\text{loc}})$ since p_4 is obviously harmonic. Finally, standard considerations on Stokes' equations show that p_3 is bounded in $C([\epsilon, T]; C^\infty)$ for all $0 < \epsilon < T$. Let us also notice, for future purposes, that we have the following bounds on ∇u_1, ∇u_2

$$\|\nabla u_2\|_{L^2(L^2)} < \infty, \quad \|\nabla u_1\|_{L^p(L^q)} < \infty \quad \text{if} \quad N = 3, \tag{3.9}$$

where $1 \le p \le 2$, $q = \frac{3p}{2p-1}$.

3.2. L^q Bounds on τ

We now prove the bounds which are stated in parts (ii) and (iii) of Theorem 2.1. We begin with the case when $N = 3$ (and $1 < p \le \frac{q}{2q-3}$, $2 \le q \le 3$) or when $N = 2$ and $1 < p \le \frac{q}{q-1}$, $2 \le q \le \infty$. In these cases, we use classical regularity results for Stokes' equations and we deduce easily bounds on ∇u_1 in $L^p(L^q)$ from the bounds obtained in Section 3.1 above, where u_1 solves (3.4) with $u_1|_{t=0} \equiv u_0 (\in I_{p,q})$. And, using once more regularity results for Stokes' equations, we obtain $\|\nabla u_2\|_{L^p(L^q)} \le C\ \|\tau\|_{L^p(L^q)}$, where C denotes various constants independent of (u, τ) and u_2 solves (3.5) with $u_2|_{t=0} \equiv 0$ so that $u = u_1 + u_2$. Therefore, we have on each interval $[0, t]$, where $t \in [0, T]$,

$$\|\nabla ug\|_{L^p(L^q)}\ \le\ C\ \big(1 + \|\tau\|_{L^p(L^q)}\big). \tag{3.10}$$

Next, we deduce from (2.2),

$$\frac{\partial}{\partial t}\Big(\frac{\|\tau\|^q}{q}\Big) + \mathrm{div}\Big\{u\frac{\|\tau\|^q}{q}\Big\} + a||\tau||^q = b\mathrm{Tr}\Big\{D(u)\cdot\tau\Big\}\ ||\tau||^{q-2};$$

hence, integrating in x, we obtain

$$\frac{d}{dt}\int ||\tau||^q \ \leq\ C\Big(\int |\nabla u|^q\Big)^{1/q}\ \Big(\int ||\tau||^q\Big)^{(q-1)/q}. \tag{3.11}$$

Therefore, we have for all $t \in [0,T]$,

$$\begin{aligned} ||\tau||(t) &\leq C\ \Big(1 + \int_0^t ||\nabla u||_{L^q}(s)ds\Big) \leq C\ \big(1 + t^{\frac{p-1}{p}}\ ||\nabla u||_{L^p(0,t;L^q)}\big) \\ &\leq C\ (1 + ||\tau||_{L^p(0,t;L^q)}) \quad \text{in view of (3.10)} \end{aligned}$$

and we deduce easily a bound on τ in $L^\infty(L^q)$ which thus imply, using again (3.10), the bound on ∇u in $L^p(L^q)$.

When $N = 2$, $q < \infty$ and $p > \frac{q}{q-1}$, the argument is slightly different. From the previous argument, we already know that τ is bounded in $L^\infty(L^q)$. Using, once more, regularity results for Stokes' equations, we have

$$||\nabla u||_{L^p(L^q)} \ \leq\ C(1 + ||(u\cdot\nabla)u||_{L^p(L^{q_*})+L^{p_*}(L^q)}),$$

where $\frac{1}{q_*} = \frac{1}{q} + \frac{1}{2}$, $\frac{q}{q-1} < p_* < p$. Then, for each $\epsilon \in (0,1)$, we may write $u = u^1_\epsilon + u^2_\epsilon$ where $||u^1_\epsilon||_{L^\infty(0,T;L^2)} \leq \epsilon$ and $u^2_\epsilon \in L^\infty(0,T.L^\infty)$. Hence, we have

$$||\nabla u||_{L^p(L^q)} \leq C(1 + \epsilon||\nabla u||_{L^p(L^q)} + C_\epsilon||\nabla u||_{L^{p_*}(L^q)})$$

and we conclude easily choosing ϵ small enough since we have already obtained above a bound on ∇u in $L^{\frac{q}{q-1}}(L^q)$.

However, the above decomposition is valid provided we show that u is equicontinuous in L^2 uniformly in $t \in [0,T]$. This fact is indeed true since, in the course of proving Theorem 2.1, our compactness analysis will show that the sequence of approximated solutions is relatively compact in $C\Big([0,T];L^2\Big)$ (recall that $N = 2$) and that analysis only uses the bounds in $L^p(L^q)$ for $1 < p \leq q/(q-1)$.

3.3. Equicontinuity in L^2 of τ

We now wish to prove that if τ_0 belongs to an equicontinuous set in L^2, then τ is equicontinuous in L^2, uniformly in $t \in [0,T]$, or in other words, the following estimate is true, uniformly in τ_0,

$$\sup_{t\in[0,T]}\int \|\tau\|^2\ 1_{(\|\tau\|\geq M)} \to 0 \quad \text{as} \quad M \to +\infty. \tag{3.12}$$

In order to prove this estimate, we introduce the solutions u_1, u_2 of (3.4),(3.5) respectively satisfying $u_1|_{t=0} \equiv u_0$ and $u_2|_{t=0} \equiv 0$. Then, given $v \in L^2(0,T;H^1)$ (H^1_0 in the case of Dirichlet boundary conditions) such that div$v \equiv 0$ and an anti-symmetric tensor $j \in L^2(0,T;L^2)$, we consider the solution τ of

$$\frac{\partial\tau}{\partial t} + v\cdot\nabla\tau + \tau\cdot j - j\cdot\tau + a\tau = b(D(u_1) + D(u_2)), \quad \tau|_{t=0} \equiv \tau_0. \tag{3.13}$$

Before we explain the meaning of (3.13), we point out that $F = -(u.\nabla)u$ is fixed in a bounded set of $L^2(\mathcal{H}^1) + L^1(L^{p,1})$, where $p \in (1,2)$ if $N = 2$ and $p = \frac{3}{2}$ if $N = 3$ and that (v,w) is fixed in a bounded set of $L^2(H^1) \times L^2(L^2)$ so that u_1 is entirely determined by u_0 and F. Besides, $u_1 \in L^1(W^{1,q})$ for some $q > 2$ ($q < \infty$ if $N = 2$, $q = 3$ if $N = 3$). And, (u_2,τ) solves a coupled affine system (for (v,j,u_1) fixed) which admits a unique solution

