Chin. Ann. of Math.
21B: 2(2000),153-164.

L_p–L_q DECAY ESTIMATES FOR HYPERBOLIC EQUATIONS WITH OSCILLATIONS IN COEFFICIENTS

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Abstract

This work is concerned with the proof of $L_p - L_q$ decay estimates for solutions of the Cauchy problem for $u_{tt} - \lambda^2(t)b^2(t) \triangle u = 0$. The coefficient consists of an increasing smooth function λ and an oscillating smooth and bounded function b which are uniformly separated from zero. The authors' main interest is devoted to the critical case where one has an interesting interplay between the growing and the oscillating part.

Keywords $L_p - L_q$ decay estimates, Wave equation, Fourier multipliers
1991 MR Subject Classification 35L70, 35L80, 35B20
Chinese Library Classification O175.27, O175.2 **Document Code** A
Article ID 0252-9599(2000)02-0153-12

§1. Introduction

To prove global existence results for the solutions of the Cauchy problem for nonlinear wave equations so-called $L_p - L_q$ decay estimates for the solutions of the linear wave equation play an essential role[3,4,7]. That is the following estimate due to Strichartz[12]: there exist constants C and L depending on p and n such that

$$\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} + \|\nabla_x u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{W_p^L(\mathbb{R}^n)}, \tag{1.1}$$

where $1 < p \le 2,\ 1/p + 1/q = 1$ and $u = u(t,x)$ is the solution to

$$u_{tt} - \triangle u = 0,\ u(0,x) = 0,\ u_t(0,x) = u_1(x) \in C_0^\infty(\mathbb{R}^n).$$

The present paper is devoted to the study of the influence of the time variable in the coefficients of the main part on $L_p - L_q$ decay estimates. To illustrate our results consider the model problem

$$u_{tt} - \exp(2t^\alpha)b^2(t) \triangle u = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \ \alpha \in \mathbb{R}, \tag{1.2}$$

where $b = b(t)$ is a 1-periodic, non-constant, smooth and positive function. The following classification for (1.2) with $\alpha \in \mathbb{R}$ holds:

Manuscript received October 18, 1999.

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(1) If $\alpha \in (-\infty, 1/2)$, then there are no $L_p - L_q$ decay estimates for the solutions.

(2) If $\alpha \in (1/2, \infty)$, then there are $L_p - L_q$ decay estimates for the solutions.

(3) If $\alpha = 1/2$ and if the spatial dimension n is sufficiently large, then we have $L_p - L_q$ decay estimates, too.

The influence of oscillations. We show that oscillations in the coefficients can have a negative influence on $L_p - L_q$ decay estimates. Therefore we take the Cauchy problem

$$u_{tt} - b^2(t) \triangle u = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \tag{1.3}$$

where $b = b(t)$ is as in (1.2) and u_0, u_1 belong to $C_0^\infty(\mathbb{R}^n)$. Due to Gronwall's inequality the following energy estimate for the solution of (1.3) holds:

$$\|u_t(t,\cdot)\|_{L_2(\mathbb{R}^n)} + \|\nabla_x u(t,\cdot)\|_{L_2(\mathbb{R}^n)} \le C \exp(C_0 t)(\|u_0\|_{W_2^1(\mathbb{R}^n)} + \|u_1\|_{L_2(\mathbb{R}^n)})$$

for all $t \in [0, \infty)$. This estimate is faraway from decay estimates. It seems to be a surprise that nevertheless this estimate is very near to the optimal one. Thus in general we cannot expect $L_p - L_q$ decay estimates for the solutions of (1.3).

Theorem 1.1.[8] *Let us consider the Cauchy problem* (1.3). *Then there are no constants* q, p, C, L, *and a nonnegative function* f *defined on* $\mathbb{N}$ *such that for every initial data* $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$ *the estimate*

$$\|u_t(m,\cdot)\|_{L_q(\mathbb{R}^n)} + \|\nabla_x u(m,\cdot)\|_{L_q(\mathbb{R}^n)} \le C f(m)(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^L(\mathbb{R}^n)}) \tag{1.4}$$

is fulfilled for all $m \in \mathbb{N}$ *while* $f(m) \to \infty$, $\ln f(m) = o(m)$ *as* $m \to \infty$, $m \in \mathbb{N}$.

Very fast oscillations dominate the increasing part. To classify the oscillations we consider the model equation

$$u_{tt} - \lambda^2(t) b^2(t) \triangle u = 0,$$

where the coefficient consists of an increasing smooth function λ, and of an oscillating smooth bounded function b which is uniformly positive.

Definition 1.1. *Let us suppose that there exists a real* $\beta \in [0,1]$ *such that the following condition is satisfied:*

$$|D_t b(t)| \le c_1(\beta) \colon \frac{\lambda(t)}{\Lambda(t)} (\ln \Lambda(t))^\beta\ ,\ t \in [T, \infty), \tag{1.5}$$

where T *is large and the function* $\Lambda = \Lambda(t)$ *is defined by* $\Lambda(t) := \int_0^t \lambda(\tau) d\tau$. *Then we call the oscillations fast oscillations, slow oscillations, respectively. If* (1.5) *is not satisfied for* $\beta = 1$, *then we call the oscillations very fast oscillations.*

It turns out that the notion of very fast oscillations gives us an exact description of a fairly wide class of equations in which the oscillating part dominates the increasing one. In [9] it is shown that one can prove a statement similar to Theorem 1.1 for the solutions of the Cauchy problems for the equations from this class. Thus, very fast oscillations may destroy $L_p - L_q$ decay estimates.

§2. Oscillations via Growth

2.1. Main Result

The goal of this paper is to prove $L_p - L_q$ decay estimates for the solutions of

$$u_{tt} - \lambda^2(t) b^2(t) \triangle u = 0,\ u(t_0,x) = u_0(x),\ u_t(t_0,x) = u_1(x) \tag{2.1}$$

with $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$ in the case of fast oscillations ($\beta = 1$ in (1.5)). Opposite to slow oscillations the oscillations feel only in this case the influence of dimension n of the spatial variables on the $L_p - L_q$ decay estimates.

We suppose $t_0 \in [T, \infty)$, T is large. All constants appearing in the next conditions are assumed to be positive.

