Chin. Ann. of Math.
21B: 2(2000),147-152.

UNILATERAL EIGENVALUE PROBLEMS FOR NONLINEARLY ELASTIC PLATES: AN APPROACH VIA PSEUDO-MONOTONE OPERATORS

LILIANA GRATIE*

Abstract

This paper considers a class of variational inequalities that model the buckling of a von Karman plate clamped on a part of its boundary and lying on a flat rigid support. The existence and bifurcation results of D. Goeleven, V. H. Nguyen and M. Thera[6] rely on the Leray-Schauder degree. Using the topological degree for pseudo-monotone operators of type (S_+), the author establishes a more general existence result for such unilateral eigenvalue problems.

Keywords Variational inequalities, Topological degree, Generalized monotone operators, Unilateral eigenvalue problem, Nonlinearly elastic plates
1991 MR Subject Classification 35A15, 35S05, 49J40
Chinese Library Classification O175.3, O176 **Document Code** A
Article ID 0252-9599(2000)02-0147-06

§1. The Topological Degree for Generalized Monotone Operators

Monotone operator theory is often an efficient tool for proving the existence of solutions to nonlinear problems. In particular, the pseudo-monotone operators introduced by Brezis[2] are very useful for studying nonlinear elliptic problems. We consider here the following classes of mappings of generalized monotone type.

(QM)=the class of quasi-monotone operators,

(S_+) =the class of operators of type (S_+),

(PM)=the class of pseudo-monotone operators,

(LS)=the class of Leray-Schauder operators (compact perturbations of the identity).

Throughout this section, X is a real reflexive Banach space with norm $\|\cdot\|$ and X^* denotes its dual space with norm still denoted by $\|\cdot\|$. We let $\langle\cdot,\cdot\rangle$ denote the pairing between X^* and X, in the sense that $\langle f,u\rangle = f(u)$ for all $f\in X^*$ and $u\in X$.

Definition 1.1. (a) *The operator $A : X \to X^*$ is of type (S_+) if any sequence $\{u_n\}$ in X that weakly converges to u in X and satisfies*

$$\limsup\langle Au_n, u_n - u\rangle \le 0 \quad \textit{as } n\to\infty \tag{1.1}$$

is in fact strongly convergent in X.

Manuscript received October 29, 1999.

*Faculty of Engineering–Brăila, Dunărea de Jos University of Galati, Calea Călărasilor 29, Brăila 6100, Romania. **E-mail:** gratie@starnets.ro

(b) *The operator $A : X \to X^*$ is pseudo-monotone if any sequene $\{u_n\}$ that weakly converges to u in X and satisfies* (1.1) *is such that*

$$\langle Au_n, u_n - u\rangle \to 0 \quad as\ n \to \infty$$

and Au_n weakly converges to Au.

(c) *The operator $A : X \to X^*$ is quasi-monotone if any sequence $\{u_n\}$ that weakly converges to u in X satisfies*

$$\limsup\langle Au_n - Au, u_n - u\rangle \geq 0 \quad as\ n \to \infty.$$

Following [12], we recall the inclusions

$$(\mathrm{LS}) \subset (S_+) \subset \mathrm{PM}) \subset (\mathrm{QM}).$$

A basic relation between quasi-monotone operators and mappings of type (S_+), due to Calvert and Web[4], is given below:

Theorem 1.1. (a) *If $A \in (S_+)$ and $B \in (\mathrm{QM})$, then $(A + B) \in (S_+)$.*

(b) *If $(A + B) \in (S_+)$ for all $A \in (S_+)$, then $B \in (\mathrm{QM})$.*

This theorem plays an essential role in proving the main result of the next section. The concept of topological degree has been defined for more and more comprehensive classes of nonlinear single-valued or multi-valued mappings arising in the operator equations. The original definitions of a degree for operators of type (S_+) by Skrypnik[13] and Browder[3] were based on Galerkin approximations, for which the classical Browder degree is defined. The approach followed here relies on the Berkovits-Mustonen[1] procedure. It relies on the Leray-Schauder degree and the Browder-Ton elliptic super-regularization.