(u_2, τ) in $L^2(H^1) \times L^2(L^2)$. In other words, the mapping $(\tau_0 \mapsto (u_2, \tau))$ is a continuous affine mapping from L^2 into $L^2(H^1) \times C(L^2)$. The existence and uniqueness follows easily from the adaptation of the energy identities derived formally in Section 2, namely

$$\begin{aligned} &\frac{d}{dt}\int \frac{|u_2|^2}{2} + \nu \int |\nabla u_2|^2 = -\int \mathrm{Tr}(D(u_2)\cdot\tau), \\ &\frac{d}{dt}\int \frac{||\tau||^2}{2} + a\int ||\tau||^2 = b\int \mathrm{Tr}\ D(u_1).\tau + Tr(D(u_2)\cdot\tau). \end{aligned} \tag{3.14}$$

The justification of (3.14) is straightforward once we observe that, given (u_1, u_2), there exists a unique solution, which is a renormalized solution and thus satisfies (3.14) by considering for instance $\frac{||\tau||^2}{1+\delta||\tau||^2}$ and letting δ go to 0_+ in view of the theory of R. J. DiPerna and P. -L. Lions (see [6]). Finally, we observe that, when $v \equiv u$ and $j \equiv \omega$, then $u \equiv u_1 + u_2$ and τ is the tensor we are really inerested in.

Next, we observe that the proof made in the first part of the preceding subsection 3.2 yields also a bound on τ in $L^\infty(L^q)$ if $\tau_0 \in L^q$. In other words, the affine mapping $(\tau_0 \mapsto \tau)$ from L^2 into $C(L^2)$ that we denote by K satisfies the following properties

$$\begin{cases} \|K(\tau_0^1 - \tau_0^2)\|_{C([0,T];L^2)} \le C\|\tau_0^1 - \tau_0^2\|_{L^2}, & \forall \tau_0^1, \tau_0^2 \in L^2, \\ \|K\tau_0\|_{C([0,T];L^q)} \le C(1 + \|\tau_0\|_{L^q}), & \forall \tau_0 \in L^q. \end{cases} \tag{3.15}$$

Note that K depends upon (u_0, u_1, v, ω) but that the constants appearing in (3.15) are uniform! These properties of K imply that we have, denoting by $\tau = K\tau_0$, for all $M, R \in (0,\infty)$ and for all $t \in [0,T]$,

$$\begin{aligned} \int \|\tau\|^2 1_{(\|\tau\| \ge M)} &\le 2\int \|K(\tau_0 1_{(\|\tau_0\|<R)})\|^2\ 1_{\|\tau\|>M} + 2\int \|K(\tau_0) - K(\tau_0 1_{(\|\tau_0\|<R)})\|^2 \\ &\le \|K(\tau_0 1_{(\|\tau_0\|<R)})\|^2_{L^q}\ \mathrm{meas}(\|\tau\| > M)^{1-2/q} + C\int \|\tau_0\|^2 1_{(\|\tau_0\|\ge R)} \\ &\le C(1 + \|\tau_0 1_{(\|\tau_0\|<R)}\|^2_{L^q})M^{2-4/q}\Big(\int \|\tau\|^2\Big) + C\int \|\tau_0\|^2 1_{(\|\tau_0\|\ge R)} \\ &\le CM^{2-4/q}(1 + \|\tau_0\|^2_{L^2})(1 + R^{(2-q)2/q}\|\tau_0\|_{L^2}) + C\int \|\tau_0\|^2 1_{(\|\tau_0\|\ge R)}. \end{aligned}$$

Hence, since $q > 2$, we deduce

$$\overline{\lim_{M\to+\infty}}\ \sup_{t\in[0,T]} \int \|\tau\|^2\ 1_{(\|\tau\|\ge M)}\ \le\ C\int \|\tau_0\|^2\ 1_{(\|\tau_0\|\ge R)}$$

and we have shown the estimate (3.12).

At this point, we also wish to notice that, u being fixed in $L^2(H^1)$ (with $\mathrm{div} u = 0$) and ω being fixed in $L^2(L^2)$, then $\tau \in L^\infty(L^2)$ is the unique solution of (1.2) and is a renormalized solution (see [6]). This is why $\tau \in C([0,\infty); L^2)$ and $\tau \in C([0,\infty); L^q)$ if $\tau_0 \in L^q$ and $\tau \in L^\infty(L^q)$. Furthermore, by general arguments easily adapted from [6], the (relative) compactness of τ in $L^2(L^2)$ together with the compactness of τ_0 in L^2 easily yield the compactness of τ in $C([0,T]; L^2)$ (for all $T \in (0,\infty)$).

§4. Compactness

4.1. Preliminaries

In this section, we consider a sequence (u^n, τ^n) of solutions of (1.1)–(1.2) satisfying, uniformly in n, the a priori bounds shown in Section 3 and corresponding to initial conditions

(u_0^n, τ_0^n) such that u_0^n converges strongly in L^2 to τ_0 while τ_0^n is equicontinuous in L^2, namely

$$\sup_n \int \|\tau_0^n\|^2 \; 1_{(\|\tau_0^n\| \geq M)} \underset{n}{\to} 0.$$

Hence, (3.12) holds uniformly in n.

We next fix $T \in (0, \infty)$ and we study the passage to the limit in Equations (1.1)–(1.2) as n goes to $+\infty$. Obviously, we may assume without loss of generality, extracting subsequences if necessary, that u^n converges weakly to $u \in L^2(0,T;H^1)$ (and weakly in $L^\infty(0,T;L^2)$) and τ^n converges weakly-* to τ in $L^\infty(0,T;L^2)$. The bounds shown on the pressure imply some bounds on $\frac{\partial u^n}{\partial t}$ that we do not wish to detail but that are enough to guarantee by standard arguments that u^n converges strongly to u in $L^2(L^p)$ for all $p < \frac{2N}{N-2}$, and in $L^\infty(L^p)$ for all $p < 2$ and thus in $L^q(L^p)$ for all $2 \leq q \leq \infty$, $p < \frac{2qN}{qN-4}$. Next, we write $u^n = u_1^n + u_2^n + u_3^n$ which solve respectively

$$\begin{cases} \frac{\partial u_1^n}{\partial t} - \nu \Delta u_1^n + \nabla p_1^n = -(u^n \cdot \nabla) u^n, \quad \operatorname{div} u^n = 0, \\ u_1^n\big|_{t=0} = 0, \end{cases} \tag{4.1}$$

$$\frac{\partial u_2^n}{\partial t} - \nu \Delta u_2^n + \nabla p_2^n \; = \; 0, \quad \operatorname{div} u_0^n = 0 \;, \quad u_2^n\big|_{t=0} = u_0^n, \tag{4.2}$$

$$\frac{\partial u_3^n}{\partial t} - \nu \Delta u_3^n + \nabla p_3^n \; = \operatorname{div}(\tau^n), \quad \operatorname{div} u_3^n = 0; \quad u_3^n\big|_{t=0} = 0. \tag{4.3}$$