For the function $\lambda \in C^\infty[T, \infty)$, we assume ($t \in [T, \infty)$)

$$c(\ln \Lambda(t))^{-c} \le d_0 \frac{\lambda(t)}{\Lambda(t)} \le \frac{\lambda'(t)}{\lambda(t)} \le d_1 \frac{\lambda(t)}{\Lambda(t)} \le C(\ln \Lambda(t))^C, \tag{2.2}$$

$$|D_t^k \lambda(t)| \le d_k \left(\frac{\lambda(t)}{\Lambda(t)} \right)^k \lambda(t) \qquad \text{for } k \ge 2,\ k \in \mathbb{N}. \tag{2.3}$$

Now the function b is not necessarily a periodic one. In addition to (1.5) with $\beta = 1$ we suppose ($t \in [T, \infty)$)

$$0 < C_0 := \inf_{[T,\infty)} b^2(t) \le C_1 := \sup_{[T,\infty)} b^2(t) < \infty, \tag{2.4}$$

$$|D_t^k b(t)| \le c_k \left(\frac{\lambda(t)}{\Lambda(t)} \ln \Lambda(t) \right)^k \quad \text{for } k \ge 2, \ \ k \in \mathbb{N}. \tag{2.5}$$

Theorem 2.1. *Let us choose a constant N satisfying $N > C_b/(4C_0^2)$, where $C_b := \sup\limits_{[T,\infty)} \Lambda(t)|D_t b(t)|/(\lambda(t) \ln \Lambda(t))$. We define*

$$r_0 := 1 + 2\kappa C_{0,0}/N + \kappa C_1 N,$$

where $C_{0,0}$ is the constant from Lemma 4.3 and $\kappa > 1$ is suitably chosen. If $\frac{(n-1)}{2}(\frac{1}{p} - \frac{1}{q}) > r_0$ and if the conditions (1.5) for $\beta = 1$, (2.2) to (2.5) are satisfied, then for every small $\varepsilon > 0$ there exists a constant $T(\varepsilon, \kappa)$ such that the decay estimate

$$\begin{aligned} &\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} + \|\nabla_x u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \\ &\le C_{n,N,\varepsilon} (\ln \Lambda(t_0))^{2n+1} \Lambda(t_0)^{2C_{0,0}/N} \frac{1}{\sqrt{\lambda(t_0)}} \\ &\quad \times \Lambda(t)^{-\frac{(n-1)}{2}(\frac{1}{p}-\frac{1}{q}) + r_0 + 2\varepsilon} \big(\lambda(t_0) \|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^L(\mathbb{R}^n)} \big) \end{aligned}$$

holds for the solution $u = u(t,x)$ to the Cauchy problem (2.1), where $t \in [t_0, \infty), t_0 \in [T(\varepsilon, \kappa), \infty)$. Here $L = [n(\frac{1}{p} - \frac{1}{q})] + 1$. The constants $C_{n,N,\varepsilon}$ and r_0 depend on the behaviour of λ and b and of its first two derivatives on the interval $[T, \infty)$, but do not depend on t_0 (see Remark 7.2).

2.2. Philosophy of the Approach

By F, F^{-1} we denote the Fourier transform, inverse Fourier transform with respect to x, respectively. Applying F to (2.1) we get

$$v_{tt} + \lambda^2(t) b^2(t) |\xi|^2 v = 0, \ v(t_0, \xi) = F(u_0), \ v_t(t_0, \xi) = F(u_1), \tag{2.6}$$

where $v = F(u)$. Setting $U = (U_1, U_2)^T := (\lambda(t)|\xi| v, D_t v)$ the differential equation from (2.6) can be transformed to the system

$$D_t U - \begin{pmatrix} 0 & \lambda(t)|\xi| \\ \lambda(t) b^2(t) |\xi| & 0 \end{pmatrix} U - \frac{D_t \lambda(t)}{\lambda(t)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U = 0. \tag{2.7}$$

Our main object of the following considerations is the fundamental solution of this system, that is, the solution $\mathcal{U} = \mathcal{U}(t,\tau,\xi) \in C^\infty([\tau,\infty)\times \mathbb{R}^n)$ of the Cauchy problem

$$D_t\mathcal{U} - \begin{pmatrix} 0 & \lambda(t)|\xi| \\ \lambda(t)b^2(t)|\xi| & 0 \end{pmatrix} \mathcal{U} - \frac{D_t\lambda(t)}{\lambda(t)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{U} = 0, \tag{2.8}$$

$$\mathcal{U}(\tau,\tau,\xi) = I \quad \text{(identity matrix)}. \tag{2.9}$$

We prove that $\mathcal{U}(t,t_0,\xi)$ can be represented in the form

$$\mathcal{U}(t,t_0,\xi) = \sum_{l=+,-} \mathcal{U}^l(t,t_0,\xi)\exp\Big(li\int_{t_0}^t \lambda(s)b(s)ds|\xi|\Big), \tag{2.10}$$

where $\mathcal{U}^-(t,t_0,\xi)$ and $\mathcal{U}^+(t,t_0,\xi)$ have connections to symbol classes (see Propositions 5.1 and 5.2). We intend to obtain the representation (2.10) in the set $\{(t,\xi)\in[t_0,\infty)\times(\mathbb{R}^n\setminus\{0\})\}$. Using this representation we obtain the solution of (2.1) in the form

$$u(t,x) = F^{-1}\Big(\frac{\lambda(t_0)}{\lambda(t)}\mathcal{U}_{11}(t,t_0,\xi)F(u_0)(\xi) + \frac{1}{\lambda(t)|\xi|}\mathcal{U}_{12}(t,t_0,\xi)F(u_1)(\xi)\Big),$$

where $\mathcal{U}_{ik}$ are the elements of $\mathcal{U}$. Following the approach of [6] necessary $L_p - L_q$ decay estimates for Fourier multipliers depending on the parameter t will be derived in Section 5.

Zones. We define the pseudodifferential zone by

$$Z_{pd}(t_0,N) := \{(t,\xi) : \xi \in \mathbb{R}^n\setminus\{0\},\ \Lambda(t)|\xi| \le \kappa N\ln\Lambda(t) \ \text{ and } \ t\ge t_0\},$$

the hyperbolic zone by

$$Z_{hyp}(t_0,N) := \{(t,\xi) : \xi \in \mathbb{R}^n\setminus\{0\},\ \Lambda(t)|\xi| \ge N\ln\Lambda(t) \ \text{ and } \ t\ge t_0\},$$

the oscillation's subzone of $Z_{hyp}(t_0,N)$ by

$$Z_{osc}(t_0,N) := \{(t,\xi) : \xi \in \mathbb{R}^n\setminus\{0\}, N\ln\Lambda(t) \le \Lambda(t)|\xi| \le \kappa N\ln^2\Lambda(t) \ \text{ and } \ t\ge t_0\}.$$

Here $\kappa > 1$ is chosen fixed. Then we define the functions $t = t_\xi$ (or $t = t(|\xi|)$) and $p_0 = p_0(t_0)$ in the following way:

$$t_\xi : \Lambda(t_\xi)|\xi| = N\ln\Lambda(t_\xi)\ , \quad p_0 : \Lambda(t_0)p_0 = N\ln\Lambda(t_0),$$

where N is some positive parameter to be fixed later. In what follows we will often write p for $|\xi|$.