Let M be a countable subset of the space X. By a classical result (see [12]), it follows that there exist a separable Hilbert space H and a compact one-to-one linear operator $B : H \to X$ such that $M \subset B(H)$ and $B(H)$ is dense in X. We further define the adjoint operator $P : X^* \to H$ by

$$\langle P(u), \nu\rangle = \langle u, B(\nu)\rangle,$$

for all $\nu \in H, u \in X^*$, where $\langle\cdot,\cdot\rangle$ stands for the inner product in H. Obviously, P is also a linear compact injection.

For a given open bounded subset $\Omega \subset X$, let

$$F_\Omega(S_+) := \{F : \overline{\Omega} \to X^*;\ F \in (S_+),\ \text{bounded and demicontinuous}\}$$

and

$$H_\Omega(S_+) := \{H_t : \overline{\Omega} \to X^*;\ 0 < t \leq 1,\ H_t \text{ bounded homotopy of class } (S_+)\}.$$

With each $F \in F_\Omega(S_+)$, we now associate the family of Leray-Schauder mappings $\{F_\lambda; \lambda > 0\}$ given by $F_\lambda(u) := u + \lambda BPF(u)$, for all $u \in \overline{\Omega}$. Note that for each fixed $\lambda > 0, F_\lambda$ maps $\overline{\Omega}$ into X and has the form $I + C_\lambda$, where $C_\lambda = \lambda BPF$ is compact. Therefore, the Leray-Schauder degree is well defined for the triplet (F_λ, Ω, y) whenever $y \notin F_\lambda(\partial\Omega)$.

Let $F \in F_\Omega(S_+)$, let $A \subset \overline{\Omega}$ be a closed subset such that $0 \notin F(A)$. Using Corollary 4.3 from [12], we get that there exists $\lambda_1 > 0$ such that $0 \notin F_\lambda(A)$ for all $\lambda \geq \lambda_1$; moreover, if $0 \notin F(\partial\Omega)$, there exists $\lambda_0 > 0$ such that $d_{LS}(F_\lambda, \Omega, 0)$ has a constant value for $\lambda \geq \lambda_0$.

It is now natural to define the S-degree as follows:

$$d_S(F, \Omega, 0) = d_{LS}(F_\lambda, \Omega, 0),$$

where $\lambda > \lambda_0$, whenever $0 \notin F(\partial\Omega)$.

In addition, for any $y_0 \in X^*\backslash F(\partial\Omega)$, we can define

$$d_S(F, \Omega, y_0) = d_S(F - y_0, \Omega, 0).$$

To verify that we have obtained a genuine topological degree, there are four axioms to be satisfied (see again [12]): existence of the solution, additivity with respect to the domain, invariance under admissible homotopies, and normalization. Furthermore, the uniqueness of the topological degree for operators of type (S_+), named the S-degree, follows from the uniqueness of the Leray-Schauder degree.

§2. A General Existence Result for Unilateral Eigenvalue Problems Involving Generalized Monotone Operators

Let K be a closed, convex cone in the real Hilbert space X, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let U be a bounded open subset of K. Consider the following nonlinear variational eigenvalue problem

$$\text{Find } (u, \lambda) \in U \times \mathbf{R}_+ \text{ such that } \langle Au - \lambda Lu + Cu - f, \nu - u \rangle \geq 0 \text{ for all } \nu \in K, \tag{2.1}$$

where f is given in X, λ is a positive parameter and A, L, C are operators satisfying the following assumptions

(H_1) $A : X \to X$ is linear, continuous, and α-coercive, i.e., there is $\alpha > 0$ such that

$$\langle Au, u \rangle \geq \alpha \|u\|^2 \quad \text{for all} \quad u \in K;$$

(H_2) $L : K \to X$ is continuous, positively homogeneous of order one, i.e.,

$$A(tu) = tA(u) \quad \text{for all} \quad u \in X \quad \text{and} \quad t > 0.$$