Obviously, u_2^n converges strongly in $C([0,T];L^2) \cap L^2(H^1)$ to some u_2 solving the same equation with u_0 in place of u_0^n. And u_3^n converges weakly in $L^2(H^1)$ and weakly $-*$ in $L^\infty(L^2)$ to some u_3 solving also (4.3) with τ in place of τ^n. Furthermore, u_3^n converges strongly to u_3 in the same $L^q(L^p)$ spaces than u^n does. And, thus u_1^n enjoys the same convergence properties to $u_1 = u - (u_2 + u_3)$. In addition, exactly as in [13] or in the preceding sections, the bounds on $(u^n \cdot \nabla) u^n$ in $L^2(\mathcal{H}') \cap L^1(L^{p_0,1})$ (with $p_0 < 2$ if $N = 2$, $p_0 = \frac{3}{2}$ if $N = 3$) and thus in $L^q(L^p)$ for $1 < q < 2$, $p = \frac{q p_0}{2(q-1)p_0 + 2 - q}$, yield bounds on $\frac{\partial u_1^n}{\partial t}$, ∇p_1^n and $D_x^2 u_1^n$ in $L^q(L^p)$ which in turn imply that ∇u_1^n converges strongly to ∇u_1 in $L^2(L^p) \cap L^1(L^r)$ for all $p < 2$, $r < \infty$ if $N = 2$ and $r < 3$ if $N = 3$.

We next claim that $\nabla u_1^n \Phi(u_1^n)$ converges strongly to $\nabla u_1 \Phi(u_1)$ in $L^2(L^2)$ for any scalar continuous function Φ such that $\Phi(z) \to 0 \quad \text{as} \quad |z| \to +\infty$. By density, it is clearly enough to show that this claim holds for any $\psi \in C_0^\infty(\mathbb{R})^N$. And the main reason for this fact is the following observation (crucial for renormalized solutions of parabolic equations) which is valid for any $\psi \in C^1(\mathbb{R}^N)$ such that $|\nabla \psi(z)| \leq C(1 + |z|^\Theta)$ on $\mathbb{R}^N$

$$-\Delta u_1^n \cdot \nabla \psi(u_1^n) = -\frac{1}{\nu}(u^n \cdot \nabla) u^n \cdot \nabla \psi(u_1^n) - \frac{1}{\nu} \nabla p_1^n \cdot \nabla \psi(u_1^n) - \frac{1}{\nu} \frac{\partial}{\partial t} \psi(u_1^n)$$

and, provided $\Theta > 0$ is small enough, the right-hand side converges weakly to

$$-\frac{1}{\nu}(u \cdot \nabla) u \cdot \nabla \psi(u_1) - \frac{1}{\nu} \nabla p_1 \cdot \nabla \psi(u_1) - \frac{1}{\nu} \frac{\partial}{\partial t} \psi(u_1) = -\Delta u_1 \cdot \nabla \psi(u_1)$$

in view of the bounds and the convergences recalled above. From this observation, we deduce easily that we have $D^2 \psi(u_1^n) \cdot (\nabla u_1^n, \nabla u_1^n) \underset{n}{\rightharpoonup} D^2 \psi(u_1) \cdot (\nabla u_1, \nabla u_1)$. We then choose $\psi(z) = (1 + |z|^2)^{m/2}$ with $m = 1 + \theta$ and find

$$(1 + |u_1^n|^2)^{\frac{m}{2} - 1} a_{ij}(u_1^n) \partial_k (u_1^n)_i \partial_k (u_1^n)_j \underset{n}{\rightharpoonup} (1 + |u_n|^2)^{\frac{m}{2} - 1} a_{ij}(u_1) \partial_k (u_1)_i \partial_k (u_1)_j, \tag{4.4}$$

where $a_{ij}(z) = \delta_{ij} - (1 - \theta) \frac{z_i z_j}{1 + |z|^2}$. Obviously, $A^n = a_{ij}\Big(u_1^n\Big)_{ij}$ is symmetric and satisfies $\theta I \leq A^n \leq I$ and A^n converges strongly to $A = \Big(a_{ij}(u_1)\Big)_{ij}$ in $L^q(L^q)$ for all $1 \leq q < \infty$.

Then, (4.4) implies

$$(1+|u_1^n|^2)^{\frac{m-2}{4}}(A^n)^{1/2}\cdot\nabla u_1^n \underset{n}{\to} (1+|u_1|^2)^{\frac{m-2}{4}}A^{1/2}\cdot\nabla u_1 \tag{4.5}$$

$\Big((A)_{ij}^{1/2}=\delta-ij-\Big(1-\sqrt{1-(1-\theta)\frac{|z|^2}{1+|z|^2}}\Big)\, z_iz_j$ with $z\le u_1\Big)$ strongly in L^2. Multiplying (4.5) by $(A)^{-1/2}$ (which is bounded), we deduce

$$(1+|u_1^n|^2)^{(m-2)/4}\nabla u_1^n \underset{n}{\to} (1+|u_1|^2)^{(m-2)/4}\nabla u_1 \quad \text{in} \quad L^2.$$

Finally, if $\Phi\in C_0^\infty(\mathbb{R}^n)$, $\Phi(u_1^n)(1+u_1^n|^2)^{-(m-2)/4}$ is bounded uniformly in n on $\mathbb{R}^N$ and converges (in $L^q(L^q)$ for all $q\in[1,\infty)$) to $\Phi(u_1)(1+|u_1|^2)$. This is enough to ensure that we have

$$\Phi(u_1^n)\nabla u_1^n \underset{n}{\to} \Phi(u_1)\nabla u_1 \tag{4.6}$$

and our claim is shown for all $\Phi\in C_0^\infty$ and such for all continuous Φ vanishing at infinity.