Lemma 2.1. *Define for $p\in(0,p_0)$, $N>0$, the function $t=t_p$ (or $t=t(p)$) as the solution to $\Lambda(t_p)p = N\ln\Lambda(t_p)$. Then*

$$\partial_p t_p = -\frac{1}{p}\frac{\Lambda(t_p)}{\lambda(t_p)}\Big(1-\frac{1}{\ln\Lambda(t_p)}\Big)^{-1}, \quad |\partial_p^k t_p| \le C_k p^{-k}\frac{\Lambda(t_p)}{\lambda(t_p)}, \quad k\ge 1.$$

Further, for p_0 small one has

$$\ln\Lambda(t_p) \le -2\ln p \le 2\ln\Lambda(t_p) \quad \textit{when} \quad p \le 2p_0 \le 1/2.$$

We define the functions $t_{\rm osc} = t_{{\rm osc},\xi}$ (or $t_{\rm osc} = t_{\rm osc}(|\xi|)$) and $p_{\rm osc} = p_{\rm osc}(t_0)$ by

$$t_{{\rm osc},\xi} : \Lambda(t_{{\rm osc},\xi})|\xi| = N\ln^2\Lambda(t_{{\rm osc},\xi}), \quad p_{\rm osc} : \Lambda(t_0)p_{\rm osc} = N\ln^2\Lambda(t_0).$$

Lemma 2.2. *Define for $p\in(0,p_{\rm osc})$, $N>0$, the function $t_{\rm osc} = t_{{\rm osc},p}$ (or $t_{\rm osc} = t_{\rm osc}(p)$) as the solution to $\Lambda(t_{{\rm osc},p})p = N\ln^2\Lambda(t_{{\rm osc},p})$. Then*

$$\partial_p t_{{\rm osc},p} = -\frac{1}{p}\frac{\Lambda(t_{{\rm osc},p})}{\lambda(t_{{\rm osc},p})}\Big(1-\frac{1}{\ln\Lambda(t_{{\rm osc},p})}\Big)^{-1}, \quad |\partial_p^k t_{{\rm osc},p}| \le C_k p^{-k}\frac{\Lambda(t_{{\rm osc},p})}{\lambda(t_{{\rm osc},p})}, \quad k\ge 1.$$

For small $p_{\rm osc}$ *one has*

$$\ln \Lambda(t_{{\rm osc},p}) \leq -2\ln p \leq 2\ln \Lambda(t_{{\rm osc},p}) \quad when \quad p \leq 2p_{\rm osc} \leq 1/2.$$

§3. Consideration in $Z_{pd}(t_0, N)$

The system (2.8) is spherically symmetric. Therefore we start with the auxiliary system

$$D_t\mathcal{V} - \begin{pmatrix} 0 & \lambda(t)p \\ \lambda(t)b^2(t)p & 0 \end{pmatrix}\mathcal{V} - \frac{D_t\lambda(t)}{\lambda(t)}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathcal{V} = 0, \tag{3.1}$$

$$\mathcal{V}(\tau,\tau,p) = I \quad \text{(identity matrix)}, \tag{3.2}$$

where $p \in \mathbb{R}_+$. The relation $\mathcal{U}(t,\tau,\xi) = \mathcal{V}(t,\tau,|\xi|)$ brings in the dimension n and leads to one condition providing decay estimates. Using matrizant we can write $\mathcal{V} = \mathcal{V}(t,t_0,p)$ explicitly:

$$\mathcal{V}(t,t_0,p) = I + \sum_{j=1}^{\infty}\int_{t_0}^{t} A(t_1,p)\int_{t_0}^{t_1} A(t_2,p)\cdots\int_{t_0}^{t_{j-1}} A(t_j,p)dt_j\cdots dt_1, \tag{3.3}$$

where

$$A(t,p) := \begin{pmatrix} 0 & \lambda(t)p \\ \lambda(t)b^2(t)p & 0 \end{pmatrix} + \frac{D_t\lambda(t)}{\lambda(t)}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using the considerations from [10] and the definition of $Z_{pd}(t_0,N)$ one can prove

$$\|\mathcal{V}(t,t_0,p)\| \leq \frac{\lambda(t)}{\lambda(t_0)}\min\{\Lambda(t)^{\kappa C_1 N}, p^{-(1+\varepsilon)\kappa C_1 N}\} \quad \text{for all} \quad p = |\xi|,\ (t,\xi)\in Z_{pd}(t_0,N),$$

with C_1 from (2.4). The definition of $Z_{pd}(t_0,N)$ implies here that $\Lambda(t) \leq p^{-(1+\varepsilon)}$ for all $t \geq T = T(\varepsilon)$, $\varepsilon > 0$ arbitrary. In particular one can also prove that

$$\|\mathcal{V}(t_p,t_0,p)\| \leq \frac{\lambda(t_p)}{\lambda(t_0)}p^{-(1+\varepsilon)C_1 N} \quad \text{for all} \quad p = |\xi|,\ (t_\xi,\xi)\in Z_{pd}(t_0,N).$$

To estimate derivatives $D_p^k\mathcal{V}$ we use (3.3). For $k \geq 1$ we have

$$\begin{aligned} D_p^k\mathcal{V}(t,t_0,p) &= \int_{t_0}^{t} D_pA(t_1,p)\int_{t_0}^{t_1} D_pA(t_2,p)\cdots\int_{t_0}^{t_{k-1}} D_pA(t_k,p)dt_k\cdots dt_1 \\ &+ \sum_{j=k+1}^{\infty}\sum_{\substack{k_1+\cdots+k_j=k \\ 0\leq k_1\leq 1,\cdots,0\leq k_j\leq 1}}\frac{k!}{k_1!\cdots k_j!}\int_{t_0}^{t} D_p^{k_1}A(t_1,p)\cdots\int_{t_0}^{t_{j-1}} D_p^{k_j}A(t_j,p)dt_j\cdots dt_1. \end{aligned}$$