(H_3) $C : K \to X$ is contiunous, positively homogeneous of order three, i.e.,

$$C(tu) = t^3 C(u) \quad \text{for all} \quad u \in X \quad \text{and} \quad t > 0,$$

and satisfies

$$\langle Cu, u \rangle > 0 \quad \text{for all } u \in K\backslash\{0\}.$$

If $f = 0$, then (2.1) has the trivial solution $u = 0$, which, for a plate problem, corresponds to an equilibrium without buckling. When $f \neq 0$ and λ increases from zero onward, buckling occurs and we are interested in the modeling of this phenomenon. In [6], all the existence and bifurcation results depend on arguments using the Leray-Schauder degree. The present analysis improves on [6], by means of a more general topological degree.

Let $F(u, \lambda) = Au - \lambda Lu + Cu - f$ where the sum $(-\lambda L + C)$ is a quasi-monotone operator. Since A is an operator of type (S_+), so is F by Theorem 1.1. Let $K_r = \{x \in K; \|x\| \leq r\}$. Then the topological degree $d_S(F, K_r, 0)$ is well defined.

Theorem 2.1.[8] *Under hypotheses* (H_1), (H_2) and (H_3), there exists $r_0 = r_0(\lambda, f) > 0$ such that, for each $r \geq r_0$,

$$d_S(F, K_r, 0) = 1.$$

Proof. Let U be a bounded open set in X such that (2.1) has no solutions on ∂U. Since the S-degree of F at 0 relative to U is well defined, we may consider the homotopy of type

(S_+)

$$H_\lambda(t,u) = Au + Cu - t(\lambda Lu + f).$$

We claim that there exists $r_0 > 0$ such that the problem

$$\text{Find } (u,\lambda) \in U \times \mathbf{R}_+ \text{ such that } \langle H_\lambda(t,u), v-u\rangle \ge 0, \quad \forall v \in K$$

has no solutions on $\partial U \times K_r$ for $r \ge r_0$ and $t \in [0,1]$, where $K_r = \{x \in K; \|x\| \le r\}$. Indeed, suppose the contrary. Then, we can find sequences $\{u_n\}$ and $\{t_n\}$ such that $\|u_n\| \to \infty$ and

$$\langle Cu_n + Au_n, v - u_n\rangle \ge t_n\langle \lambda Lu_n + f, v - u_n\rangle, \quad \forall v \in K.$$

Taking $v = 0$, we obtain

$$\langle Cu_n + Au_n, u_n\rangle \le t_n\langle \lambda Lu_n + f, u_n\rangle, \quad \forall v \in K. \tag{2.2}$$

We prove that there exists $\varepsilon > 0$ such that $\langle Cu_n, u_n\rangle \ge \varepsilon\|u_n\|^4$ for all $n \in \mathbf{N}$. Otherwise, setting $v_n = \frac{u_n}{\|u_n\|}$, we would obtain $\langle Cv_n, v_n\rangle \to 0$ as $n \to \infty$. Since we may assume that $v_n \to v_0 \in K$, we have $\langle Cv_0, v_0\rangle = 0$ by the strong continuity of C, and therefore $v_0 = 0$ by virtue of (H_3).

Using (2.2), (H_1) and (H_3), we get

$$t_n\lambda\langle Lu_n, u_n\rangle \ge \alpha\|u_n\|^2 - t_n\langle f, u_n\rangle.$$

We may asume that $t_n \to t_0 \in [0,1]$. Dividing by $\lambda\|u_n\|^2$ and letting $n \to \infty$ we obtain

$$t_0\langle Lv_0, v_0\rangle \ge \frac{\alpha}{\lambda} > 0,$$

which is a contradiction for λ small enough, because $L(v_0, v_0)$ is a constant. Using again (2.2), hypotheses (H_1), (H_2), (H_3) and the previous estimate, we get

$$\varepsilon\|u_n\|^4 + \alpha\|u_n\|^2 \le \langle Cu_n + Au_n, u_n\rangle \le |\lambda|\|Lu_n\|\,\|u_n\| + \|f\|\,\|u_n\|$$

and dividing by $\|u_n\|^4$, we infer that

$$\varepsilon + \|u_n\|^{-2}(\alpha - \lambda\|L\|) - \|f\|\,\|u_n\|^{-4} \le 0.$$

Taking the limit as $n \to \infty$, we obtain that $\varepsilon \le 0$, which is a contradiction.