Remark. Let us observe that this fact is true in particular for the classical Navier-Stokes equations when $N=3$, a fact that we were not aware of and which indicates some potential in the idea of renormalizing Navier-Stokes equations.

4.2. Main Compactness Result

Theorem 4.1. *If τ_0^n converges strongly to τ_0 in L^2, then τ^n converges strongly to τ in $C([0,T];L^2)$. And (u,τ) solves* (1.1)–(1.2)–(1.3).

Remarks. (i) As we pointed out at the end of Section 3, it is enough to show that τ^n converges to τ in $L^2(0,T;L^2)$.

(ii) We wish to show that the assumption on τ_0^n is in general necessary in order to ensure not only that τ^n converges strongly in L^2 to τ but also that (u,τ) solves (1.1)–(1.2). This claim can be checked by a simple construction of oscillating solution of (1.1)–(1.2) although we shall really consider oscillating solutions of (1.1)–(1.2) with small force terms going to 0 uniformly (it is possible to modify the construction in order to get rid of force terms but we shall not do so here). The idea of the construction is to set, in the periodic case (with periods equal to 1), $u_n(x,t)=\frac{1}{n}v(nx,t)$, $\tau_n(x,t)=\sigma(nx,t)$, for $n\ge 1$, where (v,σ) solve

$$\begin{cases} -\nu\Delta v+\nabla p=\mathrm{div}\sigma\ , \quad \mathrm{div} v=0,\\ \frac{\partial\sigma}{\partial t}+v\cdot\nabla\sigma+\sigma\cdot\omega-\omega\cdot\sigma+a\sigma=bD(v). \end{cases} \tag{4.7}$$

Solving (4.7) for an arbitrary initial condition $\sigma|_{t=0}\equiv\sigma_0\in L^2$ follows from a similar but simpler proof than the one of Theorem 2.1 and one also easily shows that if $\sigma_0\in L^q$, for any $q<\infty$, $\sigma\in C([0,\infty);L^q)$ and $v\in C([0,\infty);W^{1,q})$. Then, obviously, (u_n,τ_n) solve (1.1)–(1.2) with a force term f_n given by $f_n=\frac{1}{n}\big(\frac{\partial v}{\partial t}+v\cdot\nabla v\big)\ (nx,t)$ which goes to 0 in $C([0,T];L^q)$ $(\forall T<\infty,\ \forall q<\infty)$ if $\sigma_0\in L^q$ for all $q<\infty$. As we shall see below, it is possible to build some examples of smooth solutions of (4.7). Next, we observe that τ_n converges weakly and not strongly to $\big(\int\sigma dx\big)$ and similarly $\omega(u_n)$ and $\tau_n\cdot\omega(u_n)-\omega(u_n)\cdot\tau_n$ converge weakly to 0 and $\int\sigma\cdot\omega-\omega\cdot\sigma dx$ respectively. And the question of (weakly) passing to the limit amounts to checking that $\big(\int\sigma\cdot\omega-\omega\cdot\sigma\ dx\big)$ vanishes.

This is not the case in general, proving thus our claim; in order to see that, one may choose $N=2$, $v=(0,u(x_1,t))$, $\tau=\tau(x_1,t)$, $\omega=\frac{\partial u}{\partial x_1}(x_1,t)=-\frac{1}{\nu}\sigma_{12}$ and then (4.7), in this case, reduces to

$$\begin{cases} \frac{\partial\sigma_{11}}{\partial t}+\frac{2}{\nu}\ \sigma_{12}^2+a\sigma_{11}=0,\\ \frac{\partial\sigma_{12}}{\partial t}-\frac{1}{\nu}\ (\sigma_{11}-\sigma_{22})\sigma_{12}+a\sigma_{12}=-\frac{b}{2\nu}\ \sigma_{12},\\ \frac{\partial\sigma_{22}}{\partial t}-\frac{2}{\nu}\ \sigma_{12}^2+a\sigma_{22}=0, \end{cases}$$

which obviously admits solutions such that $\sigma_{12} \not\equiv 0$. Then,

$$\int \sigma \cdot \omega - \omega\sigma dx = \begin{pmatrix} \frac{2}{\nu}\int \sigma_{12}^2 dx & -\frac{1}{\nu}\int(\sigma_{11}-\sigma_{22})\sigma_{12}\ dx \\ -\frac{1}{\nu}\int(\sigma_{11}-\sigma_{22})\sigma_{12}\ dx & \frac{2}{\nu}\int \sigma_{12}^2 dx \end{pmatrix}$$

and thus does not vanish in general.

The fact that (u, τ) solves (1.1)–(1.2)–(1.3) is a trivial consequence of the strong convergence of τ^n. Indeed, if we try to pass to the limit in (1.1)–(1.2), the only terms we have to worry about are the nonlinear ones. Two of those are easily handled, namely div $(u^n \otimes u^n)$ and div $(u^n\tau^n)$.

Indeed, we saw that u^n converges strongly in various spaces including $L^2(L^2)$ and this fact allows to pass to the limit in those terms. Also, once we know that τ^n converges strongly in $L^2(L^2)$, then there is no difficulty to recover energy identities when $N = 2$ or energy inequalities when $N = 3$ (using the bounds and the convergences already shown). The only really difficult term is the term $\tau^n \cdot \omega^n - \omega^n \cdot \tau^n$ which involves products of weakly convergent tensors and has no weak continuity property in general as shown in the remark above. And this is the term which makes necessary to obtain the strong L^2 compactness of τ^n when τ_0^n is compact.

In order to prove this compactness assertion, we introduce various quantities that measure the possible losses of compactness: we may assume without loss of generality (extracting subsequences if necessary) that $||\tau^n||^2$ converges weakly in $L^1(L^1)$ to $||\tau||^2 + \eta$ where $\eta \in L^\infty(L^1) \geq 0$ (we need to use here the equicontinuity of τ^n in L^2 shown in Section 3.3); that $\tau^n \cdot \omega^n - \omega^n \cdot \tau^n$ converges weakly in $L^1(L^1)$ to $\tau \cdot \omega - \omega \cdot \tau + \beta$ where $\beta \in L^2(L^1)$ (using again the equicontinuity of τ^n in L^2); that $D(u^n) \cdot \tau^n$ converges weakly in $L^1(L^1)$ to $D(u) \cdot \tau + \alpha$ where $\alpha \in L^2(L^1)$ (using once more the equicontinuity of τ^n in L^2); that, when $N = 2$, $|\nabla u^n|^2$ converges weakly in the sense of measures to $|\nabla u|^2 + \mu$ where μ is a nonnegative bounded measure; and, finally, that, when $N = 3$, $|\nabla u_3^n|^2$ converges in the sense of measure to $|\nabla u_3|^2 + \mu$, where μ is a nonnegative bounded measure.