Careful calculations lead to

$$\begin{aligned} \|D_p^k\mathcal{V}(t,t_0,p)\| &\leq \min\Big\{C_k\Big(\kappa N\ln\Lambda(t) + \ln\frac{\lambda(t)}{\lambda(t_0)}\Big)^k\Lambda(t)^{\kappa C_1 N},\ C_{k,\varepsilon}p^{-\varepsilon k-(1+\varepsilon)\kappa C_1 N}\Big\} \\ &\quad\times p^{-k}\frac{\lambda(t)}{\lambda(t_0)}, \quad p = |\xi|, \quad (t,\xi)\in Z_{pd}(t_0,N); \\ \|D_p^k\mathcal{V}(t_p,t_0,p)\| &\leq \min\Big\{C_k\Big(N\ln\Lambda(t_p) + \ln\frac{\lambda(t_p)}{\lambda(t_0)}\Big)^k\Lambda(t_p)^{C_1 N},\ C_{k,\varepsilon}p^{-\varepsilon k-(1+\varepsilon)C_1 N}\Big\} \\ &\quad\times p^{-k}\frac{\lambda(t_p)}{\lambda(t_0)}, \quad p = |\xi|, \quad (t_\xi,\xi)\in Z_{pd}(t_0,N); \end{aligned}$$

respectively. To estimate $D_t^l D_p^k\mathcal{V}(t,t_0,p)$ we use (3.1) and induction principle on l. But we restrict ourselves to a direct consequence of the estimates for $D_t^l D_p^k\mathcal{V}(t,t_0,p)$.

Proposition 3.1. *For every given positive number ε and for every k and α the following estimates hold for (t,ξ), $(t_\xi,\xi)\in Z_{pd}(t_0,N)$:*

$$\|D_t^k D_\xi^\alpha \mathcal{U}(t,t_0,\xi)\| \le \min\Big\{C_{k,\alpha}\Big(\kappa N\ln\Lambda(t)+\ln\frac{\lambda(t)}{\lambda(t_0)}\Big)^{|\alpha|}\Lambda(t)^{\kappa C_1 N},$$
$$C_{k,\alpha,\varepsilon}|\xi|^{-\varepsilon|\alpha|-(1+\varepsilon)\kappa C_1 N}\Big\}\,|\xi|^{-|\alpha|}\frac{\lambda(t)}{\lambda(t_0)}\Big(\lambda(t)|\xi|+\frac{\lambda(t)}{\Lambda(t)}\Big)^k(\ln\Lambda(t))^{\max\{k-1,0\}},$$
$$\|D_t^k D_\xi^\alpha \mathcal{U}(t_\xi,t_0,\xi)\| \le \min\Big\{C_{k,\alpha}\Big(N\ln\Lambda(t_\xi)+\ln\frac{\lambda(t_\xi)}{\lambda(t_0)}\Big)^{|\alpha|}\Lambda(t_\xi)^{C_1 N},$$
$$C_{k,\alpha,\varepsilon}|\xi|^{-\varepsilon|\alpha|-(1+\varepsilon)C_1 N}\Big\}|\xi|^{-k}\frac{\lambda(t_\xi)}{\lambda(t_0)}\Big(\lambda(t_\xi)|\xi|+\frac{\lambda(t_\xi)}{\Lambda(t_\xi)}\Big)^k(\ln\Lambda(t_\xi))^{\max\{k-1,0\}}$$

for all $t_0\ge T$. Here $C_{k,\alpha}$ and $C_{k,\alpha,\varepsilon}$ are independent of t_0 while T is independent of k and α.

§4. Consideration in Oscillation's Subzone

In $Z_{\rm osc}(t_0,N)$ we will carry out only one step of diagonalization of Subsection 2.1.7 (see [13]). As it is noted there the next steps of perfect diagonalization are useless. Indeed, when $\beta=1$ in (1.5) we lose the large parameter in a neighborhood of $t=t_\xi$. This large parameter helps to get an asymptotic expansion and to appeal to Brenner's lemma[1].

One step of diagonalization allows in the study of Fourier multipliers to apply Hardy-Littlewood theorem not only in $Z_{pd}(t_0,N)$, but in $Z_{\rm osc}(t_0,N)$, too. This will be the other strategy to study the critical case $\beta=1$ in (1.5). To carry out one step of perfect diagonalization in $Z_{\rm osc}(t_0,N)$ and in general more steps in the remaining part of $Z_{hyp}(t_0,N)$ we need the following classes of symbols (cf. [11] and [13]).

Definition 4.1. *For given real numbers m_1,m_2,m_3 and for positive N we denote by $S_{t_0,N}\{m_1,m_2,m_3\}$ the set of all symbols $a=a(t,\xi)\in C^\infty(Z_{hyp}(t_0,N))$ satisfying*

$$|D_t^l D_\xi^\alpha a(t,\xi)|\le C_{l,\alpha}|\xi|^{m_1-|\alpha|}\lambda(t)^{m_2}\Big(\frac{\lambda(t)}{\Lambda(t)}\ln\Lambda(t)\Big)^{m_3+l},\quad (t,\xi)\in Z_{hyp}(t_0,N)$$

for all multi-indices α and all l, where the constants $C_{l,\alpha}$ are independent of t_0.

Let us define the matrices

$$M^{-1}(t):=\frac{1}{\sqrt{\lambda(t)b(t)}}\begin{pmatrix}1 & 1\\ -b(t) & b(t)\end{pmatrix},\quad M(t):=\frac12\sqrt{\frac{\lambda(t)}{b(t)}}\begin{pmatrix}b(t) & -1\\ b(t) & 1\end{pmatrix}. \tag{4.1}$$

Setting $\mathcal{U}=M^{-1}\mathcal{W}$, by some calculations we transform system (2.8) into

$$D_t\mathcal{W}-\mathcal{D}(t,\xi)\mathcal{W}+B(t,\xi)\mathcal{W}=0, \tag{4.2}$$

where

$$\mathcal{D}(t,\xi):=\begin{pmatrix}\tau_1(t,\xi) & 0\\ 0 & \tau_2(t,\xi)\end{pmatrix},\quad B(t,\xi):=-\frac12\frac{D_t(\lambda(t)b(t))}{\lambda(t)b(t)}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$\tau_1(t,\xi):=-\lambda(t)b(t)|\xi|+\frac{D_t\lambda(t)}{\lambda(t)},\quad \tau_2(t,\xi):=\lambda(t)b(t)|\xi|+\frac{D_t\lambda(t)}{\lambda(t)}.$$

This represents the diagonalization mod $S_{t_0,N}\{0,0,1\}$. The next lemma shows that we can carry out one step of perfect diagonalization in $Z_{\rm osc}(t_0,N)$. For the proof see [11, 13].