Using now the homotopy invariance property of the S-degree, we get

$$d_S(F_\lambda, K_r, 0) = d_S(H_\lambda(1,\cdot), K_r, 0) = d_S(H_\lambda(0,\cdot), K_r, 0) = d_S(A + C, K_r, 0).$$

Define another homotopy by $G(t,u) = Au + tCu$. We claim that, for each $r > 0$, the problem

$$\text{Find } u \in U \text{ such that } \langle G(t,u), v-u\rangle \ge 0 \text{ for all } v \in K$$

has no solution on ∂K_r for $t \in [0,1]$. Indeed, suppose the contrary. Then there exist $r > 0, s \in [0,1]$, and $y \in \mathbf{K}$ with $\|y\| = r$, such that

$$\langle Ay + sCy, v - y\rangle \ge 0 \text{ for all } v \in K.$$

For $v = 0$, we get

$$\langle Ay + sCy, y\rangle \le 0,$$

and by hypotheses (H_1) and (H_3), it follows that $\alpha\|y\|^2 \le 0$. This yields $y = 0$, a contradiction. Therefore

$$d_S(F, K_r, 0) = d_S(A + C, K_r, 0) = d_S(G(1,\cdot), K_r, 0) = d_S(G(0,\cdot), K_r, 0) = d_S(A, K_r, 0).$$

Since A is coercive, $Au = 0$ has a solution and thus

$$d_S(F, K_r, 0) = 1.$$

Hence the proof is complete.

We are now in a position to prove a general existence result for nontrivial solutions.

Theorem 2.2. *Assume that assumptions* (H_1), (H_2) *and* (H_3) *hold and let* $f \in X$ *be fixed. If there exists* $u_0 \in K$ *such that* $\langle f, u_0 \rangle > 0$, *then for each* $\lambda \in \mathbf{R}_+$, *there exists a nontrivial solution* $u(\lambda) \in K$ *of the problem* (2.1).

Proof. The existence of a solution for (2.1) follows from Theorem 2.1 and from the existence property of the S-degree. For zero to be a solution, it is necessary that $\langle f, \nu \rangle \le 0$ for all $\nu \in K$, and thus $u(\lambda) \ne 0$.

§3. Application to Variational Inequalities of Von Karman Type

For a nonlinearly elastic plate with unilateral conditions, subjected to a body force of density f, the equilibrium of the plate is governed by a variational inequality of type (2.1), and if there exists $u_0 \in K$ such that $\langle f, u_0 \rangle > 0$, then we may apply the above theorem to get the existence of an equilibrium for any $\lambda \in \mathbf{R}_+$.

Let there be given a thin plate, identified with the closure of a bounded, open subset Ω of $\mathbf{R}^2$, with a boundary $\partial\Omega$ of class C^1. Assume that the plate is clamped on a part Γ_0 of its boundary $\partial\Omega$ and simply supported on the remaining part of the boundary. Define the space

$$X := \Big\{ u \in H^2(\Omega) : u = 0 \ \text{ on } \ \Gamma, \ \frac{\partial u}{\partial n} = 0, \ \text{ a.e. on } \ \Gamma_0 \Big\},$$

and let the set K of admissible displacements be the closed convex cone of X defined by

$$K := \{ u \in X : u \ge 0 \ \text{ a.e. } \ \text{in} \ \ \Omega \}.$$

The equilibrium of a nonlinearly elastic plate subjected to unilateral conditions is governed by the following variational inequalities

$$\text{Find } u \in K \text{ and } \lambda \in \mathbf{R}_+ \ \text{ such that } \langle u - \lambda Lu + Cu - f, \nu - u \rangle \ge 0 \ \text{ for all } \ \nu \in K, \quad (3.1)$$

where L is a linear operator describing the lateral loading in the plane of the plate, C is a "cubic" nonlinear operator generalizing that introduced in the von Karman nonlinear theory of plates (see [5, Chapter 5]), f is the density of the vertical force, λ is a positive parameter measuring the magnitude of the lateral loading, and u is the unknown transverse displacement. Applying Theorem 2.2 with $A = I$, we obtain the existence of solutions for (3.1), within the theory of pseudo-monotone operators of type (S_+), which is more appropriate for the study of variational inequalities.