We next claim that we have

$$\mathrm{Tr}(\alpha) + \nu\mu = 0, \quad |\alpha| \leq C\eta, \quad |\beta| \leq C\eta, \tag{4.8}$$

where, here and below, C denotes various positive constants.

The first equality in (4.8) is an easy consequence of energy considerations and we only make the proof when $N = 3$ (the adaptations to the case when $N = 2$ being straightforward). Recalling that u_3^n and u_3 solve the Stokes' equation (4.3), we deduce easily

$$\begin{cases} \frac{\partial}{\partial t}\Big(\frac{|u_3^n|^2}{2}\Big) - \nu\Delta\Big(\frac{|u_3^n|^2}{2}\Big) + \nu|\nabla u_3^n|^2 + \mathrm{div}(u_3^n p_3^n) = \mathrm{div}(u_3^n\tau^n) - \mathrm{Tr}(\tau^n \cdot D(u^n)), \\ \frac{\partial}{\partial t}\Big(\frac{|u_3|^2}{2}\Big) - \nu\Delta\Big(\frac{|u_3|^2}{2}\Big) + \nu|\nabla u_3|^2 + \mathrm{div}(u_3 p_3) = \mathrm{div}(u_3\tau) - \mathrm{Tr}(\tau \cdot D(u)), \end{cases}$$

and passing to the limit using the bounds and convergence properties already shown, we obtain

$$\frac{\partial}{\partial t}\Big(\frac{|u_3|^2}{2}\Big) - \nu\Delta\Big(\frac{|u_3|^2}{2}\Big) + \nu|\nabla u_3|^2 + \nu\mu + \mathrm{div}(u_3 p_3) = \mathrm{div}(u_3\tau) - \mathrm{Tr}(\tau \cdot D(u)) - Tr(\alpha).$$

The equality between $(\mathrm{Tr}(\alpha))$ and $\nu\mu$ follows. Let us notice that this equality implies that $\mu \in L^2(L^1)$. Next, we prove the bounds on α and β contained in (4.8). It is clearly enough to show that we have

$$|\alpha| \leq C\sqrt{\eta}\sqrt{\mu}, \quad |\beta| \leq C\sqrt{\eta}\sqrt{\mu}. \tag{4.9}$$

This is almost immediate except for a technical detail in the case when $N = 3$ that we now explain. First of all, we claim that

$$\begin{cases} D(u_1^n)\cdot\tau^n \underset{n}{\rightharpoonup} D(u_1)\cdot\tau, \quad D(u_2^n)\cdot\tau^n \underset{n}{\rightharpoonup} D(u_2)\cdot\tau, \\ \tau^n\cdot\omega^n(u_1^n)-\omega^n(u_1^n)\cdot\tau^n \underset{n}{\rightharpoonup} \tau\cdot\omega(u_1)-\omega(u_1)\cdot\tau, \\ \tau^n\cdot\omega^n(u_2^n)-\omega^n(u_2^n)\cdot\tau^n \underset{n}{\rightharpoonup} \tau\cdot\omega(u_2)-\omega(u_2)\cdot\tau. \end{cases}$$

The "u_2^n" terms are immediate since u_2^n converges to u_2 strongly in $L^2(H^1)$. For the "u_1^n" terms, we deduce from the results shown in Section 4.1 that

$$\begin{cases} \Phi(u_1^n)D(u_1^n)\cdot\tau^n \underset{n}{\rightharpoonup} \Phi(u_1)D(u_1)\cdot\tau, \\ \Phi(u_1^n)\{\tau^n\cdot\omega^n(u_2^n)-\omega^n(u_1^n).\tau^n\} \underset{n}{\rightharpoonup} \Phi(u_1)\{\tau\cdot\omega(u_1)-\omega(u_1)\cdot\tau\} \end{cases} \tag{4.10}$$

for any $\Phi\in C_0(\mathbb{R}^N)$. We then choose $\Phi=\varphi\Big(\frac{\cdot}{M}\Big)$ where $\varphi\in C_0^\infty(\mathbb{R}^N)$, $0\le\varphi\le 1$ on $\mathbb{R}^N$, $\varphi\equiv 1$ on B_1 and we conclude observing that

$$\begin{aligned} \iint\Big(1-\varphi\Big(\frac{u_1^n}{M}\Big)\Big)\ |\nabla u_1^n|\ \|\tau^n\|\ dxdt &\le \Big(\iint\ |\nabla u_1^n|^2\ dxdt\Big)^{1/2}\iint\ \|\tau^n\|^2\ 1_{(\|u_1^n|\ge M)}\ dxdt \\ &\le\ C\ \sup_{[0,T]}\ \Big(\int\|\tau^n\|^2\ 1_{(u_1^n|\ge M)}\ dx\Big) \end{aligned}$$

and this bound goes to 0 as M goes to $+\infty$, uniformly in n, in view of the equicontinuity of τ^n in L^2 shown in Section 3.3. We want to point out here that the result of Section 4.1 is not really necessary for the above conclusion. However, we kept it here since it is interesting in its self. Indeed, in the above computation (4.10), we can replace $\Phi(u_1^n)D(u_1^n)$ by $\Phi(|\nabla u_1^n|)D(u_1^n)$ and use compactness of $D(u_1^n)$ in $L^p(L^p)$ for all $p<2$. Then we conclude by using the equicontinuity of τ^n in L^2.

Therefore, we have shown the following facts

$$\begin{cases} D(u_3^n)\cdot\tau^n \underset{n}{\rightharpoonup} D(u_3)\cdot\tau+\alpha, \\ \tau^n\cdot\omega^n(u_3^n)-\omega^n(u_3^n)\cdot\tau^n \underset{n}{\rightharpoonup} \tau\cdot\omega(u_3)-\omega(u_3)\cdot\tau+\beta \end{cases} \tag{4.11}$$

and we complete the proof of (4.8) or (4.9) by observing that we have

$$\begin{cases} (D(u_3^n)-D(u_3))\cdot(\tau^n-\tau) \underset{n}{\rightharpoonup} \alpha, \\ (\tau^n-\tau)\cdot(\omega(u_3^n)-\omega(u_3))-(\omega(u_3^n)-\omega(u_3))\cdot(\tau^n-\tau) \underset{n}{\rightharpoonup} \beta, \\ \|\tau^n-\tau\|^2 \underset{n}{\rightharpoonup} \eta, \quad |\nabla(u_3^n-u_3)|^2 \underset{n}{\rightharpoonup} \mu. \end{cases}$$

Obviously, when $N=2$, the final observation with u_3^n replaced by u^n suffices to proves (4.9).