Lemma 4.1. *There exist matrix-valued functions $N_1(t,\xi) \in S_{t_0,N}\{0,0,0\}$, $R_1(t,\xi) \in S_{t_0,N}\{-1,-1,2\}$ such that the following operator-valued identity holds:*

$$(D_t - \mathcal{D}(t,\xi) + B(t,\xi))N_1(t,\xi) = N_1(t,\xi)(D_t - \mathcal{D}(t,\xi) - R_1(t,\xi)),$$

where $N_1(t,\xi)$ is invertible and belongs together with its inverse matrix $N_1^{-1}(t,\xi)$ to $S_{t_0,N}\{0,0,0\}$ provided T is sufficiently large and $t_0 \in [T,\infty)$.

Some calculations give for $t \in [T,\infty)$, $T = T(\kappa)$,

$$\|N^{(1)}(t,\xi)\| \le \frac{C_b}{4N\kappa C_0^2} \quad \text{for all } (t,\xi) \in Z_{hyp}(t_0,N), \tag{4.3}$$

with constants C_0 from (2.4) and C_b from Theorem 2.1. Consequently, the condition for N from Theorem 2.1 implies the invertibility of $N_1 := I + N^{(1)}$.

In the following we have to derive estimates for the solution of (2.8), (2.9). Using the spherical symmetry we first devote ourselves to (3.1), (3.2). Our approach makes it necessary to distinguish the cases $|\xi| \le p_0$ and $|\xi| \in [p_0, p_{\rm osc}]$.

Let $E_2 = E_2(t,r,\xi)$ be the matrix-valued function

$$E_2(t,r,\xi) = \frac{\lambda(t)}{\lambda(r)} \begin{pmatrix} \exp(-i|\xi|\int_r^t \lambda(s)b(s)ds) & 0 \\ 0 & \exp(i|\xi|\int_r^t \lambda(s)b(s)ds) \end{pmatrix} \tag{4.4}$$

for $t,r \ge \max\{t_0,t_\xi\}$. We denote $\mathcal{R}_1(t,r,\xi) := E_2(r,t,\xi)R_1(t,\xi)E_2(t,r,\xi)$ and

$$Q_1(t,r,\xi) := \sum_{j=1}^{\infty} i^j \int_r^t \mathcal{R}_1(t_1,r,\xi)dt_1 \int_r^{t_1} \mathcal{R}_1(t_2,r,\xi)dt_2 \cdots \int_r^{t_{j-1}} \mathcal{R}_1(t_j,r,\xi)dt_j. \tag{4.5}$$

4.1. The Case $|\xi| < p_0$.

The starting point of our consideration is the remark that the matrix-valued function

$$M^{-1}(t)N_1(t,p)E_2(t,t_p,p)(I + Q_1(t,t_p,p))N_1(t_p,p)^{-1}M(t_p)\mathcal{V}(t_p,t_0,p),$$

which is defined in $Z_{\rm osc}(t_0,N)$ ($p = |\xi|$, $(t,\xi) \in Z_{\rm osc}(t_0,N)$) and where $\mathcal{V}(t_p,t_0,p)$ is regarded as the value of $\mathcal{V}(t,t_0,p)$ on $t = t_p$, solves (3.1) and coincides with $\mathcal{V}(t_p,t_0,p)$ at $t = t_p$. Hence it coincides with $\mathcal{V}(t,t_0,p)$ everywhere in its domain of definition, that is,

$$\mathcal{V}(t,t_0,p) = M^{-1}(t)N_1(t,p)E_2(t,t_p,p)(I + Q_1(t,t_p,p))N_1(t_p,p)^{-1}M(t_p)\mathcal{V}(t_p,t_0,p).$$

For $\mathcal{V}(t_p,t_0,p)$ and its derivatives $D_t^k D_p^l \mathcal{V}(t_p,t_0,p)$ we can use Proposition 3.1 if we replace there ξ or $|\xi|$ by p and α by l. Lemmas 2.1 and 4.1 help to estimate $N_1(t,p), N_1(t_p,p)^{-1}$, respectively. Matrices $E_2(t,t_p,p), M^{-1}(t)$ and $M(t_p)$ are given explicitly by (4.1) and (4.4). Taking account of $E_2(t,t_p,p) = E_2(t,t_0,p)E_2(t_0,t_p,p))$ it is enough to use the following lemma to estimate $E_2(t_0,t_p,p)$.

Lemma 4.2. *For every positive number ε and every l the following estimates hold:*

$$\left|\partial_p^l \exp\left(ip\int_{t_p}^t \lambda(s)b(s)ds\right)\right| \le \min\{C_{l,\varepsilon}p^{-(1+\varepsilon)l}, C_l\Lambda(t)^l\}, \quad p = |\xi|,\ (t,\xi) \in Z_{pd}(t_0,N),$$

where the constants C_l and $C_{l,\varepsilon}$ are independent of $t_0 \in [T,\infty)$.

It remains to derive estimates for $Q_1(t,t_p,p)$, $\mathcal{R}_1(t,t_p,p)$, respectively.

Lemma 4.3. *The matrix-valued function $\mathcal{R}_1 = \mathcal{R}_1(t,t_p,p)$ satisfies, for every l,k and p, $p = |\xi|$, in $Z_{hyp}(t_0,N) \cap \{|\xi| < p_0\}$, the estimate*

$$\|\partial_t^k \partial_p^l \mathcal{R}_1(t,t_p,p)\| \le C_{k,l}(\lambda(t)p)^k \Lambda(t)^l \frac{\lambda(t)\ln^2\Lambda(t)}{\Lambda^2(t)p},$$

where the constants $C_{k,l}$ *are independent of* $t_0 \in [T,\infty)$.

Corollary 4.1. *The matrix-valued function* $\mathcal{R}_1 = \mathcal{R}_1(t,t_p,p)$ *satisfies, for every* $l,k \le M$ *and every* p , $p = |\xi|$, *in* $Z_{osc}(t_0,N) \cap \{|\xi| < p_0\}$, *the estimate*

$$\|\partial_t^k \partial_p^l \mathcal{R}_1(t,t_p,p)\| \le C_{M,N,\varepsilon}(\lambda(t)p)^k p^{-(1+\varepsilon)l} \frac{\lambda(t)\ln^2\Lambda(t)}{\Lambda^2(t)p},$$

where the constants $C_{M,N,\varepsilon}$ *are independent of* $t_0 \in [T,\infty)$.

The following lemma gives an estimate for $\partial_p^r Q_1(t,t_p,p)$. The proof one can find in [11].