For a thorough account of variational inequalities arising in contact problems in elasticity, involving plates and shells, see [10].

Remark 2.1. It is also possible to study the bifurcation for nonlinear eigenvalue problems modeled by variational inequalities of von Karman type, using the topological degree for pseudo-monotone operators (see [11]). As for the bifurcation for nonlinear equations, there are two main approaches for the bifurcation problem of variational inequalities, namely, variational and topological approaches. The problem of global bifurcation for variational inequalities of von Karman type is studied in [9], using the index jump condition.

Remark 2.2. The variational formulation of engineering problems often leads to variational inequalities, which are noncoercive. The lack of coerciveness is due to boundary conditions that are insufficiently blocking up or to the presence of a destabilizing term depending on a parameter, as is the case for unilateral buckling in elasticity. In recent years, engineers and mathematicians have focused their attention on noncoercive inequalities, using several different approaches, such as the theory of recession functions, the Leray-Schauder theory, and the critical point theory. A new recession notion, the set of asymptotic "bad" directions is especially useful for general noncoercive variational or hemivariational inequalities. Proving that this set is empty shows that the respective inequality has at least one solution. In [7], the recession analysis is applied to von Karman noncoercive inequalities.

References

[1] Berkovits, J. & Mustonen, V., On the degree theory for mappings of monotone type [J], *Nonlinear Anal.,* **10**(1986), 1373–1383.

[2] Brezis, H., Equations et inequations non-lineaires dans les espaces vectoriels en dualite [J], *Ann. Institute Fourier Grenoble,* **18**(1968), 115–176.

[3] Browder, F. E., Degree of mapping for nonlinear mappings of monotone type [J], *Proc. Nat. Acad. Sci. USA*, **80**(1983), 1771–1773.

[4] Calvert, B. & Webb, J. R. L., An existence theorem for quasimonotone operators [J], *Atii. Acad. Nat. Lincei,* **50**(1972), 362-368.

[5] Ciarlet, P. G., Mathematical elasticity, Volume II: Theory of plates [M], Elsevier, North-Holland, Amsterdam, 1997.

[6] Goeleven, D., Nguyen, V. H. & Thera, M., Nonlinear eigenvalue problem governed by variational inequality of von Karman type—a degree theoretic approach [J], *Topol. Methods in Nonlinear Anal.,* **2**(1993), 253–276.

[7] Gratie, L. & Pascali, D., Recession methods for noncoercive variational inequalities [J], *Libertas Math.,* **17**(1997), 155–164.

[8] Gratie, L., Unilateral eigenvalue problems for nonlinearly elastic plates and pseudo-monotone operators [J], *C. R. Acad. Sci. Paris,* **327:**I(1998), 959–964.

[9] Gratie, L. & Pascali, D., The index jump principle and bifurcation results for nonlinearly elastic plates [J], 1999 (to appear).

[10] Khludnev, A. M. & Sokolowski, J., Modelling and control in solid mechanics [M], Birkhhauser Verlag, 1997.

[11] Le, V. K. & Schmitt, K., Global bifurcation in variational inequalities: Applications to obstacle and unilateral inequalities [M], A. M. S. **123**, Springer-Verlag, New York, 1997.

[12] Pascali, D., Variational and topological methods in nonlinear analysis—Topological degree for monotone mappings [M], Lecture Notes, New York University, Courant Institute, 1997.

[13] Skrypnik, I. V., Nonlinear elliptic equations of higher order [M], Naukova Dumka, Kiev, 1973.