We next conclude this section by a formal argument that explains the main idea of the proof of Theorem 4.1. First of all, passing to the limit in (1.2), we obtain

$$\frac{\partial\tau}{\partial t}+\operatorname{div}(u\tau)+\tau\cdot\omega-\omega\cdot\tau+\beta+a\tau=b\ D(u). \tag{4.12}$$

On the other hand, if we pass to the limit in (2.2) (ignoring the issue of the integrability of $u^n\|\tau^n\|^2\cdots$) we obtain formally

$$\frac{\partial}{\partial t}\Big(\frac{\|\tau|^2}{2}+\frac{\eta}{2}\Big)+\operatorname{div}\Big\{u\Big(\frac{\|\tau|^2}{2}+\frac{\eta}{2}\Big)\Big\}+a(|\tau\|^2+\eta)=b\mathrm{Tr}(Du\cdot\tau)+b\ \mathrm{Tr}(\alpha). \tag{4.13}$$

Multiplying (4.12) by τ, taking the trace and comparing with (4.13) easily yields the following equation for η,

$$\frac{\partial\eta}{\partial t}+\operatorname{div}(u\eta)+2a\eta=2b\mathrm{Tr}(\alpha)-2\mathrm{Tr}(\beta\cdot\tau), \tag{4.14}$$

hence, using (4.8), we deduce

$$\frac{\partial \eta}{\partial t} + \operatorname{div}(u\eta) \;\leq\; C\eta(1+\|\tau\|), \quad \eta \geq 0 \tag{4.15}$$

while $\eta|_{t=0} \equiv 0$ since we assumed that τ_0^n converges strongly in L^2 to τ_0.

Then, formally, we conclude that $\eta \equiv 0$, thus proving the compactness of τ^n in $L^2(L^2)$, by integrating by parts (4.15) over Ω and using Gronwall's inequality. Notice that this is purely formal since we do not know that $\|\tau\| \in L^1(L^\infty)$. This question and the lack of bounds allowing to "control" the term $u^n\|\tau^n\|^2$ are the two main difficulties we shall overcome in the next two sections.

4.3 Proof in the L^q Case with $q>3$

We first complete the proof of Theorem 4.1 in the case when we have a bound on τ^n in $L^\infty(0,T;L^q)$ for some $q>3$. Let us recall that we have indeed obtained such a bound when $N=2$, τ_0^n is bounded in L^q and u_0^n is bounded in $I_{p,q}$ for some $p \in \left(1, \frac{q}{q-1}\right)$. In fact, it is possible to push the argument below to the case when $q=3$, so it may apply to three-dimensional situations, but since we shall have to do a more general argument of a similar spirit in the final subsection below, we shall not present this technical extension here.

This bound of course implies that $\tau \in L^\infty(L^q)$ (in fact $C(L^q)$ and that $\eta, \beta \in L^\infty(L^{q/2})$. Then, we may pass to the limit in (2.2) and we recover (4.14)–(4.15) rigorously. We still need to conclude that (4.15) implies that $\eta \equiv 0$. At this stage, it is worth considering a more general situation than the one considered in Theorem 4.1, namely the situation when we do not assume anymore that τ_0^n converges strongly in L^2 to 0. Then, everything we did before is still valid (we just need to replace u_3^n by $u_3^n + u_2^n$ in the definition of μ and in the proof of (4.8). And we still obtain (4.15), except that we do not have anymore that $\eta|_{t=0} \equiv 0$: instead, we find $\eta|_{t=0} \equiv \eta_0 \geq 0$ and $\eta_0 \equiv 0$ if and only if τ_0^n converges strongly in L^2 to τ_0.

We then claim that we have for all $t \geq 0$,

$$\eta(X(t,x)) \leq \eta_0(x)\exp\Big\{C\int_0^t (1+\|\tau\|(X(s,x),s))ds\Big\} \quad \text{a.e.} \tag{4.16}$$

where X is the unique a.e. flow in the sense of R. J. DiPerna and P. L. Lions (see [6]) associated to the particle paths, i.e. solution of

$$\dot{X} = u(X,t), \quad X(0,x) = x. \tag{4.17}$$

The inequality (4.16) follows directly from (4.15) and the results shown in [6]. And we recall that X leaves invariant the Lebesgue measure. In particular, for each t,

$$\Big\| \int_0^t \|\tau\|\big(X(s,x),s\big)ds \Big\|_{L^2} \leq \int_0^t \|\tau(s)\|_{L^2} ds < \infty.$$

Hence, $\exp\Big\{C\int_0^t (1+\|\tau\|(X(s,x),s))ds\Big\} < \infty$ a.e. and (4.16) implies that $\eta(X(t,x),t) = 0$ a.e. if $\eta_0 \equiv 0$. Using once more the invariance of Lebesgue measure, we deduce that $\eta(x,t) \equiv 0$ a.e. in x for all $t \geq 0$ and the proof is complete.

It is possible to make another argument which only uses the theory of renormalized solutions of transport equations of [6] without using the associated flows (the two aspects are completely equivalent as shown in [6] $\cdots$) and consists in introducing the unique renormalized solution of

$$\frac{\partial \psi}{\partial t} + \operatorname{div}(u\psi) = C(1+\|\tau\|), \quad \psi|_{t=0} = 0 \tag{4.18}$$

and checking as in [6] that $\hat{\eta} = \eta e^{-\psi}$ is a renormalized solution of

$$\frac{\partial \hat{\eta}}{\partial t} + \mathrm{div}(u\hat{\eta}) \leq 0. \tag{4.19}$$

Hence, we have for all $t \geq 0$,

$$\int \eta(x,t)\exp(-\psi(x,t))dx \ \leq \ \int \eta_0(x)dx. \tag{4.20}$$

In particular, if $\eta_0 \equiv 0$ then $\eta e^{-\psi} \equiv 0$ and we conclude easily since $\psi \in C(L^2)$.