Lemma 4.4. *The matrix-valued function* $Q_1 = Q_1(t,t_p,p)$ *satisfies, for every* $r \le M$ *and every* p, $p = |\xi|$, *in* $Z_{osc}(t_0,N) \cap \{|\xi| < p_0\}$, *the estimate*

$$\|\partial_p^r Q_1(t,t_p,p)\| \le C_{M,N,\varepsilon} p^{-\varepsilon - r - 2\kappa C_{0,0}/N}, \tag{4.6}$$

where the constants $C_{M,N,\varepsilon}$ *are independent of* $t_0 \in [T,\infty)$. *The constant* $C_{0,0}$ *is taken from Lemma* 4.3 $(k = l = 0)$.

Proposition 4.1. *The fundamental solution* $\mathcal{V} = \mathcal{V}(t,t_0,p)$ *to the Cauchy problem, that is, the solution to* (3.1), (3.2), *satisfies for all* r, $r \le M$, $t_0 \in [T,\infty)$, $p = |\xi|$, *in* $Z_{\rm osc}(t_0,N) \cap \{|\xi| < p_0\}$ *the estimates*

$$\|\partial_p^r \mathcal{V}(t,t_0,p)\| \le C_{M,N,\varepsilon} \frac{\sqrt{\lambda(t_p)\lambda(t)}}{\lambda(t_0)} p^{-\varepsilon - r - 2\kappa C_{0,0}/N - \kappa C_1 N}.$$

The following proposition is a direct consequence of the previous one.

Proposition 4.2. *The fundamental solution* $\mathcal{U} = \mathcal{U}(t,t_0,\xi)$, *that is, the solution to* (2.8), (2.9), *satisfies for all* α, $|\alpha| \le M$, $t_0 \in [T,\infty)$, *for* (t,ξ), $(t_{\rm osc},\xi) \in Z_{\rm osc}(t_0,N) \cap \{|\xi| < p_0\}$ *the estimates*

$$\|\partial_\xi^\alpha \mathcal{U}(t,t_0,\xi)\| \le C_{N,M,\varepsilon} \frac{\sqrt{\lambda(t_\xi)\lambda(t)}}{\lambda(t_0)} |\xi|^{-\varepsilon - |\alpha| - 2\kappa C_{0,0}/N - \kappa C_1 N},$$

$$\|\partial_\xi^\alpha \mathcal{U}(t_{\rm osc},t_0,\xi)\| \le C_{N,M,\varepsilon} \frac{\sqrt{\lambda(t_\xi)\lambda(t_{\rm osc})}}{\lambda(t_0)} |\xi|^{-\varepsilon - |\alpha| - 2\kappa C_{0,0}/N - \kappa C_1 N}.$$

4.2. The case $|\boldsymbol{\xi}| \in [\boldsymbol{p_0}, \boldsymbol{p_{\rm osc}}]$.

The considerations in this part are similar to those in the other part of $Z_{\rm osc}(t_0,N)$. We have seen in the previous subsection that the estimates for $\partial_t^k \partial_p^l \mathcal{R}_1(t,t_p,p)$ and $\partial_p^r Q_1(t,t_p,p)$ are determined essentially by the behaviour of $\Lambda(t)$ at $t = t_p$. This brings powers of p. Now we are in the position that there is no influence from $Z_{pd}(t_0,N)$. Thus the behaviour of $\Lambda(t)$ at $t = t_0$ is important. But this leads to the constant $(\ln\Lambda(t_0))^{2n+1}\Lambda(t_0)^{2C_{0,0}/N}$ in Theorem 2.1. Following the approach of the previous subsection one can prove the next lemma.

Lemma 4.5. *For every* r, $r \le M$, *the matrix-valued function* $Q_1 = Q_1(t,t_0,p)$ *fulfills the estimates*

$$\|\partial_p^r Q_1(t,t_0,p)\| \le C_M (\ln\Lambda(t_0))^M \Lambda(t_0)^{2C_{0,0}/N},$$

where C_M *is independent of* $t_0 \in [T,\infty)$.

To complete the estimates in $Z_{\rm osc}(t_0,N)$ we write for $p = |\xi|$, $p \in [p_0, p_{\rm osc}]$,

$$\mathcal{V}(t,t_0,p) = M^{-1}(t)N_1(t,p)E_2(t,t_0,p)(I + Q_1(t,t_0,p))N_1^{-1}(t_0,p)M(t_0),$$

and apply Lemma 4.5.

Proposition 4.3. *The fundamental solution* $\mathcal{V} = \mathcal{V}(t, t_0, p)$ *to the Cauchy problem, that is, the solution to* (3.1), (3.2), *satisfies for all* r, $r \le M$, $t_0 \in [T, \infty)$, $p = |\xi|$, *in* $Z_{\rm osc}(t_0, N) \cap \{|\xi| \in [p_0, p_{\rm osc}]\}$ *the estimates*

$$\|\partial_p^r \mathcal{V}(t, t_0, p)\| \le C_M (\ln \Lambda(t_0))^M \Lambda(t_0)^{2C_{0,0}/N} \sqrt{\frac{\lambda(t)}{\lambda(t_0)}} p^{-r}.$$

Proposition 4.4. *The fundamental solution* $\mathcal{U} = \mathcal{U}(t, t_0, \xi)$ *to the Cauchy problem, that is, the solution to* (2.8), (2.9), *satisfies for all* α, $|\alpha| \le M$, $t_0 \in [T, \infty)$, *for* (t, ξ), $(t_{\rm osc}, \xi) \in Z_{\rm osc}(t_0, N) \cap \{|\xi| \in [p_0, p_{\rm osc}]\}$ *the estimates*

$$\|\partial_\xi^\alpha \mathcal{U}(t, t_0, \xi)\| \le C_M (\ln \Lambda(t_0))^M \Lambda(t_0)^{2C_{0,0}/N} \sqrt{\frac{\lambda(t)}{\lambda(t_0)}} |\xi|^{-|\alpha|},$$

$$\|\partial_\xi^\alpha \mathcal{U}(t_{\rm osc}, t_0, \xi)\| \le C_M (\ln \Lambda(t_0))^M \Lambda(t_0)^{2C_{0,0}/N} \sqrt{\frac{\lambda(t_{\rm osc})}{\lambda(t_0)}} |\xi|^{-|\alpha|}.$$

§5. Consideration in Remaining Part of the Hyperbolic Zone

It remains to estimate the fundamental solution $\mathcal{U} = \mathcal{U}(t, t_0, \xi)$ in the remaining part of $Z_{hyp}(t_0, N)$, that is, in $\{(t, \xi) : \xi \in \mathbb{R}^n \setminus \{0\},\ \Lambda(t)|\xi| \ge N \ln^2 \Lambda(t) \text{ and } t \ge t_0\}$. The lower bound $\ln^2 \Lambda(t)$ ensures that we have now a large parameter. Thus we can carry out further steps of perfect diagonalization. A corresponding result to Lemma 4.1 holds for

$$(D_t - \mathcal{D}(t, \xi) + B(t, \xi)) N_M(t, \xi) = N_M(t, \xi)(D_t - \mathcal{D}(t, \xi) - F_M(t, \xi) - \mathcal{R}_M(t, \xi)).$$

This allows us to apply a hyperbolic-type approach which gives an asymptotic expansion of the amplitudes (see [11,13]).

5.1. The Case $|\boldsymbol{\xi}| \le \boldsymbol{p}_{\rm osc}$.

Here one feels the influence of $Z_{\rm osc}(t_0, N)$. Using Propositions 4.2 and 4.4 and the representation for $\mathcal{U}(t, t_0, \xi)$ we arrive at the next result, where we use (2.2) to obtain

$$\sqrt{\lambda(t_\xi)} \le C_\varepsilon |\xi|^{-1/2-\varepsilon}.$$

Proposition 5.1. *The fundamental solution* $\mathcal{U} = \mathcal{U}(t, t_0, \xi)$ *to the Cauchy problem, that is, the solution to* (2.8), (2.9), *can be represented in the form*

$$\mathcal{U}(t, t_0, \xi) = \sum_{l=+,-} \mathcal{U}^l(t, t_0, \xi) \exp\Big(li \int_{t_0}^t \lambda(s) b(s) ds |\xi|\Big), \tag{5.1}$$

where the matrix-valued amplitudes $\mathcal{U}^-, \mathcal{U}^+$ *satisfy for all* α, $|\alpha| \le (M-1)/2$,

$$\|\partial_\xi^\alpha \mathcal{U}^\pm(t, t_0, \xi)\| \le C_{M,N,\varepsilon} (\ln \Lambda(t_0))^M \Lambda(t_0)^{2C_{0,0}/N} \sqrt{\frac{\lambda(t)}{\lambda(t_0)}} |\xi|^{-\frac{1}{2}-\varepsilon-|\alpha|-2\kappa C_{0,0}/N-\kappa C_1 N}$$

in the remaining part of $Z_{hyp}(t_0, N)$ *for* $|\xi| \le p_{\rm osc}$.

5.2. The Case $|\boldsymbol{\xi}| \ge \boldsymbol{p}_{\rm osc}$.

Proposition 5.2. *The fundamental solution* $\mathcal{U} = \mathcal{U}(t, t_0, \xi)$ *to the Cauchy problem can be represented in the form* (5.1), *where* $\mathcal{U}^-, \mathcal{U}^+$ *satisfy for all* α, $|\alpha| \le (M-1)/2$,

$$\|\partial_\xi^\alpha \mathcal{U}^\pm(t, t_0, \xi)\| \le C_{M,N} \sqrt{\frac{\lambda(t)}{\lambda(t_0)}} |\xi|^{-|\alpha|}$$

in the remaining part of $Z_{hyp}(t_0, N)$ *for* $|\xi| \geq p_{\rm osc}$.

§6. Summary

Theorem 6.1. *Let* $v = v(t, \xi)$ *be the solution to the Cauchy problem*

$$D_t^2 v - \lambda^2(t) b^2(t) |\xi|^2 v = 0, \ v(t_0, \xi) = v_0(\xi), \ D_t v(t_0, \xi) = v_1(\xi).$$

It can be written as

$$v(t, \xi) = \sum_{\substack{l=+,- \\ k=0,1}} a_k^l(t, t_0, \xi) v_k(\xi) \exp\Big(li \int_{t_0}^t \lambda(s) b(s) ds |\xi|\Big),$$

$$D_t v(t, \xi) = \sum_{\substack{l=+,- \\ k=0,1}} b_k^l(t, t_0, \xi) v_k(\xi) \exp\Big(li \int_{t_0}^t \lambda(s) b(s) ds |\xi|\Big),$$

where

$$\begin{aligned} a_0^l(t, t_0, \xi) &= \mathcal{U}_{11}^l(t, t_0, \xi) \lambda(t_0)/\lambda(t), \\ a_1^l(t, t_0, \xi) &= \mathcal{U}_{12}^l(t, t_0, \xi)/(|\xi| \lambda(t)), \\ b_0^l(t, t_0, \xi) &= \mathcal{U}_{21}^l(t, t_0, \xi) |\xi| \lambda(t_0), \\ b_1^l(t, t_0, \xi) &= \mathcal{U}_{22}^l(t, t_0, \xi). \end{aligned}$$

The amplitude functions satisfy the following estimates:

(1) $\lambda(t)|\xi| \, |a_k^l(t, t_0, \xi)| + |b_k^l(t, t_0, \xi)| \leq C_\varepsilon \lambda(t)^{1-k} |\xi|^{1-k-\kappa N C_1 - \varepsilon}, \quad (t, \xi) \in Z_{pd}(t_0, N)$;

(2) $\lambda(t)|\xi| \, |a_k^l(t, t_0, \xi)| + |b_k^l(t, t_0, \xi)| \leq C_{0,N,\varepsilon} \lambda(t_0)^{1-k} \sqrt{\frac{\lambda(t)}{\lambda(t_0)}} \Lambda(t_0)^{2C_{0,0}/N} |\xi|^{-\varepsilon - r_0}, \quad (t, \xi) \in Z_{\rm osc}(t_0, N)$;

(3) $|\partial_\xi^\alpha a_k^l(t, t_0, \xi)| \leq C_{M,N,\varepsilon} \frac{\lambda(t_0)^{1-k}}{\lambda(t)} \sqrt{\frac{\lambda(t)}{\lambda(t_0)}} (\ln \Lambda(t_0))^M \Lambda(t_0)^{2C_{0,0}/N} |\xi|^{-k-\varepsilon-r_0-|\alpha|}$,

$$|\partial_\xi^\alpha b_k^l(t, t_0, \xi)| \leq C_{M,N,\varepsilon} (\lambda(t_0)|\xi|)^{1-k} \sqrt{\frac{\lambda(t)}{\lambda(t_0)}} (\ln \Lambda(t_0))^M \Lambda(t_0)^{2C_{0,0}/N} |\xi|^{-\varepsilon-r_0-|\alpha|}$$

for $|\alpha| \leq (M-1)/2$ *in the remaining part of* $Z_{hyp}(t_0, N)$. *Here* $\varepsilon > 0$ *and* $\kappa > 1$ *can be chosen arbitrarily,* $C_{0,0}$ *and* C_1 *are the constants from Lemma* 4.3 *and* (2.4). *The constant* r_0 *is defined by*

$$r_0 := 1/2 + 2\kappa C_{0,0}/N + \kappa C_1 N.$$

§7. Fourier Multipliers

Theorem 6.1 and (2.6) yield the representation of the solution of (2.1) by the aid of Fourier multipliers. To get $L_p - L_q$ decay estimates for these Fourier multipliers we divide our consideration into two steps in accordance with two completely different ideas: Hardy-Littlewood inequality[2] and Littman lemma[5].