4.4 Proof in the General Case

We first introduce a few more quantities: extracting subsequences if necessary, we may assume that we have the following weak convergences for each $\delta \in (0,1)$,

$$\frac{\|\tau^n\|^2}{(1+\delta\|\tau^n\|^2)^i} \underset{n}{\rightharpoonup} \frac{\|\tau\|^2}{(1+\delta\|\tau\|^2)^i} + \eta_\delta^i, \ 0 \leq \eta_\delta^i \leq 1/\delta \ \text{ a.e. for } \ i=1,2, \tag{4.21}$$

$$\frac{D(u^n)\cdot\tau^n}{(1+\delta\|\tau^n\|^2)^2} \underset{n}{\rightharpoonup} \frac{D(u)\cdot\tau}{(1+\delta\|\tau\|^2)^2} + \alpha_\delta, \quad \alpha_\delta \in L^2(L^2), \tag{4.22}$$

$$\|\tau^n\|^2 + 1 \underset{n}{\rightharpoonup} N^2, \quad N \in L^\infty(L^2), \quad N = (\|\tau\|^2 + \eta + 1)^{1/2} \ \text{ a.e.} \tag{4.23}$$

We next observe that we have

$$\frac{\partial}{\partial t}(\|\tau^n\|^2) + \mathrm{div}(u^n\|\tau^n\|^2) + 2a\|\tau^n\|^2 = 2b\mathrm{Tr}(D(u^n)\cdot\tau^n);$$

hence, letting n go to $+\infty$, we deduce easily

$$\frac{\partial N}{\partial t} + \mathrm{div}(u\ N) + a\frac{N^2-1}{N} = F \in L^2(L^2), \tag{4.24}$$

where $F = (b\mathrm{Tr}(D(u)\cdot\tau + \alpha))N^{-1}$, $|F| \leq C(1+\|D(u)\|)$. Indeed, let $\phi \in C_0^\infty(\mathbb{R})$ be such that $\phi \equiv 1$ on $[0,1]$ and $\phi_M(\cdot) = M\phi(\frac{\cdot}{M})$. Hence we deduce (using the fact that we have renormalized solutions $\cdots$)

$$\begin{aligned}&\frac{\partial}{\partial t}\phi_M(\|\tau^n\|^2+1) + \mathrm{div}(u^n\phi_M(\|\tau^n\|^2+1)) + 2a\phi_M'(\|\tau^n\|^2+1)\|\tau^n\|^2 \\ &= 2b\phi_M'(\|\tau^n\|^2+1)\mathrm{Tr}(D(u^n)\cdot\tau^n).\end{aligned}$$

Passing to the limit in n, we deduce that

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{\phi_M(\|\tau^n\|^2+1)} + \mathrm{div}(u\overline{\phi_M(\|\tau^n\|^2+1)}) + 2a\overline{\phi_M'(\|\tau^n\|^2+1)\|\tau^n\|^2} \\ &= 2b\overline{\phi_M'(\|\tau^n\|^2+1)\mathrm{Tr}(D(u^n)\cdot\tau^n)}.\end{aligned}$$

Then taking the square root, we get

$$\begin{aligned}&\frac{\partial}{\partial t}N_M + \mathrm{div}(uN_M) + \frac{a}{N_M}\overline{\phi_M'(\|\tau^n\|^2+1)\|\tau^n\|^2} \\ &= \frac{b}{N_M}\overline{\phi_M'(\|\tau^n\|^2+1)\mathrm{Tr}(D(u^n)\cdot\tau^n)},\end{aligned} \tag{4.25}$$

where we denote by $N_M = \overline{\phi_M(\|\tau^n\|^2+1)}^{1/2}$. Next, using the equicontinuity of $\|\tau^n\|^2$, we deduce easily that N_M converges strongly to N in L^2 and we conclude easily.

On the other hand, we have

$$\frac{\partial}{\partial t}\Big(\frac{\|\tau^n\|^2}{1+\delta\|\tau^n\|^2}\Big) + \mathrm{div}\Big(u^n\frac{\|\tau^n\|^2}{1+\delta\|\tau^n\|^2}\Big) + 2a\frac{\|\tau^n\|^2}{(1+\delta\|\tau^n\|^2)^2} = 2b\frac{\mathrm{Tr}(D(u^n)\cdot\tau^n)}{(1+\delta\|\tau^n\|^2)^2}, \tag{4.26}$$

$$\frac{\partial}{\partial t}\Big(\frac{\|\tau\|^2}{1+\delta\|\tau\|^2}\Big)+\mathrm{div}\Big(u\frac{\|\tau\|^2}{1+\delta\|\tau\|^2}\Big)+2a\frac{\|\tau\|^2}{(1+\delta\|\tau\|^2)^2}+2\mathrm{Tr}\Big(\frac{\beta\cdot\tau}{(1+\delta\|\tau\|^2)^2}\Big)$$
$$=2b\frac{\mathrm{Tr}(D(u)\cdot\tau)}{(1+\delta\|\tau\|^2)^2}. \tag{4.27}$$

Hence, letting n go to $+\infty$ in (4.26) yields

$$\frac{\partial}{\partial t}\Big(\frac{\|\tau\|^2}{1+\delta\|\tau\|^2}+\eta^1_\delta\Big)+\mathrm{div}\Big(u\Big[\frac{\|\tau\|^2}{1+\delta\|\tau\|^2}+\eta^1_\delta\Big]\Big)+2a\frac{\|\tau\|^2}{(1+\delta\|\tau\|^2)^2}\Big)+2a\eta^2_\delta$$
$$=2b\frac{\mathrm{Tr}(D(u)\cdot\tau)}{(1+\delta\|\tau\|^2)^2}=2b\mathrm{Tr}(\alpha_\delta),$$

and comparing with (4.27), we finally obtain

$$\frac{\partial}{\partial t}\eta^1_\delta+\mathrm{div}(u\eta^1_\delta)+2a\eta^2_\delta=2b\mathrm{Tr}(\alpha_\delta)-2\mathrm{Tr}\Big(\frac{\beta\cdot\tau}{(1+\delta\|\tau\|^2)^2}\Big).$$

Next, we use (4.24) and deduce the following equation

$$\frac{\partial}{\partial t}(\eta^1_\delta/(1+N))+\mathrm{div}[u(\eta^1_\delta/(1+N))]+\frac{2a\eta^2_\delta}{1+N}=2b\mathrm{Tr}(\alpha_\delta)(1+N)^{-1}$$
$$-2\mathrm{Tr}\Big(\frac{\beta\cdot\tau}{(1+\delta\|\tau\|^2)^2}\Big)(1+N)^{-1}-\Big(F-a\frac{N^2-1}{N}\Big)\eta^1_\delta(1+N)^{-2}. \tag{4.28}$$