Let us choose a function $\psi \in C^\infty(\mathbb{R}^n)$ satisfying $\psi(\xi) \equiv 0$ for $|\xi| \leq 1/2$, $\psi(\xi) \equiv 1$ for $|\xi| \geq 3/4$ and $0 \leq \psi(\xi) \leq 1$. Moreover, we define

$$K(t) := (4N \ln^2 \Lambda(t))/\Lambda(t).$$

Generalizing the approach of [6] one can prove the next two results.

Theorem 7.1 (Application of Hardy-Littlewood Inequality). *Let us consider Fourier multipliers depending on the parameter $t \in [t_0, \infty), t_0 \in [T, \infty)$, which are defined by*

$$F^{-1}\Big(e^{i\int_{t_0}^{t} \lambda(s)b(s)ds|\xi|}(1-\psi(\xi/K(t)))|\xi|^{-2r}a_0(t,t_0,\xi)F(u_0)(\xi)\Big),\ u_0 \in C_0^\infty(\mathbb{R}^n).$$

Suppose that the following assumption is satisfied for $a_0(t,t_0,\xi)$:

$$|a_0(t,t_0,\xi)| \le C_{N,\varepsilon}|\xi|^{-\varepsilon-r_0}, \quad r_0 = \frac{1}{2} + 2\kappa C_{0,0}/N + \kappa C_1 N,$$

in $Z_{pd}(t_0,N) \cup Z_{osc}(t_0,N)$, where $C_{0,0}$ and C_1 are the constants from Lemma 4.3 and (2.4). Then we have the decay estimate

$$\begin{aligned}&\Big\|F^{-1}\Big((1-\psi(\xi/K(t)))|\xi|^{-2r}a_0(t,t_0,\xi)F(u_0)(\xi)\Big)\Big\|_{L_q(\mathbb{R}^n)}\\ &\le C_r C_{N,\varepsilon}\Lambda(t)^{2r+r_0+2\varepsilon-n(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L_p(\mathbb{R}^n)}\end{aligned}$$

provided that

$$\begin{aligned}&1 < p \le 2, \quad \frac{1}{p}+\frac{1}{q}=1,\\ &0 \le 2r < n\Big(\frac{1}{p}-\frac{1}{q}\Big) - 2\varepsilon - r_0.\end{aligned}$$

Theorem 7.2 (Application of Littman Lemma). *Let us consider*

$$F^{-1}\Big(e^{i\int_{t_0}^{t} \lambda(s)b(s)ds|\xi|}\psi(\xi/K(t))|\xi|^{-2r}a_0(t,t_0,\xi)F(u_0)(\xi)\Big), \quad u_0 \in C_0^\infty(\mathbb{R}^n).$$

Suppose that the following assumption is satisfied for $a_0(t,t_0,\xi)$:

$$|\partial_\xi^\alpha a_0(t,t_0,\xi)| \le C_{M,N,\varepsilon}|\xi|^{-\varepsilon-|\alpha|-r_0},$$

r_0 as above, in $\{(t,\xi) : \xi \in \mathbb{R}^n \setminus \{0\},\ \Lambda(t)|\xi| \ge N\ln^2\Lambda(t)$ and $t \ge t_0\}$. Then we have the decay estimate

$$\begin{aligned}&\Big\|F^{-1}\Big(e^{i\int_{t_0}^{t} \lambda(s)b(s)ds|\xi|}\psi(\xi/K(t))|\xi|^{-2r}a_0(t,t_0,\xi)F(u_0)(\xi)\Big)\Big\|_{L_q(\mathbb{R}^n)}\\ &\le C_r C_{M,N,\varepsilon}\Lambda(t)^{2r+r_0+2\varepsilon-n(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L_p(\mathbb{R}^n)}\end{aligned}$$

provided that

$$\begin{aligned}&1 < p \le 2, \quad \frac{1}{p}+\frac{1}{q}=1,\\ &\frac{(n+1)}{2}\Big(\frac{1}{p}-\frac{1}{q}\Big) \le 2r < n\Big(\frac{1}{p}-\frac{1}{q}\Big) - 2\varepsilon - r_0.\end{aligned}$$

Remark 7.1. It is clear how we feel the dimension n in the critical case. If

$$r_0 := 1/2 + 2C_{0,0}/N + C_1 N < (n-1)/2,$$

then there exist a $\kappa > 1$ and suitable p and q such that the assumptions of Theorems 7.1 and 7.2 are satisfied.

Remark 7.2. At the end of this paper we want to remember the constants from Theorem 2.1. First we have to choose a constant $N > C_b/(4C_0^2)$, for example $N = \gamma C_b/(4C_0^2)$ with $\gamma \ge \gamma_0 > 1$, γ_0 fixed. Then $C_{0,0}/N$ can be estimated uniformly for all $\gamma \ge \gamma_0$ by a constant

$C = C(\gamma_0, C_b, C_0)$. Thus we are motivated to choose $\gamma = \gamma_0$ because of the term $C_1 N$. Now we are in a position to fix n, p, q and $\kappa > 1$ satisfying

$$(n-1)(1/p - 1/q)/2 > r_0 := 1 + 2\kappa C_{0,0}/N + \kappa C_1 N.$$

Finally with a positive small ε we guarantee that the last inequality remains true if we add 2ε to the right-hand side. In this way one can choose the constants needed for our approach and can follow all considerations represented in this paper. The exponent $2n+1$ in $(\ln \Lambda(t_0))^{2n+1}$ follows with Littman's lemma and the condition $|\alpha| \leq (M-1)/2 = n$ in Propositions 5.1 and 5.2. Consequently,

$$C_{M,N,\varepsilon} = C_{n,N,\varepsilon}.$$

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