At this point, we wish to let δ go to 0_+ and we admit temporarily that η^1_δ and η^2_δ converge in $L^\infty(L^1)$ to η, while α_δ converges to α in $L^2(L^1)$. We then observe that, obviously, $-N^2\le\eta^1_\delta\le N^2$, hence $\eta^1_\delta/(1+N)$ converges to $\frac{\eta}{1+N}$ in, say, $L^2(L^2)$. We may then let δ go to 0_+ and we find

$$\frac{\partial}{\partial t}\Big(\frac{\eta}{1+N}\Big)+\mathrm{div}\Big(u\Big(\frac{\eta}{1+N}\Big)\Big)+2a\Big(\frac{\eta}{1+N}\Big)$$
$$=2b\frac{\mathrm{Tr}(\alpha)}{1+n}-2\frac{\mathrm{Tr}(\beta\cdot\tau)}{1+N}-\Big(F-a\frac{N^2-1}{N}\Big)\eta(1+N)^{-2}.$$

We next use (4.8) and we deduce

$$\frac{\partial}{\partial t}\Big(\frac{\eta}{1+N}\Big)+\mathrm{div}\Big(u\Big(\frac{\eta}{1+N}\Big)\Big)+2a\frac{\eta}{1+N}\le C(1+\|\tau\|+\|D(u)\|)\frac{\eta}{1+N}. \tag{4.29}$$

In other words, we have obtained, in full generality, a differential inequality which plays the same role as (4.15) except that η is replaced by $\frac{\eta}{1+N}$ and $\|\tau\|$ by $\|\tau\|+\|D(u)\|$. We may now copy the proof made in Section 4.3 and we conclude.

Therefore, there only remains to show our claims on η^1_δ, η^2_δ and α_δ which are easy consequences of the equicontinuity of τ^n in L^2 obtained in Section 2.3. Indeed, it suffices to observe that we have for $i=1,2$ and for all $M\in(0,\infty)$,

$$\Big|\frac{\|\tau^n\|^2}{(1+\delta\|\tau^n\|^2)^i}-\|\tau^n\|^2\Big|\le\|\tau^n\|^2 1_{(\|\tau^n\|\ge M)}+C(M)\delta,$$
$$\Big|\frac{D(u^n)\cdot\tau^n}{(1+\delta\|\tau^n\|^2)^2}-D(u^n)\cdot\tau^n\Big|\le|D(u^n)\cdot\tau^n|1_{(\|\tau^n\|\ge M)}+C(M)\delta|D(u^n)|$$
$$\le|D(u^n)|[\|\tau^n\|1_{\|\tau^n\|\ge M)}]+C(M)\delta|D(u^n)|.$$

And we conclude easily.

References

[1] Astarita, G. & Morruci, G., Principles of non-Newtonian fluid mechanics [M], Mc Graw Hill, New-York, 1974.

[2] Bird, R. B., Curtiss, C. F., Amstrong, R. C. & Hassager, O., Dynamics of polymeric liquids, Kinetic Theory [M], Vol. 2, 2nd edition, Wiley, New-York, 1987.

[3] Bird, R. B., Curtiss, C. F., Amstrong, R. C. & Hassager, O., Dynamics of polymeric liquids, Kinetic Theory [M], Vol. 2, 2nd edition, Wiley, New-York, 1987.

[4] Chemin, J. Y. & Masmoudi, N., About liperpan of regular solutions of equations related to viscoelastic fluids [R], Preprint.

[5] Coifman, R., Lions, P. -L., Meyer, Y. & Semmes, S., Compensated-compactness and Hardy spaces [J], *J. Math. Pures. Appl.,* **72**(1993), 247–286.

[6] DiPerna, R. & Lions, P. -L., Ordinary differential equations, Sobolev spaces and transport theory [J], *Invent. Math.,* **98**(1989), 511–547.

[7] Fernández-Gora, E., Gillén, F. & Ortega, R., Some theoretical results concerning non Newtonian fluids of the Oldroyd kind [J], *Ann. Scuola. Norm. Sup. Pisa Q Sci. 4,* **26**(1998), 1–29.

[8] Guillopé, C. & Saut, J. C., Existence results for the flow of viscoelastic fluids with a differential constitutive law [J], *Nonlinear Anal.,* **15:**9(1990), 849–869.

[9] Guillopé, C. & Saut, J. C., Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type [J], *Math. Mod. Numer. Anla.,* **24:**3(1990), 369–401.

[10] Halin, P., Lielens, G., Keenings, R. & Legat, V., The lagrangian particle method for macroscopic and micro-macro viscoelastic flow computations [J], *J. Non-Newtonian Fluid Mechanics,* **79**(1998), 387–403.

[11] Joseph, D. D., Fluid dynamics of viscoelastic liquids [M], Springer-Verlag, 1990.

[12] Joseph, D. D., Renardy, M. & Saut, J. C., [J], *Arch. Rat. Mech. Anal.,* **87**(1985), 213–251.

[13] Lions, P. L., Mathematical topics in fluid mechanics [M], Vol 1, Incompressible models, Clarendon Press, Oxford Science Publications, Oxford, 1996.

[14] Lions, P.L., Mathematical topics in fluid mechanics [M], Vol. 2, Compressible models, Clarendon Press, Oxford Science Publications, Oxford, 1998.

[15] Lions, P. L., Compacité des solutions des équations de Navier-Stokes compressibles isentropiques [J]. *C. R. Acad. Sci. Paris,* **317**(1993), 115–120.

[16] Oldroyd, J. G., On the formulation of rheological equations of state [J], *Proc. Roy. Soc. London,* **200A**(1950), 523–541.

[17] Oldroyd, J. G., Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids [J], *Proc. Roy. Soc. London,* **245A**(1958), 278–297.

[18] Ortega, R. R., Thesis [M], University of Seville (Spain), 1995.

[19] Öttinger, H. C., Stochastic processes in polymeric fluids [M], Tools and examples for developing simulation algorithms, Springer, Berlin, 1996.

[20] Öttinger, H. C. & Ormela, M., Dynamics and thermodynamics of complex fluids I [J], *Phys. Rev. E.,* **56**(1997), 6620–6632; II [J], *Phys. Rev. E.,* **56**(1997), 6633–6655.

[21] Renardy, M., Hrusa, W. J. & Nohel, J. A., Mathematical Problems in Viscoelasticity [M], Longman, London, 1987.

[22] Walters, K. (ed.), Rheometry: industrial applications [M], J. Wiley and Sons, 1980.